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Abstract
Segmentation is a core strategy in modern marketing, and age-specific segmentation based on the age of the consumers is very common in practice. Age-specific segmentation enables the change of the segments composition during time and can be studied only by means of dynamic advertising models. Here we assume that a firm wants to optimally promote and sell a single product in an age-segmented market and we model the awareness of this product using an infinite dimensional Nerlove-Arrow goodwill as a state variable. Assuming an infinite time horizon, we use some dynamic programming techniques in infinite dimension to characterize both the optimal advertising effort and the optimal goodwill path in the long run. An interesting feature of the optimal advertising effort is an anticipation effect with respect to the segments considered in the target market, due to time evolution of the segmentation. We analyze this effect in two different scenarios: in the first, the decision makers can choose the advertising flow directed to different age segments at different times, while in the second they can only decide the activation level of an advertising medium with a given age-spectrum.

Keywords
M37, E22, C61, C62

JEL Codes
optimal advertising, dynamic programming, vintage capital

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Optimal investment in age-structured goodwill

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Abstract

Segmentation is a core strategy in modern marketing, and age-specific segmentation based on the age of the consumers is very common in practice. Age-specific segmentation enables the change of the segments composition during time and can be studied only by means of dynamic advertising models. Here we assume that a firm wants to optimally promote and sell a single product in an age-segmented market and we model the awareness of this product using an infinite dimensional Nerlove-Arrow goodwill as a state variable. Assuming an infinite time horizon, we use some dynamic programming techniques in infinite dimension to characterize both the optimal advertising effort and the optimal goodwill path in the long run. An interesting feature of the optimal advertising effort is an anticipation effect with respect to the segments considered in the target market, due to time evolution of the segmentation. We analyze this effect in two different scenarios: in the first, the decision-maker can choose the advertising flow directed to different age segments at different times, while in the second she/he can only decide the activation level of an advertising medium with a given age-spectrum.

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1 Introduction

Segmentation is a core strategy in modern marketing and the introduction of segmentation in dynamic advertising models is an interesting area of research
where different disciplines interact. Market segmentation is the result of partitioning the whole market into distinct consumers’ groups, each characterized by special sets of attribute values, so that their members exhibit the same needs and behaviours [19, Ch.10, p.391].

Some optimal advertising models with segmented capitals are already available in literature, for instance: segmentation is used to compare different advertising media in [15] and [16], or to analyze advertising strategies either distributed over several geographic regions in [5] and [20] or associated to products of varying vintages in [3].

Here segmentation of capitals is based on consumers’ age, as in the preliminary work [12], [16]. More precisely, we consider a firm wanting to optimally promote and sell a single product in a age-segmented market, over an infinite time horizon. The awareness of consumers about the product is described by means of an infinite dimensional Nerlove-Arrow goodwill (the state variable), depending jointly on time and age.

Note that age-segmentation often arises in practice (see [19, Ch.10, p.400]), as some products are spontaneously age-specific. Moreover age segmentation naturally calls for a dynamic framework: segments composition changes during time due to the fact that time and age vary jointly. For instance, the “16 year segment” will be the “18 year segment” in 2 years.

Models where capital shows this behaviour are known in literature as vintage capital models [2, 13], and the idea that advertising may fall within this class was already in [3]. Nonetheless, in [3] the state variable describes a continuum of goods differentiated by means of their vintage, and with new goods continuously launched onto the market, while here it describes the goodwill of a single product, differentiated by means of the age in which consumers are clustered. From the mathematical point of view the two models are similar, but the economic interpretation in terms of marketing variables is definitely different. The results in this paper may rather be considered a continuous version of those in [16], under the respect that they extend the finite and discrete segmentation in [16] to a continuum of infinitesimal segments. However, the fact that here segments are age-specific, and hence connected to time evolution, accounts in some cases for different conclusions about optimal advertising policies, as specified in Section 3.

We solve the problem in two different settings, and later compare the results (differently from [12] where only the former was analyzed). In the first setting the decision-maker chooses advertising strategies which are jointly differentiated by means of time and age. Note that this coincides with the general but too strong assumption that the controller may advertise towards each age-segment independently, using different marketing strategies to reach different segments. Indeed, since in practice the decision maker is bound to use one among advertising media (such as television, newspapers, web sites) which hits all market segments with the same message but with varying effectiveness, we analyze also a different (more realistic) scenario where the controller may use only one medium and the optimal advertising policy is characterized accordingly. The analysis allows also to introduce a preference index associated to advertising
media which is specific for age-structured goodwill.

The choice of simple (quadratic) advertising costs and (linear) revenues in the objective functional is deliberate. The focus of the paper is on comparison, which is made clear with such a simple instance of profits. Nonetheless, as we will remark several times, the extension to more general revenues and costs is possible, as shown in [12] and also in [9].

Using dynamic programming we can straightforwardly characterize the optimal advertising strategies in both scenarios, also in the long run. Moreover, we observe in both scenarios an anticipating effect in the optimal advertising policies: for the decision-maker it is convenient to invest not only in the segments that compose the target market, but also in the segment that will be in the target market in the future. Note that anticipating effects are already known in vintage capital models (see e.g. [14]), but to the best of our knowledge it is the first time that such effect is described in an advertising model. Moreover, the anticipating effects takes place here and not in [15], as age-segmentation is unavoidably connected to time evolution.

