A mathematical problem based on a property of bond duration: Two new proofs and completion of an unfinished proof *

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Abstract. The mathematical problem in this article is originally from the finance topic of bond duration. With one notable exception, available published proofs for the problem seem unsuitable for coverage in undergraduate investment courses. However, the proof in the exceptional case is unfinished from a mathematical standpoint. We provide two new proofs and complete the unfinished proof, without assuming any prior knowledge of the financial concepts involved. All these proofs require only mathematical tools that are familiar to students in undergraduate economics and finance programs.

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1 Introduction

This article considers a mathematical problem originally from the finance topic of bond duration, which is about the price sensitivity of a bond to changes in interest rates. When introduced by Macaulay [**4**] in 1938, bond duration was formulated as a weighted average of the arrival times of investment income, with a lower weighted average implying sooner payoff. Given this intuitive perspective, bond duration has remained relevant in the financial world today; it is part of routinely reported bond data.

We use Macaulay's formulation to state a mathematical problem, without assuming any prior knowledge of the financial concepts involved. The problem

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is on a nontrivial issue of whether a longer life of a bond necessarily corresponds to a greater bond duration. There have been several proofs for the problem, as published in various academic journals and conference proceedings from 1984 to 2017. With one notable exception, these proofs seem unsuitable for coverage in undergraduate investment courses. However, the proof in the exceptional case, which relies on a numerical example for its completion, is unfinished from a mathematical standpoint.

Given the pedagogic objective of this article, we first state the above mathematical problem in Section 2 and provide the context and the background, including a brief description of various published proofs, in Section 3. We then present two new proofs, including a direct proof and an induction proof, in Sections 4 and 5. Finally, we complete the above unfinished proof in Section 6.

As the individual proofs in Sections 4-6 require only mathematical tools that are familiar to students in undergraduate economics and finance programs, they are all suitable for pedagogic purposes. At the beginning of each section containing a specific proof, we also briefly indicate its contribution, as well as the key feature of the approach involved. Such proofs are intended to serve as a catalyst for others to explore alternative proofs for similar mathematical problems.

2 A mathematical problem

Given

$$D_n = \frac{w_n}{p_n}, \text{ for } n = 1, 2, 3, \dots, \tag{1}$$

$$w_n = c \sum_{h=1}^{n} \frac{h}{(1+r)^h} + \frac{n}{(1+r)^n}, \tag{2}$$

and

$$p_n = c \sum_{h=1}^{n} \frac{1}{(1+r)^h} + \frac{1}{(1+r)^n}, \tag{3}$$

prove that

$$D_{n+1} - D_n > 0, \text{ for } n = 1, 2, 3, \dots, \text{ if } c \geq r > 0.$$

Prove also that, if $r > c > 0$ instead, there exists a positive integer $\widehat{n}$ such that

$$D_{n+1} - D_n > 0, \text{ for } n = 1, 2, 3, \dots, \widehat{n} - 1,$$

and

$$D_{n+1} - D_n < 0, \text{ for } n = \widehat{n}, \widehat{n} + 1, \widehat{n} + 2, \dots$$

3 Context and background

In the above mathematical problem, D_n is the duration of a bond that matures in n periods, r is the bond yield, and c is the coupon rate.[1] Illustrations in

[1] Investors of a default-free bond will receive periodic coupon payments until the maturity date and also the face value of the bond on the maturity date. The bond

investment textbooks typically include various graphs of bond duration versus maturity (see, for example, Reilly and Brown [**6**, Chapter 18]). In such graphs, how D_n varies with n — in the current notation for ease of exposition throughout this article — depends on the relative values of c and r. If $c \geq r > 0$, D_n always increases with n. However, if $r > c > 0$ instead, the graph of D_n versus n shows a single maximum.

