Simple formulas to option pricing and hedging in the Black–Scholes model

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Abstract. For option whose striking price equals the forward price of the underlying asset, the Black-Scholes pricing formula can be approximated in closed-form. A interesting result is that the derived equation is not only very simple in structure but also that it can be immediately inverted to obtain an explicit formula for implied volatility. In this contribution we present and compare the accuracy of three of such approximation formulas. The numerical analysis shows that the first order approximations are close only for small maturities, Pólya approximations are remarkably accurate for a very large range of parameters, while logistic values are the most accurate only for extreme maturities.
Keywords. Option pricing, hedging, Taylor approximation, Pólya approximation, logistic approximation.

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1 Introduction

For the special case of at the money European call options, using suitable approximations of the standard normal distribution, the Black-Scholes pricing formula can be written in closed-form. A useful result is that the derived equation is not only very simple in structure but also that it can be immediately inverted to obtain an explicit formula for implied volatility.

This approach has been introduced by [3], [4] e [5] where first order Taylor polynomial approximation is suggested and has been taken up again in [8] where the use of Pólya approximation is proposed.

In this contribution, we derive a similar closed form invertible equation using an approximation of the standard normal distribution, based on logistic distribution.

Obviously, the accuracy of values depends on volatility level and on time to maturity. The analysis carried out highlights that first order polynomial uses very simple forms but presents close approximations only for a limited range of parameters. The Pólya and logistic distributions require some additional computations. Pólya approximations are remarkably accurate for a large range of volatility and maturity, while the logistic approximations are the best only for very large maturities.

This note is organized as follows. The next Section presents the approximation formulas to standard normal distribution and the extent of tracking errors. Section 3 contains computational evidence of Black-Scholes values obtained with the three approximation functions. Implied volatility and hedging parameters are discussed in Section 4 and 5, respectively. The final Section reports some concluding remarks.

2 First order, logistic and Pólya approximation formulas

The well known Black–Scholes formula [2] for European call option on underlying assets paying no dividends is

$$c_{BS} = S\,N(d_1) - X\,e^{-r\tau}N(d_2) \tag{1}$$

where:

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

and

$$d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau}\,.$$

In the above equations:

c_{BS} is the theoretical price of a European call option on a stock,
S is the stock price,
X is the striking price,
r is the continuously compounded rate of interest,
τ is the time to option expiration,
σ is the volatility (standard deviation of the instantaneous rate of return on the stock),
$N(\cdot)$ is the cumulative standard normal density function.

We analyze three approximations to the cumulative standard normal distribution

$$\begin{aligned} N(d) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-t^2/2}dt = \frac{1}{2} + \int_0^d \frac{e^{-t^2/2}}{\sqrt{2\pi}}dt \\ &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left\{d - \frac{d^3}{6} + \frac{d^5}{40} - \cdots + \cdots\right\}\,. \end{aligned} \tag{2}$$

The first approximation ignores in (2) all the terms beyond d (i.e. terms of third or higher order are dropped). Therefore

$$A_1(d) = \frac{1}{2} + \frac{d}{\sqrt{2\pi}} \simeq \frac{1}{2} + 0.4 \cdot d\,. \tag{3}$$

Table 1. Standard normal distribution: exact values, approximate values and relative errors (·100)

d	$N(d)$	$A_1(d)$	$100 \cdot E_{r,1}$	$A_2(d)$	$100 \cdot E_{r,2}$	$A_3(d)$	$100 \cdot E_{r,3}$
0.00	0.5000	0.50	0.0000	0.5000	0.0000	0.5000	0.0000
0.05	0.5199	0.52	0.0118	0.5213	0.2528	0.5199	0.0001
0.10	0.5398	0.54	0.0319	0.5424	0.4819	0.5398	0.0006
0.20	0.5793	0.58	0.1278	0.5843	0.8619	0.5793	0.0041
0.30	0.6179	0.62	0.3380	0.6249	1.1303	0.6180	0.0125
0.40	0.6554	0.66	0.6985	0.6639	1.2862	0.6556	0.0269
0.50	0.6915	0.70	1.2347	0.7007	1.3360	0.6918	0.0476
0.75	0.7734	0.80	3.4430	0.7818	1.0845	0.7743	0.1219
1.00	0.8413	0.90	6.9720	0.8457	0.5177	0.8431	0.2109
1.25	0.8944	1.00	11.813	0.8935	-0.0995	0.8969	0.2870
1.50	0.9332	1.10	17.875	0.9277	-0.5885	0.9363	0.3278
2.00	0.9772	1.30	33.026	0.9678	-0.9687	0.9800	0.2826
3.00	0.9987	1.70	70.230	0.9940	-0.4693	0.9992	0.0538
4.00	1.0000	2.10	110.00	0.9989	-0.1075	1.0000	0.0022

