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Constraint**

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Strategic Information Transmission with Budget Constraint

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Abstract

In this paper, I discuss a Cheap Talk model that arises during the allocation of a limited budget to multiple Senders by a Receiver with private communication. The Receiver's utility is the sum of the utilities of the Senders. Considering quadratic utility functions, I show that there is no fully revealing equilibrium with budget constraint. I also show that a higher budget facilitates information transmission to the Receiver in terms of ex-ante expected utility by considering (1) an equilibrium where only one Sender reveals truthfully, (2) a symmetric equilibrium with two intervals and (3) a commitment strategy by the Receiver where only one Sender receives his desired amount. The commitment strategy is doing better than the other two types of equilibria for budget more than a particular value. This requires us to look for equilibria with higher number of intervals which does better than the commitment strategy.

Keywords

Cheap Talk, Multiple Senders, Budget Constraint

JEL Codes

C72, D82, D83

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1 Introduction

Many social and commercial organizations generally have different branches to deal with different issues. The organization frequently faces the decision of how much of the budget or of the resources to allocate to each branch. But as many organizations do not possess adequate wealth to give the desired amount of each branch, an organization faces a task of efficiently allocating its wealth to its branches which is the classical budget allocation problem. But each branch may like to get its best choice without caring about the whole organization by misreporting its desired need. This forms the basis of the Cheap Talk setting that I set to discuss in this paper¹.

In the seminal paper *Strategic Information Transmission* by Crawford and Sobel (1982) [3], the authors described a form of communication which is costless (Cheap Talk) between an informed Sender and an uninformed Receiver regarding the state of the Nature where the players prefer different actions for given states of the Nature. The difference in preferences between the players (in other words the difference in biases) given the states of the Nature gives rise to strategic communication among players. Since then there have been numerous papers on different aspects of Cheap Talk. Gilligan and Krehbiel(1989)[6], Krishna and Morgan[8] are the main works with multiple Senders in one-dimensional state space. The paper of Krishna and Morgan [8] also discusses about the sequential communication. Farrel and Gibbons (1989)[4] and Newmann and Sansing (1993)[11] discusses Cheap Talk with multiple Receivers. Battaglini (2002)[2], Levy and Razin (2004)[12] are some of the works on Cheap Talk in multiple dimensions state and policy space. Li (2003)[9] and Frisell and Lagerloef (2007)[5] discuss the Cheap Talk with uncertain biases. The paper by Gordon (2010)[7] discusses Cheap Talk in one dimensional state space between a Sender and a Receiver where the biases are state dependent. In this paper, the author has described an equilibrium with infinite partitions of the state space without truth revelation.

My model incorporates many features of the above literature. In my model, I have

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multiple Senders with one Receiver and I have state-dependent biases that arises when there is budget constraint. I have also multiple dimensions of the state space as well as the policy space, but each Sender is only interested in his own dimension of the state space and policy space. All these features arise quite naturally if we consider the following example.

Consider the manager (she) of a water Reservoir who wants to allocate water to two farmers (he), call them farmer 1 and farmer 2. The manager corresponds to the Receiver and the farmers correspond to the Senders in the Cheap Talk literature. The manager faces with a fixed amount of water to allocate between the farmers. Each farmer's need of water is his private knowledge. I assume that the farmers have quadratic utility functions. This assumption is quite natural because a higher water than the requirement can cause flood or less water can cause drought. The manager here represents the society and hence her utility is the sum of the utilities of both the farmers. If there were no budget constraint, each farmer asks for the exact amount he needs and the manager allocates him the exact amount. But faced with a budget constraint, the manager may not allocate the required amount to each farmer. She allocates to each farmer that maximizes her utility within the budget limit and so each farmer gets a reduced amount. But then one farmer may not like to ask the exact amount he needs and will like to ask a higher amount given the other farmer is asking his exact amount and the Receiver believes both the farmers. This gives rise to strategic communication found in the Cheap Talk literature because the preferences (biases) of the players are different here, as discussed in Crawford and Sobel (1982) [3]. In my model, the biases depend upon what amounts the farmers need, that is the biases are state dependent similar to Gordon (2010)[7]. Also the bias depends upon how much budget is available, if there is sufficient budget, then there is no bias among players.

The paper by Mcgee and Yang (2009)[10] discusses a multi-sender Cheap Talk model in a multidimensional state space. In their model, the Senders have full information in some dimensions but not all dimensions of the state space similar to my model where each farmer only knows how much water he needs. In their paper, the biases are given exogenously whereas in my model the biases arise as a result of resource or budget constraints and in my model the biases are state-dependent unlike their model. Also in Mcgee and Yang [10], the policy space is one dimensional whereas the state space is

multidimensional, in my work, both the state space is multidimensional as well as the policy space. In my model, the Senders receive utilities in the dimensions they provide information i.e. their areas of expertise rather than all dimensions.

My paper considers the budget constraint problem discussed in Ambrus and Takahashi (2007)[1] where the budget constraint restricts the policy space. My model differs from their model in terms of utility functions and biases. Unlike their paper, the utility functions I consider here have state-independent biases because of the budget constraint. In my model, the utility of each Sender does not depend on the state of the other Sender and the Receiver's utility is the sum of the utilities of both the Senders.