The earliest published proof of the above property of bond duration, as provided by Hawawini [**2**] in 1984, is calculus-based; it treats n as a continuous variable. After deriving a closed-form expression of D_n, it confirms the signs of the first partial derivative of D_n with respect to n. Another calculus-based proof, as provided by Pianca [**5**], is confined to the case of $r > c > 0$; it determines the specific value of n corresponding to the single maximum of D_n in terms of c and r by using the Lambert W function.[2]

All remaining published proofs have kept n as a positive integer, as specified in the bond model. The earliest of such proofs is a direct proof, as provided by Smith [**7**] in 1988. By treating D_n as a fulcrum — which is the point of balance for the arrival times of investment income — the proof hinges on how the fulcrum moves on the timeline as n increases. If $c \geq r > 0$ and $D_n < n$ (which is true for $n > 1$), D_n always increases with n. The case of $r > c > 0$ is less obvious, as how D_n varies with n depends on the values of c, r, n, and D_n. The proof of the existence of a single maximum in the graph of D_n versus n for the latter case in [**7**] is via a numerical example. Thus, the proof in [**7**], though innovative for pedagogic purposes, is incomplete from a mathematical standpoint.

The direct proof by Kojić and Lukač [**3**] also requires only algebraic tools to confirm that D_n always increases with n if $c \geq r > 0$. However, as the expression of $D_{n+1} - D_n$ in [**3**] is in terms of c, r, n, and D_{n+1}, how D_n varies with n if $r > c > 0$ is not easily discernible. The proof of a single maximum in the graph of D_n versus n for the case of $r > c > 0$ has turned out to be a major task in [**3**]; there is a separate section just to rule out the existence of multiple extrema in the graph.

The induction proof for $c \geq r > 0$ by Feng and Kwan [**1**] — which keeps n as a positive integer and can be extended to accommodate $r > c > 0$ as well — requires only familiar algebraic tools. Unlike the algebraic approach in [**3**], as the expression of $D_{n+1} - D_n$ in [**1**] is in terms of c, r, and n only, how D_n varies with n is easily recognizable. The proof in [**1**] can potentially fill the void left by the unfinished proof in [**7**]. Its pedagogic appeal as compared to the calculus-based approach in [**2**] is that there is no need for any digressions to

yield — which is a shorter version of the term "yield to maturity of the bond" — is investors' required rate of return each period, such as every six months, for holding the bond. The coupon rate is the coupon payment each period as a proportion of the face value of the bond.

[2] The Lambert W function is named after Johann Heirich Lambert (1728-1777). By definition, it is the multivalued function $W(z)$ that satisfies the equation $z = W(z)\exp[W(z)]$ for any complex number z. Equivalently, it can also be defined as the inverse function of $f(W) = W\exp(W)$.

justify the substitution of a continuous variable for the positive integer n there. However, as the preparation for the induction proof itself in [**1**] requires some tedious algebraic steps, improvements to the approach are still warranted.

This article originated from two assignments for new finance Ph.D. students in a graduate course during the Fall Term of 2020. The assignments were intended to provide different proofs for the part of the above mathematical problem where $c \geq r > 0$. A student in the class completed a direct proof, which requires only familiar algebraic tools. Another student derived the closed-form expression of D_n in terms of c, r, and n, as reported originally by Macaulay [**4**] without any accompanying analytical details. The latter student also completed an induction proof that is simpler than that in [**1**]. The subsequent collaboration of the instructor and the above two students has led to this article. It covers a direct proof and an induction proof, now extended to the case of $r > c > 0$ as well. It also completes the unfinished proof in [**7**] for $r > c > 0$.

4 A direct proof

The direct proof in this section uses the same closed-form expressions of $\sum_{h=1}^{n}(1+r)^{-h}$ and $\sum_{h=1}^{n} h/(1+r)^{-h}$ in Feng and Kwan [**1**]. The contribution here is in reducing the equivalent of $D_{n+1} - D_n$ (for its sign only) to an analytically convenient form, for which the arithmetic mean and geometric mean inequality is applicable for the completion of the proof.

4.1 Some Preliminary Algebraic Steps

Given how D_n, w_n, and p_n are defined in equations (1)-(3), as

$$D_{n+1} = \frac{(1+r)w_{n+1}}{(1+r)p_{n+1}} = \frac{c + c\sum_{h=1}^{n} \frac{h+1}{(1+r)^h} + \frac{n+1}{(1+r)^n}}{c + c\sum_{h=1}^{n} \frac{1}{(1+r)^h} + \frac{1}{(1+r)^n}} = \frac{c + p_n + w_n}{c + p_n},$$

we can write

$$D_{n+1} - D_n = \frac{w_{n+1}}{p_{n+1}} - \frac{w_n}{p_n} = 1 - \frac{cw_n}{p_n(c + p_n)}. \tag{4}$$