The mathematical technique involves the rephrasing of the original problem in a Hilbert space, so that Dynamic Programming in infinite dimension may be performed. For Dynamic Programming in infinite dimensions the reader is referred e.g. to the book [1] and, for boundary control with economic applications to e.g. [3], [7], [11], [10] and the references therein. We mention that the treating of the problem in an infinite dimensional setting, besides naturally aggregating data in time over the range of all possible ages, allows to derive general properties which apply to a variety of economic problems, provides compact analytic formulas for equilibrium points and, without further theoretical work, allows extensions of the results in this paper to more general advertising costs (not necessarily quadratic, for some examples see [12]), and to more general concave revenues (not necessarily linear) where usually analytic formulas are lacking, in the spirit of [8].

The paper is organized as follows: in Section 2 we describe the model with an age-specific segmentation, also in comparison to classical non-vintage models. In Section 3 we solve the problem using some dynamic programming techniques in infinite dimension in a case where the decision-maker chooses a control variable which is differentiated by means of time and age jointly. In Section 4, we describe a model where the decision-maker may use only one advertising medium and the optimal advertising policy is characterized accordingly. In Section 5, we summarize the results of the paper.

2 The Model

We assume a firm sells a good in a segmented market and wants to organized an advertising campaign to support that good. The market is segmented using age $a$ as a demographic variable, which we assume to be continuously varying in the age interval $[0, \omega]$. We use as state variable a Nerlove-Arrow type goodwill $G$, which summarizes the past investment in advertising, and which describes
the market segmentation in terms of age. More precisely $G (t, a)$ represents the goodwill value at the time $t \in [0, +\infty)$ for the consumers of age $a \in [0, \omega]$. The goodwill evolution is described (as in [16]) by the following partial differential equation

$$\partial_t G (t, a) + \partial_a G (t, a) = -\delta G (t, a) + u (t, a)$$

(1)

where $u (t, a) \geq 0$ is the control variable for the decision-maker and represents the advertising effort at time $t$ addressed to the consumers’ segment of age $a$. With such a structure, the decision-maker can control at each time $t$ the advertising flow directed towards the consumers of the segment $a$. This coincides with the general but strong assumption that the controller may advertise towards each age-segment independently, using different marketing strategies to reach consumers belonging to different segments. Indeed, as already underlined in [16] and [15], in a more realistic scenario a firm can only activate an advertising medium, like a newspaper or a TV channel, which hits with a different effectiveness all the segments. This case when the firm advertises using an advertising channel, the motion equation (1) is differently modelled as follows

$$\partial_t G (t, a) + \partial_a G (t, a) = -\delta G (t, a) + \gamma (a) v (t)$$

(2)

where $v (t) \geq 0$ is the control variable for the decision maker and represents the activation level at time $t$ of an advertising medium with a given spectrum $\gamma (a)$. The function $\gamma (a)$ depends on the advertising channel and does not depend on time, while the decision maker can only activate this advertising channel using the control function $v (t)$. The difference between the two scenarios is clear if we compare the two feasible sets for the controls, which are $L^2 ([0, +\infty) \times [0, \omega]; [0, +\infty))$ for (1), while it becomes $L^2 ([0, +\infty); [0, +\infty))$ for (2). In the following we will study separately and then compare the two different motion settings, described by (1) which we call the segment specific scenario, and by (2), which we call the single medium scenario.

For both the equations (1) and (2), we assume the boundary conditions

$$G (t, 0) = 0 \text{ for all } t \in [0, +\infty)$$

(3)

and the initial condition

$$G (0, a) = g (a) \geq 0 \text{ for all } a \in [0, \omega].$$

(4)

Respectively, the goodwill value for the segment of age 0 must be always zero and the initial goodwill values of the different age segments is given.

Moreover, we assume that the profit flow is linear in the goodwill for each age $a \in [0, \omega]$ and it is described by $t \mapsto \pi (a) G (t, a)$, where $\pi (a) > 0$ represents the revenue rate of the segment $a$ (net of the advertising costs). Hence, the whole profit flow at the time $t$ is given by the function

$$t \mapsto \int_0^t \pi (a) G (t, a) da.$$
This assumption is quite familiar in dynamic advertising model when closed form solutions are desired (see e.g. [15] and the references therein).

To the revenues we need to subtract advertising costs, which are differently shaped for the two scenarios. In the case of the segment specific scenario, the costs are assumed also segment specific, that is, the advertising costs flow is

\[ t \mapsto \int_0^\infty \frac{\kappa(a)}{2} u^2(t, a) \, da \]

where \( \kappa(a) > 0 \) is the advertising cost parameter associated with the age segment \( a \). Note that \( \kappa \) accounts for differentiation with respect to age segments, but it is stationary in time.

On the other hand, in the single medium scenario, the advertising costs flow is simpler

\[ t \mapsto \frac{c}{2} v^2(t) \]

where \( c > 0 \) represents an advertising cost parameter associated with the activation of the advertising medium. Note that a connection between the two scenarios is made by choosing \( c = \langle \kappa, \gamma^2 \rangle = \int_0^\infty \kappa(a) \gamma^2(a) \, da \), so that the two advertising costs coincide. However, from a practical point of view, \( c \) is better thought as an exogenous parameter of the model associated to the advertising medium.

Note also that, in both cases the advertising costs, although defined in a different way, are assumed to be quadratic, as it often happens in literature (see e.g. [18, p. 103]). Indeed, a quadratic function represents a simple instance of an increasing and convex function with good analytical tractability. Nonetheless, the abstract model introduced in the next section applies to more general convex costs (see for instance, [12]).