The first order Taylor polynomial (3) is very close to the standard normal distribution only for d values between about ± 0.50, but then starts to diverge.

The second approximation is based on logistic distribution

$$F(d) = \frac{1}{1+e^{-kd}} \qquad k > 0\,. \tag{4}$$

The value $k = \pi/\sqrt{3}$ is often used; with such a constant the maximum value of the differenze $F(d) - N(d)$ is about 0.0228, attained when $d = 0.07$ (see [6]). It can be proved that the constant $k = \pi/\sqrt{3.41}$ leads to a better approximation, with a maximum difference of 0.009. Therefore, as second approximation to $N(d)$, we select

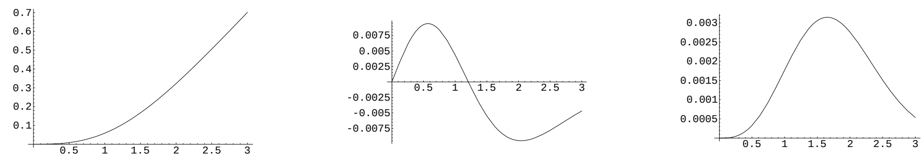
$$A_2(d) = \frac{1}{1+e^{-kd}} \qquad , \qquad k = \pi/\sqrt{3.41}\,. \tag{5}$$

The third approximation is the distribution suggested by Pólya in [7] to approximate $N(d)$, i.e.

$$A_3(d) = \frac{1}{2}\left[1 + \sqrt{1 - e^{-2d^2/\pi}}\right] \qquad d \geq 0\,. \tag{6}$$

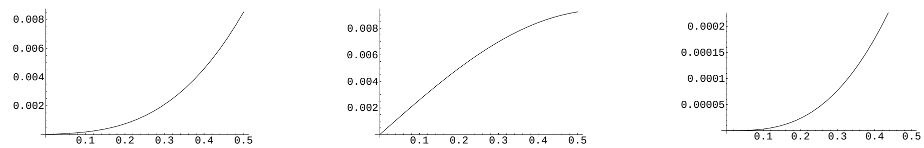
The Pólya approximation is very accurate for a wide range of possible values for d; the maximum absolute error is 0.003, when $d = 1.6$.

The behavior of the difference $A_j - N(d)$ $(j = 1, 2, 3)$ is shown in Figures 1 and 2, for d values ranging from 0.0 to 3.0 and from 0.0 to 0.5, respectively. The extent of tracking errors is presented in Table 1 for d values ranging from 0.0 to 4.



(a) $A_1(d) - N(d)$ (b) $A_2(d) - N(d)$ (c) $A_3(d) - N(d)$

Fig. 1. Differences $A_j(d) - N(d)$, $d \in [0, 3]$



(a) $A_1(d) - N(d)$ (b) $A_2(d) - N(d)$ (c) $A_3(d) - N(d)$

Fig. 2. Differences $A_j(d) - N(d)$, $d \in [0, 0.5]$

The columns $100 \cdot E_{r,j}$ $(j = 1, 2, 3)$ report the relative errors

$$E_{r,j} = \frac{A_j(d) - N(d)}{N(d)} \tag{7}$$

of the approximation functions, multiplied by 100. The valued obtained with first order approximation are always lower than true values; the corresponding relative errors are increasing with d and for $d \geq 1.0$ tend to rise more and more. This is in part due to the fact that the approximation $A_1(d)$ is not a distribution function.