Thus, to prove that

$$D_{n+1} - D_n > 0, \text{ for } c \geq r > 0$$

is equivalent to proving

$$p_n(c + p_n) - cw_n > 0, \text{ for } c \geq r > 0.$$

The proof here requires the closed-form expressions

$$\sum\nolimits_{h=1}^{n} \frac{1}{(1+r)^h} = \frac{1}{r}\left[1 - (1+r)^{-n}\right] \tag{5}$$

and

$$\sum\nolimits_{h=1}^{n} \frac{h}{(1+r)^h} = \frac{1}{r^2}\left[(1+r) - (1+r+nr)(1+r)^{-n}\right] \tag{6}$$

from equations (B4) and (B8), respectively, in Feng and Kwan [**1**, Appendix B].

Letting $k_n = (1+r)^{-n}$, we can write

$$p_n = \frac{c}{r}(1-k_n) + k_n = \frac{c}{r} + \left(1 - \frac{c}{r}\right) k_n \tag{7}$$

and

$$w_n = \frac{c}{r^2}\left[(1+r) - (1+r+nr)k_n\right] + nk_n. \tag{8}$$

It follows that

$$\begin{aligned} p_n(c+p_n) &= \frac{c^2}{r} + \frac{c^2}{r^2} + \left(\frac{c}{r} - \frac{c^2}{r^2}\right) k_n + \left(c + \frac{c}{r} - \frac{c^2}{r} - \frac{c^2}{r^2}\right) k_n + \left(1 - \frac{c}{r}\right)^2 k_n^2 \\ &= \frac{c^2}{r} + \frac{c^2}{r^2} + \frac{c}{r^2}\left(r^2 + 2r - cr - 2c\right) k_n + \left(1 - \frac{c}{r}\right)^2 k_n^2 \end{aligned}$$

and

$$p_n(c+p_n) - cw_n = \frac{c}{r^2}\left[r(1+r) + (c-r)(nr-1)\right] k_n + \left(1 - \frac{c}{r}\right)^2 k_n^2. \tag{9}$$

For analytical convenience, let $q = c - r$, which is non-negative for $c \geq r$, and write

$$p_n(c+p_n) - cw_n = \frac{[(1+r)^n(nr-1)+1]q^2 + (1+r)^n r^2(n+1)q + (1+r)^{n+1}r^2}{r^2(1+r)^{2n}}.$$

As $r > 0$, $n \geq 1$, and $(1+r)^n > 0$, to reach $p_n(c+p_n) - cw_n > 0$, a sufficient condition is

$$(1+r)^n(nr-1) + 1 \geq 0.$$

If $nr \geq 1$, the positive sign of $p_n(c+p_n) - cw_n$ is obvious. If $nr < 1$ instead, let us write the same sufficient condition as

$$(1-nr)(1+r)^n \leq 1.$$

4.2 The Arithmetic Mean and Geometric Mean Inequality

The arithmetic mean of any positive numbers can never be less than their geometric mean. For a set of n positive numbers denoted as $x_1, x_2, \ldots, x_n$, the corresponding inequality

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

is known as the arithmetic mean and geometric mean inequality. We now apply it to a set of $n+1$ positive numbers, which specifically includes a positive number $1 - nr$ and n positive numbers, with each being $1 + r$. As

$$\sqrt[n+1]{(1-nr)(1+r)^n} \leq \frac{1}{n+1}[(1-nr) + n(1+r)] = 1,$$

we have

$$(1-nr)(1+r)^n \leq 1^{n+1} = 1,$$

thus confirming the positive sign of $p_n(c+p_n) - cw_n$. As n can be any positive integer, this result implies directly that

$$D_{n+1} - D_n > 0, \text{ for } c \geq r > 0,$$

thus completing a part of the proof.