Now, summarizing all our assumptions, we can formulate the problem in the two different scenarios. In the segment specific scenario the decision-maker wants to organize an advertising campaign (choosing an advertising function \( u(t, a) \in L^2([0, +\infty) \times [0, \omega]; [0, +\infty)) \)) in order to maximize the functional

\[ J_s[u, g] = \int_0^{+\infty} \int_0^\infty e^{-\rho t} \left[ \pi(a) G(t, a) - \kappa(a) u^2(t, a)/2 \right] \, da \, dt. \quad (5) \]

The functional \( J_s \) (\( s \) is for segment specific) is a function of the control variable \( u \) and, through the trajectory \( G(t, a) \), of the initial goodwill \( g \). In the short run (i.e. with finite horizon) the problem falls within the family of the age-structured control systems and can be studied using the necessary condition introduced in [13] (for a very clear introduction of this kind of models, we refer to the book [17]). In this paper we study the problem with infinite horizon. We do so by means of dynamic programming techniques in infinite dimensions. Moreover we study equilibrium points of the system (here the use of dynamic programming proves particularly effective), mentioning that the technique may be extended to more general models (e.g. models with general concave cost functional [12]), and even with non-linear revenues (see [8]).
In the single medium scenario the decision-maker wants to choose an advertising strategy \( v(t) \in L^2([0, +\infty); [0, +\infty]) \) in order to maximize the functional

\[
J_m[v; g] = \int_0^{+\infty} e^{-rt} \left[ \int_0^{a} \pi(a) G(t, a) da - c v^2(t)/2 \right] dt
\]

(6)

Again \( J_m \) (medium) is a function of the control variable \( v \) and, through the trajectory \( G(t, a) \), of the initial goodwill \( g \).

This comparison between different scenarios is described also in [15], but in that work the market segmentation does not evolve in time. Here, the evolution of the segmentation requires a different approach to the problem.

In order to better understand the content of next sections, it is interesting to analyze the classical instance of this model, where the market is not segmented. There we have a single goodwill \( G(t) \) whose evolution is described by an ordinary differential equation:

\[
\dot{G}(t) = -\delta G(t) + u(t)
\]

(7)

which is the motion equation introduced by Nerlove and Arrow in their seminal paper [21]. In this instance of the problem, the objective functional is

\[
J[u(t)] = \int_0^{+\infty} e^{-\rho t} \left[ p G(t) - \frac{k}{2} [u(t)]^2 \right] dt.
\]

(8)

Using the phase-space analysis it is simple to obtain that the optimal advertising investment is constant

\[
u^*(t) = \frac{p}{k} (\delta \rho)
\]

(9)

and the equilibrium goodwill is

\[
G_{\infty}^*(t) = \frac{p}{\delta k} (\delta \rho)
\]

(10)

One of the aims of this paper is to study the existence of this equilibrium function in both the segment specific and single medium scenario. Moreover, we want to analyze the dependence of the optimal equilibrium point on the parameters characterizing the segmentation, i.e. \( \pi(a) \), \( \kappa(a) \), and \( \gamma(a) \).

2.1 Analytical setting

Here we rephrase the original control problem for either the partial differential equation (1) or (2) as a problem for a suitable ordinary differential equation, but in an infinite dimensional setting. The two problems appear to fall into the same class of abstract problems, as it will be specified immediately, and may be treated together with the tools of dynamic programming in Hilbert spaces.
Indeed, we consider the space of square integrable functions of variable $a$ as the state space of the control problem, that is

$$L^2 \equiv L^2([0,\omega] ; \mathbb{R}) = \{ f : [0,\omega] \to \mathbb{R} : \| f \|_2 < +\infty \},$$

where $\| f \|_2 = \left( \int_0^\omega |f(a)|^2 \, da \right)^{\frac{1}{2}}$, and $(f,g)$ is the scalar product of $f$ and $g$ in $L^2$. Then the state variable $G(t)$ is that function of $L^2$ such that $G(t)(a) = G(t,a)$. Hence, if $f'$ denotes the distributional derivative of $f$, we set $H^1 = \{ f \in L^2 : f' \in L^2 \}$, and we introduce the differential operator $A$ with domain $D(A) = \{ f \in H^1 : f(0) = 0 \}$ defined by

$$A : D(A) \subset L^2 \to L^2, \quad Af(a) = -f'(a) - \delta f(a),$$

The control space $U$ and the control operator $D : U \to L^2$ are chosen differently in the two different scenarios:

- in the case of the segment specific scenario, $U = L^2$, we interpret the control $u(t)$ as that function of $L^2$ such that $u(t)(a) = u(t,a)$, so that the control operator $D : L^2 \to L^2$ is the identity function $Du = u$;
- in the case of the single medium scenario, we choose $U = \mathbb{R}$ and define the control operator $D : \mathbb{R} \to L^2$ as the multiplication function $Du = u\gamma(\cdot)$.

In both cases the abstract reformulation of the problem is a control system for an ordinary differential equation in the Hilbert space $L^2$:

$$\begin{cases}
\dot{G}(t) = AG(t) + Du(t) & t > 0 \\
G(0) = g \in L^2.
\end{cases} \quad (11)$$

Note that the boundary condition is enclosed into the definition of the domain $D(A)$ of the operator $A$.