The accuracy of approximation $A_2(d)$ is inferior than $A_1(d)$ for $d \leq 0.5$, while becomes excellent for d in a neighborhood of 1.25 (here the difference $A_2(d) - N(d)$ chances sign).

The relative errors related to Pólya approximation are always positive, i.e. the approximation $A_3(d)$ consistently overestimates $N(d)$; nevertheless, the values obtained with Pólya technique are so precise that we can state that on the whole $A_3(d)$ is the best approximation.

3 Comparison of Black-Scholes Values

In this section we focus on options that are at the money forward, where $S = e^{-r\tau}X$. Note that many transactions in the over the counter market are quoted and executed at or near at the money forward. Moreover, empirical evidence shows (see [1]) that the implied volatility of the at the money options is superior to any combination of all the implied volatilities.

For at the money options the value of d_1 and d_2 simplify in:

$$d_1 = \frac{\sigma\sqrt{\tau}}{2} \qquad\qquad d_2 = -\frac{\sigma\sqrt{\tau}}{2} \tag{8}$$

and the Black-Scholes formula becomes

$$c_{BS} = p_{BS} = S\left[N\left(\tfrac{\sigma\sqrt{\tau}}{2}\right) - N\left(-\tfrac{\sigma\sqrt{\tau}}{2}\right)\right] = \\ = S\left[2N\left(\tfrac{\sigma\sqrt{\tau}}{2}\right) - 1\right] \tag{9}$$

where p_{BS} is the theoretical price of a European put option at the same maturity and exercise price.

In the special case of at the money options, the first order approximation for $N(d_1)$ is

$$N_{A_1}(d_1) = \frac{1}{2} + 0.4 \cdot d_1 = \frac{1}{2} + 0.2\,\sigma\sqrt{\tau}, \tag{10}$$

and substituting in (9) we have

$$c_{A_1} = 0.4 \cdot S\,\sigma\,\sqrt{\tau}. \tag{11}$$

The logistic approximation for $N(d_1)$ is

$$N_{A_2}(d_1) = \frac{1}{1 + e^{-k\sigma\sqrt{\tau}/2}}\,, \tag{12}$$

and the logistic approximation for at the money call is

$$c_{A_2} = S\left(\frac{2}{1 + e^{-k\sigma\sqrt{\tau}/2}} - 1\right). \tag{13}$$

The Pólya approximation for $N(d_1)$ is

$$N_{A_3}(d_1) = \frac{1}{2}\left[1 + \sqrt{1 - e^{-\sigma^2\tau/(2\pi)}}\right] \tag{14}$$

and the Pólya approximation for at the money call is

$$c_{A_3} = S\sqrt{1 - e^{-\sigma^2\tau/(2\pi)}}\,. \tag{15}$$

Table 2 compares the call option values obtained by the first-order Taylor polynomial, the logistic and the Pólya approximation formulas (respectively relations (11), (13) and (15)) to the Black-Scholes model (1). The volatility is $\sigma = 0.30$, the stock price is $S = 100$ and the expiration data going out to 100 years. To have at the money options, the strike price X is set for each time to option expiration τ so that $X = 100\,e^{r\tau}$.

First order approximation performs well good for short times to expiration ($\tau \leq 1$), which are the most interesting cases in practice.

For $\tau \leq 3$, the logistic approximation presents relative errors greater than 6 %. For example, when $\tau = 1/12$ we have $d_1 = 0.3 \cdot \sqrt{1/12} \cdot 0.5 = 0.0433013$; the exact value for is $N(d_1) = 0.517269$, the logistic approximation is $A_2(d_1) =$

Table 2. At the money call option values for different time to expiration, with $S = 100$ and $\sigma = 0.3$

