Value functions for multi-attribute decision problems
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Abstract. Multi-attribute decision problems involve making the correct decision when there is a list of objectives to be optimized. We consider the case where a value function is specified over the attribute of the decision problem, as is typically done in the deterministic phase of a decision analysis. The preference relation between alternatives associated to the value function take into account the preference relation on each attribute and "inter-attribute" relations. The main aim of this paper is to study this preference relation when the multivariate value function satisfies certain properties. In particular we consider supermodularity and Schur-increasingness properties.

Keywords. Multi-attribute decision problem, value function, Schur-increasing functions, supermodular functions.

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1 Introduction
The multi-attribute decision theory is a rapidly developing area both from the point of view of theoretical developments and of empirical applications. Most of its papers were issued in the recent ten years. The interest in this area is explained by the fact most real world decision-making problems are often characterized by the existence of multiple, sometimes conflicting criteria. So in decision situations, not a single but a set of aspects has to be taken in consideration, making the scope of decision making from single criteria problems to more general formulations.

Several models have been proposed as an approach to various types of multi-attribute problems. We suppose that the decision is deterministic and we define the agent’s preferences by a total order over possible outcomes. We also assume that the preference order can be captured by an order-preserving real valued value function. A traditional approach is to use a function that is a simple weight sum where each weight represents the importance given by the decision maker to a particular attribute. Despite its simplicity this approach suffer a major drawback as we can show that using a additive aggregation operator is
equivalent to assuming the independence of criteria.
In this note we consider some classes of value functions that can represent the
importance of a criterion and interaction between criteria.

2 Preliminaries

In this paper we assume that the attribute are real numbers and so we are inter-
ested in functions of several real variables.
We endowed \( \mathbb{R}^n \) with the usual product order. With this order \( \mathbb{R}^n \) become a
lattice and we denote the supremum and the infimum of \( x \) and \( y \) by \( x \lor y \) and
\( x \land y \) respectively.
We also consider the concept of majorization arising as a measure of diversity of
the components of a \( n \)-dimensional vector. Majorization has been comprehen-
sively treated by [1] and [3].
We aim to formalize the idea that the components of a vector \( x \) are less “spread
out” or ”more nearly equal” than the components of \( y \). For a vector \( x \in \mathbb{R}^n \) we
denote its elements ranked in descending order as

\[
x(1) \geq x(2) \geq \ldots \geq x(n)
\]

Thus \( x(1) \) is the largest of the \( x_i \)’s, while \( x(n) \) is the smallest.

**Definition 1.** The vector \( y \) is said to majorize the vector \( x \), which is denoted
as \( x \preceq y \), if

\[
\sum_{i=1}^{k} x(i) \leq \sum_{i=1}^{k} y(i) \quad \text{for} \quad k = 1, 2, \ldots, n - 1 \quad \text{and} \quad \sum_{i=1}^{n} x(i) = \sum_{i=1}^{n} y(i).
\]

Majorization is a partial ordering among vectors, which applies only to vec-
tors having the same sum. It is a measure of the degree to which the vector
elements differ. For example it can be easily shown that all vectors of sum \( s \)
majorize the uniform vector \( u = (\frac{s}{n}, \ldots, \frac{s}{n}) \). Intuitively, the uniform vector is
the vector with minimal differences between elements, so all vectors majorize it.
Formally, this follows from the fact that for any vector \( x \) of sum \( s \),

\[
\sum_{i=1}^{k} x(i) \geq \frac{k}{n} s.
\]

3 Some class of multivariate value functions

Let us now recall some properties for real functions defined on \( \mathbb{R}^n \).

**Definition 2.** Let \( f \) be a function \( \mathbb{R}^n \rightarrow \mathbb{R} \):

i) \( f \) is componentwise convex if is convex in each variable. \( f \) is componentwise
concave if \( -f \) is componentwise convex.
ii) $f$ is supermodular if when $x, y \in \mathbb{R}^n$

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$$  \hspace{1cm} (4)

$f$ is submodular if $-f$ is supermodular.

iii) $f$ is Schur-increasing if $f(x) \leq f(y)$ when $x \preceq y$. $f$ is Schur-decreasing if $-f$ is Schur-increasing.

Schur increasing functions thus preserve majorization. We note also that a Schur increasing or decreasing function must be a symmetric function. Moreover a symmetric convex function is Schur increasing [3].

A Schur decreasing value function consider the attribute as symmetric and prefer attributes that are less "spread out". There is a similarity between multi-attribute decision making and decision theory under uncertainty. In fact if $x$ and $y$ are interpreted as random variables with all state equally probable the condition $x \preceq y$ is equivalent to the statement that $x$ is less risky than $y$ in the sense of Rothschild and Stiglitz. Hence in this context Schur decreasing is a property of risk aversion.

The following proposition characterize supermodular functions. As it is well known supermodular (and submodular) functions play a central role in modellig concordance between random vectores (see [2]).

**Proposition 1.** Let $f$ be a function $\mathbb{R}^n \rightarrow \mathbb{R}$

i) $f$ is supermodular if and only if

$$f(x + h + k) - f(x + k) \geq f(x + h) - f(x)$$  \hspace{1cm} (5)

for all $h, k$ with $h, k \geq 0$ and $h \perp k$.

ii) is supermodular and componentwise convex if and only if

$$f(x + h + k) - f(x + k) \geq f(x + h) - f(x)$$  \hspace{1cm} (6)

for all $h, k$ with $h, k \geq 0$.

The property (5) is named increasing difference and it transfers the supermodularity condition to one involving the linear structure of $\mathbb{R}^n$.

Intuitively increasing differences says that there must be "complementarity" between an arbitrary pair of positive vectors. So a supermodular value function has the interpretation of a "complementarity" condition of the attributes.

It is worth noting that supermodularity condition is only a "inter-attribute" relation.

We present some examples of functions which are members of the above classes.

**Proposition 2.** If $\varphi$ is a convex and increasing function $\mathbb{R} \rightarrow \mathbb{R}$

i) The following functions are componentwise convex and supermodular:
1. \( f(x_1, \ldots, x_n) = f(\sum_{i=1}^{n} x_i) \)
2. \( f(x_1, \ldots, x_n) = f(\prod_{i=1}^{n} x_i) \)
3. \( f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i) \)

ii) The function \( f(x_1, \ldots, x_n) = \sum_{i=1}^{n} f(x_i) \) is Schur-increasing and supermodular.

We close by considering the class of functions Schur-increasing and supermodular.

**Proposition 3.** Let \( f \) be a function \( \mathbb{R}^n \to \mathbb{R} \). Then the following properties are equivalent:

i) \( f \) is Schur-increasing and supermodular

ii) \( f(x_1, \ldots, x_n) = \sum_{i=1}^{n} f(x_i) \) with \( f \) convex.

### 4 Concluding remarks

In this short overview we have examined multi-attribute decision problems where a value function is specified over the attributes of a deterministic decision problem.

We concentrate on developing the theory in a finite dimensional Euclidean space but one suspects that some results will go through in Riesz spaces i.e. in not necessarily finite dimensional vector spaces which has a lattice structure. This is a potentially fruitful area for future work.

We have studied particular classes of value functions that are usually considered in the area of decision theory under uncertainty. In particular the Schur-increasing functions are used to compare situations according to their level of heterogeneity while supermodular functions are used to characterize risk aversion. So this note points out a similarity between decision under uncertainty and multi-attribute decision making problems, two areas which have been developed in an almost completely independent way.

### References