The technique is very well known and may be found in classical books on evolution equations such as [22] or the more recent [6]. In [22], [6] it is also proved the operator $A$ is the generator of a strongly continuous semigroup of operators $\{ e^{tA} \}_{t \geq 0}$ with

$$[e^{tA}f](a) = e^{-\delta t}f(a - t)\chi_{[t,\omega]}(a), \quad (12)$$

for all $f \in L^2$, $a \in [0,\omega]$. Then, by means of variation of constants formula, the trajectory is given by the following function in $L^2$

$$G(t) = e^{tA}g + \int_0^t e^{(t-s)A}Du(s)ds. \quad (13)$$

As far as the objective functional is concerned, we assume that the marginal profit function $\pi$ is in $L^2$, so that the running revenue $R$ can be described as

$$R : L^2 \to \mathbb{R} \quad G \mapsto R(G) = \int_0^\omega \pi(a) \, G(a) \, da = \langle \pi, G \rangle \quad (14)$$

while the running advertising cost $C : U \to \mathbb{R}$ is different in the two scenarios:
for the segment specific scenario one has

\[ C : L^2 \to \mathbb{R} \quad u \mapsto C(u) = \int_0^\omega \frac{\kappa(a) u^2(t)}{2} da = \frac{1}{2} \langle B_{\kappa} u, u \rangle \]  \hspace{1cm} (15)

where \( \kappa \in L^\infty([0, \omega]; \mathbb{R}), 0 < \varepsilon \leq \kappa(a) \leq \bar{\kappa} \), and where \( B_{\kappa} \) is the multiplication operator \( B_{\kappa} : L^2 \to L^2, u(\cdot) \mapsto \kappa(\cdot) u(\cdot) \);

for the single medium scenario

\[ C : \mathbb{R} \to \mathbb{R} \quad u \mapsto C(u) = \frac{1}{2} cu^2. \]  \hspace{1cm} (16)

In both cases, the general objective functional, \( J_s \) or \( J_m \), can be described as

\[ J[u, g] = \int_0^{+\infty} (R(G(t)) + C(u(t))) \, dt \]

by choosing the cost function \( C \) as in (15) or (16), respectively. In any case, \( J \) is concave with respect to the control-state variables. Moreover, it is structurally similar to (8), but the functions \( R \) and \( C \) hide data aggregated with respect to the \( a \) variable. The control problem is that of maximizing (19) over the set of admissible controls

\[ U = L^p([0, +\infty); U) \]

\[ = \left\{ u : [0, +\infty) \to L^2 : \int_0^{+\infty} e^{-\rho t} \|u(t)\|_{L^p}^p \, dt < +\infty \right\} \]

where \( \| \|_{L^p} \) is either the \( L^2 \) norm, or the euclidean norm in \( \mathbb{R} \), while the value function of the optimal control problem is

\[ V(g) = \sup_{u \in U} J[u, g]. \]

Finally we define a function which will be frequently used in the following sections

\[ \tilde{\pi}(a) \equiv \int_0^\omega e^{-(\rho+\delta)(\sigma-a)} \pi(\sigma) d\sigma. \]  \hspace{1cm} (17)

The function \( \tilde{\pi} \) represents the discounted marginal demand with respect to the \( a \)-segment goodwill. Indeed the segment of age \( a \) becomes of age \( \sigma \) after a time \( \sigma - a \), hence the discounted demand of the segment \( a \), which becomes of age \( \sigma \) after a time \( \sigma - a \), is

\[ e^{-(\rho+\delta)(\sigma-a)} \pi(\sigma). \]

In the meantime the unit of goodwill is exponentially decreased and amounts to \( e^{-\delta(\sigma-a)} \). Therefore the discounted demand of a unit of goodwill of the segment \( a \) for being of age \( \sigma \) after a time \( \sigma - a \) is the integrand in (17)

\[ e^{-(\rho+\delta)(\sigma-a)} \pi(\sigma) \]

It is easy to show that if \( \pi \) is in \( L^2 \), then \( \tilde{\pi} \) has a derivative in the sense of distributions and that \( \tilde{\pi}' \) is also in \( L^2 \).
3 Segment specific scenario

Here we study in more detail the case of segment specific scenario. The variation of constant formula (13) describing the trajectory can be made more explicit as

\[ G(t, a) = e^{-\Delta t}g(a - t)\chi_{[t, \omega]}(a) + \int_{0}^{\min\{t, \omega\}} e^{-\delta s}u(t - s, a - s)ds. \] (18)

while the objective functional is better specified as follows

\[ J_s[u, g] = \int_{0}^{+\infty} e^{-rt} \left[ \langle \pi, G(t) \rangle - \frac{1}{2} \langle B_u u, u \rangle \right] dt. \] (19)

We want to compute optimal strategies and optimal trajectories of the system, and their behaviour in the long run. We start by proving that, due to the linearity of the revenue, the objective functional may be written so as to separate the dependence on \( u \) and \( g \). As a consequence, the optimal control and the value function may be computed explicitly.

**Theorem 1** Assume \( g, \pi \in L^2 \). Then \( J_s[u, g] \) is a linear functional in \( u \) and \( g \). Consequently, the optimal value function can be written as,

\[ V_s(g) = \langle \pi, g \rangle + \frac{1}{2\rho} \left[ B_u^2 \langle \pi, \pi \rangle \right] \] (20)

In particular, it is an affine function of \( g \), so that \( V_s \) is Fréchet-differentiable, with gradient

\[ \nabla V_s(g) = \pi \in L^2. \]

**Proof.** The proof is performed completely in [12] for a general (not necessarily quadratic) cost function. Another source is the proof of (31) contained in the Appendix of [4].