τ	c_{BS}	c_{A_1}	$100 \cdot E_{r,1}$	c_{A_2}	$100 \cdot E_{r,2}$	c_{A_3}	$100 \cdot E_{r,3}$
1/52	1.6596	1.6641	0.2724	1.7692	6.6077	1.6596	0.0003
1/12	3.4539	3.4641	0.2965	3.6817	6.5963	3.4539	0.0014
2/12	4.8830	4.8990	0.3278	5.2043	6.5814	4.8831	0.0028
3/12	5.9785	6.0000	0.3591	6.3711	6.5666	5.9788	0.0042
6/12	8.4470	8.4853	0.4532	8.9979	6.5222	8.4477	0.0084
9/12	10.3357	10.3923	0.5472	11.0056	6.4781	10.3370	0.0126
1	11.9235	12.0000	0.6413	12.6907	6.4341	11.9255	0.0168
2	16.7996	16.9706	1.0177	17.8517	6.2605	16.8055	0.0333
3	20.4988	20.7846	1.3944	21.7472	6.0902	20.5089	0.0495
5	26.2684	26.8328	2.1485	27.7814	5.7595	26.2898	0.0812
10	36.4744	37.9473	4.0383	38.2932	4.9866	36.5313	0.1560
20	49.7665	53.6656	7.8348	51.5816	3.6473	49.9096	0.2876
30	58.8686	65.7267	11.6498	60.3648	2.5416	59.1023	0.3969
50	71.1156	84.8528	19.3168	71.7384	0.8758	71.5118	0.5571
75	80.6069	103.923	28.9257	80.2290	-0.4688	81.1457	0.6684
100	86.6386	120.000	38.5065	85.5402	-1.2677	87.2504	0.7062

0.518408, an error of 0.001149. The Black Scholes value is $c_{BS} = 3.45386$ and the approximate call value is $c_{A_2} = 3.86169$, a relative error of 0.0659626.

Given the close tracking that the Pólya approximation has to the standard normal distribution it is no surprise that the relative errors of c_{A_3} are at all times very small and are not perceptible in correspondence to short maturities. Only for $\tau = 75$ the logistic approximation presents a lower error, but the case $\tau = 75$ is more an academic curiosity than a real-world phenomenon.

4 Implied standard deviation

One of the most widely used application of Black-Scholes formula is the estimation of volatility (instantaneous standard deviation) of the rate of return on the underlying stock, using the market prices of the option and the stock. The implied volatility is the value of standard deviation σ that perfectly explains the option price, given all other variables.

It is a widely documented phenomenon that implied volatility is not constant as other parameters varied. For example, the implied volatility from options with different maturities should not combined to provide a single estimate, because they reflect different perception on short-run versus long-run volatility, a kind of *term structure of volatility*.

Computing the implied volatility from Black-Scholes formula requires the solution of non linear equation and hence the practice has been to use a iterative procedure. However, for at the money European option this cumbersome numerical procedure can be avoided: namely, the formulas (11), (13) and (15) can be easily inverted to provide reasonably accurate estimates of implied volatility. With simple algebra we obtain the following analytical estimates of implied standard deviation:

$$ISD_{A_1} = \frac{2.5 \cdot c_{me}}{S\sqrt{\tau}} \tag{16}$$

$$ISD_{A_2} = \frac{-2\log[(S - c_{me})/(S + c_{me})]}{k\sqrt{\tau}} \tag{17}$$

$$ISD_{A_3} = \sqrt{-\frac{2\pi\log[1 - (c_{me}/S)^2]}{\tau}} \tag{18}$$

where c_{me} is the market price of the call option.

To assess the accuracy of these approximations, consider an at the money forward option ($X = Se^{r\tau}$) with $S = 100$. The option expires in six months and the market price of the option is $c_{me} = 6$. The "true" implied standard deviation, computed by Newton method is $ISD = 0.2129$; the estimates obtained with the approximation formulas are: $ISD_{A_1} = 0.2121$, $ISD_{A_2} = 0.1997$, $ISD_{A_3} = 0.2129$.

Note that, in this example, the approximated volatilities (16) and (18) prove to be very accurate, while the approximation (17) entails a small error. In any case, all three estimates (16), (17) and (18) may also be used to obtain a good starting point for a more accurate numerical procedure in order to compute the exact implied standard deviation, even for very deep out (or in) the money options.

5 Option hedging parameters

The key purpose in option portfolios management is hedging to eliminate or to reduce risk. Hedging requires the measurement and the monitoring of the sensitivity of the option value to changes in the parameters. These sensitivities can be compute by partial derivatives which, for at the money options, have not cumbersome expressions.