For the proof where $r > c > 0$, let us write equation (9) as

$$p_n(c+p_n) - cw_n = \frac{1}{(1+r)^{2n}} \left\{ \frac{c(1+r)^n}{r^2} \left[r(1+r) + (c-r)(nr-1) \right] + \left(1 - \frac{c}{r}\right)^2 \right\}.$$

As $c < r$, the term

$$\frac{c(1+r)^n}{r^2} \left[r(1+r) + (c-r)(nr-1) \right],$$

which starts for small n as being positive when $nr < 1$ — thus corresponding to $p_n(c+p_n) - cw_n > 0$ — decreases monotonically as n increases. Eventually, there will be a value of n that makes this term negative enough to result in

$$p_n(c+p_n) - cw_n < 0.$$

Let us label this specific value of n as $\widehat{n}$. For $n \geq \widehat{n}$, D_n will decrease with increasing n. Thus, there exists a positive integer $\widehat{n}$ such that

$$D_{n+1} - D_n > 0, \text{ for } n = 1, 2, 3, \ldots, \widehat{n} - 1,$$

and

$$D_{n+1} - D_n < 0, \text{ for } n = \widehat{n}, \widehat{n} + 1, \widehat{n} + 2, \ldots$$

4.3 The Exclusion of $D_{n+1} = D_n$

The exclusion of the strict equality of D_{n+1} and D_n in the above proof requires an explanation. For $r > c > 0$, the strict equality requires $nr > 1$ on the right hand side of equation (9) as a necessary condition. With c treated as an unknown variable, its values satisfying the condition of

$$p_n(c+p_n) = cw_n$$

can be solved in terms of r and n via equation (9). This is equivalent to solving the quadratic equation

$$\alpha c^2 + \beta c + r^2 = 0$$

for c, where

$$\alpha = (nr-1)(1+r)^n + 1$$

and

$$\beta = \left[(2 - nr + r)(1+r)^n - 2 \right] r.$$

The two roots are

$$c = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha r^2}}{2\alpha}. \tag{10}$$

For a set of given values of c, r, and n, the strict equality of D_{n+1} and D_n requires that one of the two roots of c in equation (10) match exactly the given value of c.

If a positive integer m is a perfect square, $\sqrt{m}$ is also a positive integer; otherwise, $\sqrt{m}$ is not a rational number. A positive rational number s can be expressed as the ratio of two coprime positive integers. If either integer is not a perfect square, $\sqrt{s}$ is not a rational number and its decimal equivalent will have an infinite number of decimals. Given how α and β are defined under the condition of $nr > 1$, the term $\beta^2 - 4\alpha r^2$ under the square root is always a rational number. However, it is highly unlikely that the two coprime integers in the ratio are both perfect squares to ensure that $\sqrt{\beta^2 - 4\alpha r^2}$ (for a positive $\beta^2 - 4\alpha r^2$) be a rational number. As explained below, even if it turns out that $\sqrt{\beta^2 - 4\alpha r^2}$ is a rational number, it is impossible that one of the two roots of c in equation (10) can match exactly the given value of c.

In practice, given values of c seldom go beyond a few decimal places. For values of r and n satisfying the condition of $nr > 1$, suppose that the precision in the measurement of r is only one-tenth of 1% and the corresponding n exceeds 10. Then, the precise value of $(1+r)^n$ in α and β and the precise value of $(1+r)^{2n}$ in $\beta^2 - 4\alpha r^2$ will have no fewer than 30 and 60 decimal places, respectively. Any higher precision in the measurement of r will result in many more decimal places in the precise values of $(1+r)^n$ and $(1+r)^{2n}$.

In an unlikely scenario that $\sqrt{\beta^2 - 4\alpha r^2}$ is a rational number, each of the two coprime integers involved must have a large number of digits and, accordingly, the decimal equivalent of either root of c in equation (10) must also have a large number of decimals. Thus, it is impossible to achieve an exact match between such a decimal equivalent and the given value of c. Further, any rounding errors in the numerical computation of the two root of c will render the strict equality of D_{n+1} and D_n unachievable. An implication is that, for any positive integer n, as the given value of c has only a small number of decimals in practical settings, the computed value of the right hand side of equation (9) will never be strictly zero.

The strict equality of D_{n+1} and D_n for $r > c > 0$ is also ruled out in the proofs in the next two sections. This is because all proofs here are based on the same bond model for which the closed-form expressions of D_n involved are equivalent. Their equivalence will allow equation (9) and its implications to remain applicable if the feasibility of $D_{n+1} = D_n$ is also explicitly considered in the remaining proofs.

5 An induction proof

To prepare for the induction proof in this section, we also use closed-form expressions of $\sum_{h=1}^{n}(1+r)^{-h}$ and $\sum_{h=1}^{n} h/(1+r)^{-h}$ in Feng and Kwan [**1**]. We then use equations (5) and (6) at the start of Section 4 to replicate the closed-form expression of D_n in Macaulay [**4**]. The contribution here is in using Macaulay's expression of D_n in the induction proof that follows.