**Remark 2** Note that the objective functional may be rewritten as in (20) due to the fact that the revenue is linear in the state variable. If the revenue is a general concave function, we lack an analytic formula for the value function.

As a consequence of the previous theorem, one has an analytic formula for optimal couples.

**Corollary 3** In the assumptions of Theorem 1, there exists a unique optimal strategy \( u^* \), not depending on \( t \), and given by

\[ u^*(t, a) = B_1^u(\pi)(a) \equiv \frac{\pi(a)}{\kappa(a)}, \] (21)
The associated optimal trajectory is given by
\[ G^*(t, a) = e^{-\delta t} g(a - t) \chi_{[t, \infty)}(a) + \int_0^{\min\{t, a\}} e^{-\delta s} \frac{\bar{\pi}(a - s)}{\kappa(a - s)} ds, \tag{22} \]

The last step is computing optimal couples in the long run, that is, as time \( t \) goes to infinity. Indeed the optimal control (21) is already independent of time, and what is left to compute is the optimal trajectory in the long run, that we call an equilibrium. Hereafter we give a more formal definition.

Note that the optimal feedback map is a constant map \( L^2 \to L^2 \), \( \xi \mapsto B_\bar{\pi}(\bar{\pi}) \) so that the closed loop equation is
\[ \dot{G}(t) = AG(t) + B_\bar{\pi}(\bar{\pi}) \tag{23} \]

**Definition 4** An equilibrium point for the system is any stationary solution of the closed loop equation (23), that is, a solution \( G \) in \( L^2 \) to the equation
\[ AG + B_\bar{\pi}(\bar{\pi}) = 0. \tag{24} \]

**Theorem 5** There exists a unique equilibrium point for the control system, given by
\[ G^*_\infty(a) = \int_0^a e^{-\delta s} \frac{\bar{\pi}(a - s)}{\kappa(a - s)} ds, \tag{25} \]
and it is asymptotically stable, that is, for all optimal goodwill paths (22)
\[ \lim_{t \to +\infty} G^*(t, a) = G^*_\infty(a), \quad \forall a \in [0, \omega]. \]

**Proof.** The proof is straightforward: the equation (24) may be written also as \( G^*_\infty = -A^{-1} B_\bar{\pi}(\bar{\pi}) \), as the operator \( A \) has a continuous inverse \( A^{-1} : L^2 \to L^2 \), defined by \( A^{-1} f(a) = -\int_0^a e^{-\delta s} f(a - s) ds \).

We note that in (21) the positivity of the advertising flow is satisfied, hence the solution is feasible for the original problem.

The optimal advertising effort described in (21) confirms the golden rule “marginal revenues equal to marginal costs” since from (20) one derives that \( u^* \) satisfies \( D_u J[u, g] = 0 \), while \( D_g J[u, g] = \bar{\pi} \) (with \( D_u[u, g], D_g[u, g] \) the Frechét differential in \( L^2 \) with respect to the \( u, g \) variable, respectively). Under this respect, (21) is consistent with [15, formula 2.4], where the role of \( \bar{\pi} \) is played by \( \pi/(\rho + \delta) \). At the same time it differs from [15, formula 2.4], as there the segmentation does not change with time, and optimization is performed along different segments separately, while here optimization occurs jointly in different segments, as time and age vary jointly. Such difference is enforced by acknowledging that discounted marginal demand \( \bar{\pi}(a) \) may be strictly positive even when \( \pi(a) \) is null. This phenomenon is not present in [15, formula 2.4], where marginal revenues obtained from segments out of the support of \( \pi \) are always zero. This fact has an easy but interesting interpretation: it is profitable
to invest in a segment $a$ even if $\pi(a) = 0$, as segment $a$ will enter the support of $\pi$ in the future, that is, it is profitable to increase today the awareness of potential future purchasers. This anticipation effect is known for optimal investment models with vintage capital (see e.g. [14]), and here confirmed for optimal advertising.

In the following subsection we exemplify all previous remarks by means of some examples.

### 3.1 Simulations

We here present two simple examples, where we choose a marginal revenue $\pi$, compute the associated $\pi$ and optimal strategies. We assume by simplicity that $\kappa(a) \equiv \bar{\kappa}$, that is, the marginal revenues for all the segments in the target market are the same.

#### 3.1.1 Target market $[\alpha, \omega]$

Let the target market be $[\alpha, \omega]$ with $\alpha \in (0, \omega)$, that is, the firm sells its product only to people of age greater than $\alpha$. A car would be such a product: it can only be sold to people aged above the minimum age $\alpha$ to obtain a driving licence. After rescaling the objective functional, we choose

$$\pi(a) = \chi_{[\alpha, \omega]}(a).$$  \hspace{1cm} (26)