The hedge ratio, or Delta, of an at the money call option is

$$\Delta_c = \frac{\partial c}{\partial S} = N(d_1) = \frac{1}{2} + \int_0^{\frac{\sigma\sqrt{\tau}}{2}} \frac{e^{-t^2/2}dt}{\sqrt{2\pi}}. \tag{19}$$

The three approximations for the hedge ratio are immediately obtained. For example, when $S = 100$, $\tau = 0.5$ and $\sigma = 0.3$ the true value is $\Delta_c = 0.542235$, while the approximated values are:

$$\Delta_{c,A_1} = 0.542426, \qquad \Delta_{c,A_2} = 0.544990, \qquad \Delta_{c,A_3} = 0.542239\,.$$

Factor Gamma is defined by

$$\Gamma_c = \frac{\partial^2 c}{\partial S^2} = \frac{\partial \Delta}{\partial S} = N'(d_1)\frac{\partial d_1}{\partial S} = \frac{e^{-d_1^2/2}}{\sqrt{2\pi}S\sigma\sqrt{\tau}} \tag{20}$$

and measures the change in the hedge ratio in correspondence to a small change in the price of the stock. Note that, in most practical circumstances, we have $d_1^2/2 \simeq 0$ and therefore

$$\Gamma_c \simeq \frac{0.4}{S\sigma\sqrt{\tau}} \,. \tag{21}$$

Factor Gamma indicates the cost of adjusting a hedge. Using the data of the previous example, from approximation (21) we have

$$\Gamma_c = \frac{0.4}{100 \cdot 0.3\sqrt{0.5}} = 0.0188562\,;$$

this value is accurate to the third decimal place (the correct value is 0.0187008).

Factor Vega, which is defined as the change in the price of the call for a small change in the volatility parameter, is given by

$$Vega = \frac{\partial c}{\partial \sigma} = N'(d_1)S\sqrt{\tau} \simeq 0.4 \cdot S\sqrt{\tau}\,. \tag{22}$$

Factor Omega is a measure of leverage and is given by the product of the Delta and the ratio S/c, i.e.

$$\Omega_c = \Delta_c \cdot \frac{S}{c}\,. \tag{23}$$

Factor Theta, or *time decay*, is given by

$$\Theta_c = \frac{\partial c}{\partial \tau} = \frac{-S\sigma}{2\sqrt{\tau}} N'(d_1) - rXe^{-r\tau}N(d_2) \tag{24}$$

and for at the money options it becomes

$$\Theta_{c,atm} = -S\left[\frac{\sigma e^{(-\sigma^2\tau/4)}}{2\sqrt{\tau\pi}} + r\,N(d_2)\right] \simeq -S\left[\frac{0.4\sigma}{2\sqrt{\tau}} + r\,N(d_2)\right]\,. \tag{25}$$

The time decay is usually measured over one day. The negative sign indicates that, as the time to expiration is declining, the option value is decaying.

Including $r = 0.05$ in data set of our numerical example, we find that the true value is $\Theta_c = -0.0295181$, while the approximated values are:

$$\Theta_{c,A_1} = -0.0295151, \qquad \Theta_{c,A_2} = -0.0294804, \qquad \Theta_{c,A_3} = -0.0295181\,.$$

6 Concluding remarks

Nowadays computing an exact Black-Scholes European call option value is really easy: the large use of computer programs and the great availability of free or low cost software to price options, undoubtedly, reduces the need for approximate formulas, apart from the accuracy of proposed formulas. So, one could wonder which is the usefulness of formulas derived from a first order polynomial or logistic and Pólya approximations to standard normal distributions.

Actually, the first order formulas are very simple, easy to remember and can be used to give immediately operational information about the option price and the implied volatility. But this simplicity shows marks of weakness in correspondence of high volatilities and times to maturity.

The formulas based on logistic and Pólya approximations without any doubt are more complicating looking, but can be easily programmed onto a pocket calculator. In particular the Pólya formula presents an impressive accuracy even for long-lived options; the approximations obtained with the logistic distribution present larger relative errors, nevertheless are sufficiently accurate for most practical circumstances.

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