5.1 Replication of Macaulay's Expression of D_n

Given equations (5) and (6) in Section 4, we can write

$$p_n = \frac{c}{r}\left[1-(1+r)^{-n}\right]+(1+r)^{-n} = \frac{c}{r} - \frac{c-r}{r}(1+r)^{-n}$$

and

$$\begin{aligned} w_n &= \frac{c}{r^2}\left[(1+r)-(1+r+nr)(1+r)^{-n}\right]+n(1+r)^{-n} \\ &= \frac{c(1+r)}{r^2} - \frac{c}{r^2(1+r)^{n-1}} - \frac{(c-r)n}{r(1+r)^n}. \end{aligned}$$

It follows that

$$\begin{aligned} D_n &= \frac{w_n}{p_n}\cdot\frac{r^2(1+r)^n}{r^2(1+r)^n} = \frac{c(1+r)^{n+1}-c(1+r)-(c-r)rn}{-(c-r)r+cr(1+r)^n} \\ &= \frac{1+r}{r}\cdot\frac{[c(1+r)^n-c]}{c(1+r)^n-(c-r)} - \frac{(c-r)n}{c(1+r)^n-(c-r)} \\ &= \frac{1+r}{r}\cdot\frac{[c(1+r)^n-(c-r)]}{c(1+r)^n-(c-r)} - \frac{(c-r)n+(1+r)}{c(1+r)^n-(c-r)}, \end{aligned}$$

which leads to

$$D_n = \frac{1+r}{r} - \frac{(c-r)n+(1+r)}{c(1+r)^n-(c-r)}. \tag{11}$$

This is an expression equivalent to that in Macaulay [**4**, Chapter 2], page 49, but in the current notation.

5.2 Some Preliminary Algebraic Steps

Letting $a = c/r$ or, equivalently, $c = ar$, we can write

$$D_n - \frac{1+r}{r} = -\frac{(a-1)rn+(1+r)}{ar(1+r)^n-(a-1)r}$$

and

$$D_{n+1} - \frac{1+r}{r} = -\frac{(a-1)r(n+1)+(1+r)}{ar(1+r)^{n+1}-(a-1)r},$$

for $r > 0$. It follows that

$$r(D_{n+1}-D_n) = \frac{(a-1)rn+(1+r)}{a(1+r)^n-(a-1)} - \frac{(a-1)rn+(1+r)+(a-1)r}{a(1+r)^n-(a-1)+ar(1+r)^n},$$

which leads to

$$r(D_{n+1} - D_n) = \frac{A_n}{[a(1+r)^n - (a-1)][a(1+r)^n - (a-1) + ar(1+r)^n]}, \quad (12)$$

where

$$\begin{aligned} A_n &= [(a-1)rn + (1+r)][a(1+r)^n - (a-1) + ar(1+r)^n] \\ &\quad -[(a-1)rn + (1+r) + (a-1)r][a(1+r)^n - (a-1)] \\ &= [(a-1)rn + (1+r)]ar(1+r)^n - (a-1)ar(1+r)^n + (a-1)^2 r \\ &= r\{a[(a-1)(rn-1) + (1+r)](1+r)^n + (a-1)^2\}. \end{aligned}$$

Now, let

$$B_n = \frac{A_n}{r} = a[(a-1)(rn-1) + (1+r)](1+r)^n + (a-1)^2.$$

As $a(1+r)^n > a > a-1$, the denominator in the expression of $r(D_{n+1} - D_n)$ on the right hand side of equation (12) is positive. Then, B_n, A_n, and $D_{n+1} - D_n$ must be of the same sign. Thus, what needs to be confirmed here is that

$$B_n > 0, \text{ for } n = 1, 2, 3, \ldots, \text{ if } c \geq r > 0.$$

5.3 The Induction Proof Itself

For an induction proof, the base case is $n = 1$. As

$$\begin{aligned} B_1 &= a[(a-1)(r-1) + (1+r)](1+r) + (a-1)^2 \\ &= (a^2 r - a^2 + 2a)(1+r) + (a^2 - 2a + 1) \\ &= a^2 r^2 + 2ar + 1 = (ar+1)^2 > 0, \end{aligned}$$

the base case is confirmed. By the induction hypothesis, suppose that $B_n > 0$ for $n = k$; that is,