As a result of computation

$$\pi(a) = \begin{cases} 
\frac{e^{(\rho+\delta)\alpha}}{(\rho+\delta)} \left[ e^{-(\rho+\delta)\alpha} - e^{-(\rho+\delta)\omega} \right] & a \leq \alpha \\
\frac{1}{\kappa(\rho+\delta)} \left[ 1 - e^{-(\rho+\delta)(\omega-a)} \right] & a > \alpha
\end{cases} \hspace{1cm} (27)$$

while the optimal advertising effort is

$$u^*(t, a) = \begin{cases} 
\frac{1}{\kappa(\rho+\delta)} \left[ e^{(\rho+\delta)(a-\alpha)} - e^{-(\rho+\delta)(\omega-a)} \right] & a \leq \alpha \\
\frac{1}{\kappa(\rho+\delta)} \left[ 1 - e^{-(\rho+\delta)(\omega-a)} \right] & a > \alpha
\end{cases} \hspace{1cm} (28)$$

Note that $u^*$ can be seen as a product between the constant $1/\kappa(\rho+\delta)$ (which is the same constant found in (9)) and a function of $a$ which describes the revenue associated to each age segment. Moreover, even if $[0, \alpha)$ is not in the target market, the advertising flow $u^*$ directed to these segments is not equal to zero. It is optimal to anticipate the time evolution of the segmentation. In Picture 1 we present the optimal control (21) while in Picture 2 we show the optimal trajectory (22) under the assumptions that $\kappa(a) \equiv \bar{\kappa}$ and under the hypothesis that $\pi(a) = \chi_{[\alpha, \omega]}(a)$. The parameters are chosen as follows: $\alpha = 18, \omega = 70, \delta = 0.1, \rho = 0.01, \bar{\kappa} = 1, g(a) \equiv 0$. 

\[11\]
Accordingly to what the pictures show, we can prove that \( u^* (t, a) \) has a maximum at \( a = \alpha \) (for any choice of the parameters). Moreover, using (25) we can compute an explicit formula for \( G_{\infty}^* (a) \) (which we do not report here for brevity) and prove that this function is increasing and convex in \([0, \alpha]\), and decreasing in \([\alpha, \omega]\) (for any choice of the parameters).

### 3.1.2 Target market \([\alpha, \beta]\)

In this second example we consider a different target market \([\alpha, \beta]\), with \(0 < \alpha < \beta < \omega\). For instance, if the firm wants to sell, say, a sports car as a coupé, then the target market for this product is bounded from below by the driving licence age \(\alpha\), as in the previous example. Moreover we assume also an upper bound: we estimate that a coupé generally sells to younger ages (for instance, a wide majority of family men would prefer roomy cars instead of a coupé). Hence, assuming again that the marginal revenues for all the segments in the target market are the same, we choose, after scaling the objective functional, \( \pi(a) = \chi_{[\alpha, \beta]} (a) \) (29)

so that

\[
\pi(a) = \begin{cases} 
\frac{e^{-(\rho+\delta)(\alpha-a)} - e^{-(\rho+\delta)(\beta-a)}}{/(\rho + \delta)} & a < \alpha \\
\frac{1 - e^{-(\rho+\delta)(\beta-a)}}{/(\rho + \delta)} & \alpha \leq a \leq \beta \\
0 & a > \beta
\end{cases}
\] (30)

and the optimal advertising effort is

\[
u^* (t, a) = \begin{cases} 
\frac{1}{\pi(\rho + \delta)} \left[ e^{(\rho+\delta)(\alpha-a)} - e^{-(\rho+\delta)(\beta-a)} \right] & a < \alpha \\
\frac{1}{\pi(\rho + \delta)} \left[ 1 - e^{-(\rho+\delta)(\beta-a)} \right] & \alpha \leq a \leq \beta \\
0 & a > \beta
\end{cases}
\]

Here the anticipation effect of the advertising effort is less relevant (i.e. for \( a < \alpha \) the advertising effort in the first example is always greater that the advertising effort obtained here). The interpretation is that an investment in
the segments \([0, \alpha]\) is less profitable because when these segments go over the threshold \(\beta\) their demand vanishes. For the same reason the advertising effort directed to the segment \((\beta, \omega]\) is always zero. In Picture 3 we present the optimal control \((21)\) while in Picture 4 we show the optimal trajectory \((22)\) (also in this case an analytic formula is available) under the assumptions that that \(\kappa(a) \equiv \bar{\kappa}\) and under the hypothesis that \(\pi(a) = \chi_{[a, \beta]}(a)\). The parameters are chosen as follows: \(\alpha = 18, \beta = 40, \omega = 70, \delta = 0.1, \rho = 0.01, \bar{\kappa} = 1, g(a) \equiv 0\).

\[\text{Picture 3} \quad \text{Picture 4}\]

\section{Single medium scenario}

We proceed for the single medium scenario as we did in the previous case. Assuming \(\gamma \in L^2\), the variation of constants formula \((13)\) is made more explicit as

\[G_m(t,a) = e^{-\delta t}g(a-t)\chi_{[t,\omega]}(a) + \int_0^{\min(t,a)} e^{-\delta s} \gamma(a-s)v(t-s)ds. \quad (31)\]

while the objective functional is

\[J_m[v, g] = \int_0^{+\infty} e^{-\rho t} \left[ (G(t), \pi) - c\alpha^2(t)/2 \right] dt. \quad (32)\]

Similarly to what we did for the segment specific scenario, we investigate an explicit formula for the optimal strategies and trajectories.

\textbf{Theorem 6} Assume \(g, \pi, \gamma \in L^2\). Then the objective functional in the single medium scenario \((6)\) can be written as

\[J_m[v, g] = \langle \pi, g \rangle + \int_0^{+\infty} e^{-\rho t} \left[ \langle \gamma, \pi \rangle v(t) - c\alpha^2(t)/2 \right] dt. \quad (32)\]

Consequently the optimal value function can be written as,

\[V_m(g) = \langle \pi, g \rangle + \frac{\langle \gamma, \pi \rangle^2}{2pc} \quad (33)\]

(with maximum attained at \(\langle \gamma, \pi \rangle / c\)), and with \(\nabla V_m(g) = \pi \in L^2\).
Proof. The formula (32) may be derived as (20) from
\[
J_m[v, g] = \int_0^{\infty} e^{-rt} \left[ \langle \pi, G_m(t) \rangle - cv^2(t)/2 \right] dt
\]
while (33) is obtained by maximizing the right hand side in (32).

Remark 7 Note that (32) is consistent with (20) once we choose
\[
u(t; a) = v(t) \quad \text{and} \quad c = \langle B_k \gamma, \gamma \rangle.
\]
In such case the value functions of the segment specific scenario \(V_s\) and of the single medium scenario \(V_m\) satisfy the following relation
\[
V_m \leq V_s, \quad \text{with} \quad V_m = V_s \iff \gamma = B_1/\pi.
\]

Corollary 8 In assumptions of Theorem 6, the optimal control is
\[
v^*(t) = \frac{\langle \pi, \gamma \rangle}{c}
\]
while the optimal age-structured goodwill evolution is
\[
G^*_m(t, a) = e^{-\delta t}g(a - t)\chi_{[t, \omega]}(a) + \frac{\langle \pi, \gamma \rangle}{c} \int_0^{\min(t, a)} e^{-\delta s}(a - s)ds.
\]

Similarly to the case of segment specific scenario, the closed loop equation is written by means of a constant optimal feedback map
\[
R! R, v \rightarrow \frac{\langle \pi, \gamma \rangle}{c},
\]
that is
\[
\dot{G}_m(t) = AG_m(t) + \frac{\langle \pi, \gamma \rangle}{c} \gamma.
\]

An equilibrium point for the system is any stationary solution of the closed loop equation above, that is, the function \(G\) in \(L^2\) given by
\[
G_m^*(t) = -\frac{\langle \pi, \gamma \rangle}{c} \gamma = \frac{\langle \pi, \gamma \rangle}{c} \int_0^a e^{-\delta s}(a - s)ds,
\]
as better stated in the following theorem.

Theorem 9 There exists a unique equilibrium point for the control system, and this solution is given by
\[
G_m^*(a) = \frac{\langle \pi, \gamma \rangle}{c} \int_0^a e^{-\delta s}(a - s)ds,
\]
and it is asymptotically stable, that is for all optimal goodwill paths
\[
\lim_{t \to \infty} G^*_m(t, a) = G_m^*(a), \quad \forall a \in [0, \omega].
\]

An interesting observation is connected with the anticipation effect, which seems more relevant in the segment specific scenario rather than in that of single medium scenario. In both cases such anticipation takes place through the
presence in the optimal control strategy (see formulas (21) (34)) of the marginal demand \( \pi \), which anticipates the evolution of the segmentation. However, while in the segment specific scenario the whole function \( \pi \) is involved, in the single medium scenario only values in the intersection of the supports of \( \gamma (a) \) and \( \pi (a) \) are taken into account, through the inner product \( \langle \pi, \gamma \rangle \). The larger is this intersection, the higher is the optimal advertising level. An example would probably be of help. Assume that \( \gamma (a) = \chi_{[18,30]}(a) \) and \( \pi (a) = \chi_{[31,60]}(a) \).

The intersection between the support of \( \gamma \), the spectrum of the advertising medium, and the support of \( \pi \), modeling the target market, is empty. Yet, the intersection between the supports of \( \gamma \) and \( \pi (a) \) is not, and the activation level is non-zero due to such overlapping.

Another application of the results of this section is the definition of an index of preference among different advertising media. Let us assume that the decision maker wants to plan an advertising campaign using only one advertising medium. He can choose in a set of advertising media characterized by different spectra \( \gamma_1(a), \gamma_2(a), ..., \gamma_n(a) \) and different activation costs \( c_1, c_2, ..., c_n \), respectively. Using (34) and denoting as \( v_i^* \) the optimal activation level of the advertising medium \( i \), we can obtain that its optimal value can be written as

\[
J_i^* \left[ v_i^*, g \right] = \langle \pi, g \rangle + \frac{\langle \pi, \gamma_i \rangle^2}{2pc_i}
\]

Therefore, the decision maker chooses the advertising medium that maximized the preference index

\[
I_i = \frac{\langle \pi, \gamma_i \rangle^2}{c_i}
\]  

(38)

This way of comparing the effectiveness of media is consistent with the one performed in [15, formula (4.1)]. The main difference is that in (38) the inner product \( \langle \pi, \gamma_i \rangle \) is defined in the space \( L^2 \), while in [15, formula (4.1)] the inner product is considered in the space \( \mathbb{R}^n \), where \( n \) is the number of segments. As a consequence, the index defined in (38) is somehow the natural extension in infinite dimension of the one defined in [15], except for the fact that in (38) \( \pi(a) \) substitutes \( \pi (a) \).