$$B_k = a[(a-1)(rk-1) + (1+r)](1+r)^k + (a-1)^2 > 0.$$

For $n = k+1$, let us write

$$\begin{aligned} B_{k+1} &= a\{(a-1)[r(k+1) - 1] + (1+r)\}(1+r)^{k+1} + (a-1)^2 \\ &= a[(a-1)(rk-1) + (1+r)](1+r)(1+r)^k + (a-1)^2 \\ &\quad +a(a-1)r(1+r)(1+r)^k \\ &= a[(a-1)(rk-1) + (1+r)](1+r)^k + (a-1)^2 \\ &\quad +a[(a-1)(rk-1) + (1+r)]r(1+r)^k + a(a-1)r(1+r)(1+r)^k, \end{aligned}$$

which is equivalent to

$$\begin{aligned} B_{k+1} &= B_k + a[(a-1)(rk-1) + (1+r)]r(1+r)^k + a(a-1)r(1+r)(1+r)^k \\ &= B_k + ar[(a-1)(rk-1) + (1+r) + (a-1)(1+r)](1+r)^k \\ &= B_k + ar[(a-1)rk + ar + 1](1+r)^k. \end{aligned}$$

As $a \geq 1$ and $r > 0$, we always have $B_{k+1} > 0$, thus completing the induction proof.

The case of $1 > a > 0$ is where $r > c > 0$. In such a case, even if $B_k > 0$, the term $ar[(a-1)rk + ar + 1](1+r)^k$ can be negative. If this term is negative enough to cause

$$B_k + ar[(a-1)rk + ar + 1](1+r)^k < 0,$$

then we have $B_k > 0$ and $B_{k+1} < 0$. As long as $B_{k+1} < 0$, we also have

$$\begin{aligned} B_{k+2} &= B_{k+1} + ar[(a-1)(r)(k+1) + ar + 1](1+r)^{k+1} \\ &< B_k + ar[(a-1)rk + ar + 1](1+r)^k < 0. \end{aligned}$$

Once we have $B_{k+1} < 0$, we also have $B_{k+2} < 0$, $B_{k+3} < 0$, and so on. As B_n and $D_{n+1} - D_n$ have the same sign, the graph of D_n versus n will have a single maximum. That is, D_n increases with n for $n = 1, 2, 3$, and so on, but when D_n starts to decrease at some value of n, it will continue to decrease as n increases. Thus, there exists a positive integer $\widehat{n}$ such that

$$D_{n+1} - D_n > 0, \text{ for } n = 1, 2, 3, \ldots, \widehat{n} - 1,$$

and

$$D_{n+1} - D_n < 0, \text{ for } n = \widehat{n}, \widehat{n} + 1, \widehat{n} + 2, \ldots$$

6 Completion of an unfinished proof

The contribution of this section is in the completion of the unfinished proof in Smith [**7**]. To facilitate the task involved, we first replicate some key materials there, but in the current notation. As the proof hinges on how $n - D_n$ varies with n, we present two alternative approaches for its verification, with and without relying on differential calculus tools.

6.1 Replication of Some Key Materials in the Fulcrum Approach

Given how D_n is defined, as $w_n - p_n D_n = 0$, we can write

$$\sum\nolimits_{h=1}^{n} \frac{(h - D_n)c}{(1+r)^h} + \frac{n - D_n}{(1+r)^n} = 0, \text{ for } n = 1, 2, 3, \ldots,$$

which is equation (12) in Smith [**7**], page 30. With D_n being the fulcrum on the timeline, we can write the same equation as

$$\sum\nolimits_{h=1}^{k} \frac{(D_n - h)c}{(1+r)^h} = \sum\nolimits_{h=k+1}^{n} \frac{(h - D_n)c}{(1+r)^h} + \frac{n - D_n}{(1+r)^n}, \tag{13}$$

where $k < D_n < k+1$. Let

$$H = \frac{n - D_n}{(1+r)^n},$$

which is the last term on the right hand side of equation (13).