### 4.1 Simulation

Here we present two simulations. In both the examples, we consider an advertising medium with spectrum

\[
\gamma (a) = \chi_{[\eta, \vartheta]}(a)
\]

with \( 0 < \eta < \vartheta < \omega \). Hence the decision-maker can hit with an advertising message only the age segments between \( \eta \) and \( \vartheta \). Moreover, as in the previous section, we consider as target market \( [\alpha, \omega] \) in the first example and \( [\alpha, \beta] \) in the second example. However, the activation cost of this advertising medium is
coherent with the previous example (where $\kappa(a) \equiv \bar{\kappa}$) if $c = \langle \kappa, \gamma^2 \rangle$, hence we assume

$$c = \int_0^\omega \chi_{[\eta, \omega]}(a) \, da = \bar{\kappa}(\vartheta - \eta)$$

Choosing in the examples the same target markets as in the previous scenario allows us to explain better the differences between the segment specific and single medium scenarios.

### 4.1.1 Target market $[\alpha, \omega]$

Since $\pi$ is given also in this case by (27), the optimal advertising effort is

$$v^*(t) \equiv v^* = \frac{1}{\kappa(\vartheta - \eta)} \int_\eta^\theta \pi(a) \, da \quad (39)$$

In particular, if $\alpha < \eta$ (i.e. the support of the advertising medium is a subset of the support of the target market which is a reasonable assumption), then (39) becomes

$$v^*(t) \equiv \frac{1}{c(\rho + \delta)} \int_\eta^\theta \left[ 1 - e^{-(\rho + \delta) \omega} e^{(\rho + \delta) a} \right] \, da \quad (40)$$

$$\equiv \frac{1}{\bar{\kappa}(\rho + \delta)} \frac{e^{-(\rho + \delta)(\omega - \theta)} - e^{-(\rho + \delta)(\omega - \eta)}}{\bar{\kappa}(\theta - \eta)(\rho + \delta)^2} \quad (41)$$

In the following simulation the parameters are chosen as follow: $\alpha = 18, \omega = 70, \delta = 0.1, \rho = 0.01, \bar{\kappa} = 1, g(a) \equiv 0.$

In Picture 5 we represent the advertising effort directed to each segment, by means of the chosen medium. The activation level (i.e. the height of the function) is given by (40). In Picture 6, we draw the optimal trajectory, noting how its shape is affected by the shape of the advertising medium.
4.1.2 Target market \([\alpha, \beta]\)

In this second example the optimal advertising effort is

\[
v^* (t) \equiv v^* = \frac{1}{\tilde{\kappa}(\vartheta - \eta)} \int_{\eta}^{\vartheta} \pi(a) da
\]

When the support of the advertising medium is "greater" than the support of the target market (i.e. if \(\beta < \eta\) in our example), then the advertising medium is not activated. It happens because it is not convenient to invest in segment-a goodwill when this segment is not (and it will not be) in the support of the target market. On the contrary, when the support of the advertising medium is "less" than support of the target market (i.e. \(\vartheta < \alpha\) in our example) the advertising medium is activated, because the segments hint by the advertising message will be in the target market in the future. It is again an example of the anticipation effect we have already described. Assuming that the support of the advertising medium is a subset of the support of the target market (i.e. \([\eta, \vartheta] \subseteq [\alpha, \beta]\) in our example) we obtain that

\[
v^* (t) \equiv v^* = \frac{1}{\tilde{\kappa}(\vartheta - \eta)} \int_{\eta}^{\vartheta} \left[ 1 - e^{-(\rho+\delta)(\beta-a)} \right] / da
\]

This formula is very close to (40), the unique difference is due to the highest segment which is in the target market. In (40) this value is \(\omega\), while in (42) this value is \(\beta\). Performing a simulation we obtain some pictures which are very close to Picture 5 and Picture 6. The unique difference is in the high of the function in Picture 5 which is connected with the intersection between the support of the advertising medium and the function \(\pi\). This notice is in accord with the results about the preference index (38) which gives us the following information: the wider is the intersection between the support of \(\pi\) and the support of the advertising medium, the more profitable the advertising medium is.

5 Conclusions

In this paper we present an extension to a continuous segmentation of the results described in [15], in the particular case where segmentation occurs with respect to consumers’ age.

We solve the problem in two different settings. In the first (the segment-specific scenario) the decision-maker chooses advertising strategies which are jointly differentiated by means of time and age, in the second (the single-medium scenario) the controller is bound to use only one advertising medium and the advertising policy represents the intensity with which such medium is activated.

Using dynamic programming in infinite dimensional spaces we can straightforwardly characterize the optimal advertising strategies in both scenarios, also
in the long run. In the analytic formula describing equilibrium points we recognize the standard golden rule marginal revenues equal to marginal costs, through the use of the discounted marginal demand function, which takes into account that revenues evolve along with time. The optimal equilibrium point for the state variable goodwill represents the optimal shape that is desirable for the decision-maker in the long run.

A phenomenon arising in both scenarios is an anticipating effect in the optimal advertising policies: for the decision-maker it is convenient to invest not only in the segments that compose the target market, but also in the segment that will be in the target market in the future.

The anticipating effect takes place here and not in [15], as age-segmentation is unavoidably connected to time evolution. Moreover, it is greater in the first scenario, due to the fact that all values of the discounted marginal demand contribute to form the optimal strategy, while in the second scenario only those intersecting the support of the spectrum of the advertising medium are taken into account.

The analysis of the second scenario allows also to introduce a preference index associated to advertising media which is specific of age-structured goodwill.

The analytical tractability of the model is strongly connected with the choice of a demand function which is linear in the state variable, although we believe that the technique can be extended to more general revenues, which is nowadays a work in progress.

References