While maintaining the same fulcrum D_n on the timeline, let us increase n to $n+1$ and replace H with

$$K = \frac{(n+1-D_n)c}{(1+r)^{n+1}} + \frac{n+1-D_n}{(1+r)^{n+1}} = \frac{[(n-D_n)+1](1+c)}{(1+r)^{n+1}}.$$

This substitution will lead to the imbalance of the two sides of equation (13). To regain the balance requires a movement of the fulcrum on the timeline. If $K > H$, we have $D_{n+1} > D_n$, where D_{n+1} is the new fulcrum on the timeline; if $K < H$, we have $D_{n+1} < D_n$ instead.

As $n - D_n > 0$, for $n > 1$, the condition for $D_{n+1} > D_n$, when simplified as

$$\frac{[(n-D_n)+1](1+c)}{1+r} > n - D_n,$$

will lead to

$$\frac{1+c}{n-D_n} > r - c. \tag{14}$$

Likewise, the condition for $D_{n+1} < D_n$ can be stated as

$$\frac{1+c}{n-D_n} < r - c. \tag{15}$$

Smith [**7**] has shown that $D_{n+1} > D_n$ for $c \geq r > 0$, as inequality (14) always holds. For the case of $r > c > 0$, we prove below that $n - D_n$ — which is positive for $n > 1$ — increases with n. Given such an analytical property, there will be a value of n beyond which inequality (15) is satisfied. As soon as this specific value of n is reached, we have $D_{n+1} < D_n$ instead. Thus, for $r > c > 0$, there is a single maximum in the D_n versus n graph.

6.2 A Calculus-Based Proof

To prove that $n - D_n$ increases with n, we first write

$$n - D_n = n - \frac{1+r}{r} + \frac{(c-r)n + (1+r)}{c(1+r)^n - (c-r)}, \tag{16}$$

by using the closed-form expression of D_n in Macaulay [**4**], as replicated in equation (11) of Section 5.[3] We then substitute a continuous variable x for the positive integer n in equation (16) and define

$$f(x) = x - \frac{1+r}{r} + \frac{(c-r)x + (1+r)}{c(1+r)^x - (c-r)},$$

[3] Smith [**7**] has also replicated Macaulay's expression of D_n in his equation (8) on page 29 there. However, the same equation has not been used analytically in his unfinished proof.

where c and r are constants satisfying the condition of $r > c > 0$. Implicitly, we have $f(n) = n - D_n$, for $n = 1, 2, 3, \ldots$

The first-order derivative of $f(x)$ is

$$\frac{d}{dx} f(x) = \frac{g(x)}{[c(1+r)^x - (c-r)]^2},$$

where

$$\begin{aligned} g(x) &= [c(1+r)^x - (c-r)]^2 + [c(1+r)^x - (c-r)](c-r) \\ &\quad -[(c-r)x + (1+r)]c(1+r)^x \ln(1+r) \\ &= c(1+r)^x\{c(1+r)^x + (r-c) + [(r-c)x - (1+r)]\ln(1+r)\}. \end{aligned}$$

For $df(x)/dx > 0$, we need $g(x) > 0$. We have

$$\begin{aligned} g(1) &= c(1+r)\{c(1+r) + (r-c) + [(r-c) - (1+r)]\ln(1+r)\} \\ &= c(1+c)(1+r)[r - \ln(1+r)] > 0, \end{aligned}$$

as $r > \ln(1+r)$ for $r > 0$. As both $(1+r)^x$ and $(r-c)x$ increase with x for $r > c > 0$, so does $g(x)$. Thus, we confirm that $g(x) > 0$ and $df(x)/dx > 0$ for all $x \geq 1$. Letting $x = 1, 2, 3, \ldots$, we have $f(1) < f(2) < f(3) < \cdots$ or, equivalently, $1 - D_1 < 2 - D_2 < 3 - D_3 < \cdots$, confirming that $n - D_n$ increases with n for $r > c > 0$.

6.3 An Alternative Proof

We can also prove that $n - D_n$ increases with n without using any calculus tools. The proof here starts with equation (4) in Section 4. As

$$\frac{cw_n}{p_n(c + p_n)} > 0,$$

equation (4) implies that

$$D_{n+1} - D_n < 1, \text{ for } n \geq 1.$$

Rearranging the terms in the inequality

$$D_{n+1} - D_n + n < 1 + n$$

leads to

$$n - D_n < (n+1) - D_{n+1}, \text{ for } n \geq 1.$$

As n is an integer, this inequality implies that $n - D_n$ increases with n. Notice that, in the proof here, whether $c \geq r > 0$ or $r > c > 0$ is irrelevant.

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