The paper by Gordon (2010)[7] discusses a model of one Sender and one Receiver in one dimensional state space and the bias is state dependent. The author shows the existence of an equilibrium with infinite intervals without full revelation when there is outward bias. In my model, I have outward bias more specifically upward bias if we fix the state of the other Sender. I conjecture the existence of an equilibrium with infinite intervals without full revelation in my work similar to his paper.

Now I discuss the results I have obtained in my model. First I show that with a budget constraint there is no full revelation of the information. With a limited budget, if a Sender (Sender is the farmer in the above example and Receiver is the manager) needs a high amount, given he knows the other Sender's need, he would like to get his desired amount by asking for a higher amount than his need. The higher is the need of each Sender, the higher he will like to deviate. Since in our model, each Sender does not know the other Sender's need, it needs to be proved whether there is full revelation with a budget constraint and I have proved that in fact there is no full revelation which means each Sender has some states where he wants to deviate.

Then I show that we have interval partition like the general cheap talk literature which means the state space of each Sender is partitioned into intervals. I can not use the technique for one dimensional state space to show for interval partition as I have multiple dimensional state space, I use a different technique to show for interval partition in our model.

Then I discuss the effect of budget with two types of equilibria: in the first type of equilibrium, only one Sender reveals fully and the other Sender partitions his state

into intervals and in the second type of equilibrium, each Sender partition the state space into intervals and I consider symmetric equilibrium where the intervals for both the Senders are same. I calculate the ex-ante expected utilities and show the effect of budget by demonstrating that higher budget facilitates more information in terms of ex-ante expected utility. This is quite intuitive because if the budget constraint is relaxed, it is more probable that a Sender receives his required amount and hence the less he would like to deviate. I also consider a commitment strategy by the Receiver where she gives to one Sender his desired amount and the rest she gives to the other Sender depending on his need. I showed that this commitment strategy is doing better than the above two type of equilibria for a budget more than 1.05 and we have the question whether there is an equilibrium which does better than the commitment strategy.

When I discuss about future research in this model, I conjecture that there exists equilibria with higher number of intervals which does better than the commitment strategy of the Receiver. As budget increases, the number of intervals may need to increase to do better than commitment. I mention certain properties of the equilibrium with infinite intervals like at which point it converges and how the intervals are separated. Even if the existence of equilibrium with higher number of intervals may be proved, the determination of the interval points remain challenging to calculate because in the indifference conditions, we may come across cubic equations.

2 The Model

Consider two Senders S_1 and S_2 and a Receiver R . Each Sender S_i (i denotes both 1, 2) observes his state $\theta_i \in \Theta_i = [0, 1]$. The realization of θ_i is drawn from a prior distribution F_i with a cdf f_i over Θ_i . I assume that F_i is a uniform distribution. Only S_1 observes the state θ_1 while only S_2 observes the state θ_2 . S_i has to report a message m_i to R about his knowledge of the state θ_i . For simplicity, I assume that θ_1 and θ_2 are independent. Let $y_i \in \mathbb{R}^+$ (real non-negative numbers) be the action taken for the state θ_i . The Receiver faces a budget constraint $y_0 \in \mathbb{R}^+$ which says that $y_1 + y_2 \leq y_0$. I assume that the value of y_0 is a common knowledge.

The utilities of the players are given by if the realized (true) states are θ_1 and θ_2 ,

$$U^{S_1}(y_1, \theta_1) = -(y_1 - \theta_1)^2$$

$$U^{S_2}(y_2, \theta_2) = -(y_2 - \theta_2)^2$$

$$U^R(y_1, y_2, \theta_1, \theta_2) = -(y_1 - \theta_1)^2 - (y_2 - \theta_2)^2$$

S_i sends a message $m_i \in M$ where M has at least as many elements as in $[0, 1]$ so that S_i can distinguish all the states of Θ_i . For the analysis here, I assume $M = [0, 1]$.

The Receiver's actions after hearing the messages m_1 and m_2 with budget constraint y_0 are $y_1(m_1, m_2, y_0) \geq 0$ and $y_2(m_1, m_2, y_0) \geq 0$ and they satisfy

$$y_1(m_1, m_2, y_0) + y_2(m_1, m_2, y_0) \leq y_0$$

To find the optimal action choice of the players in the states (θ_1, θ_2) with the budget constraint y_0 , consider Figure 1.

Case 1 ($\theta_1 + \theta_2 \leq y_0$):

We are in the region EDF , the best choice for S_1 is $y_1 = \theta_1$, for S_2 is $y_2 = \theta_2$ and for R is $y_1 = \theta_1$, $y_2 = \theta_2$.

Case 2 ($\theta_1 + \theta_2 \geq y_0$ and $\theta_2 - \theta_1 \geq y_0$):

We are in the region AEG , the best choice of R is $y_1 = 0$ and $y_2 = y_0$, best choice of S_1 is $y_1 = \theta_1$ if $\theta_1 \leq y_0$ and $y_1 = y_0$ if $\theta_1 \geq y_0$, best choice of S_2 is $y_2 = y_0$.

Case 3 ($\theta_1 + \theta_2 \geq y_0$ and $\theta_2 - \theta_1 \leq y_0$ and $\theta_1 - \theta_2 \leq y_0$):

We are in the region $GEFHB$, the best choice of the Receiver is $y_1 = \theta_1 - \frac{\theta_1 + \theta_2 - y_0}{2}$ and $y_2 = \theta_2 - \frac{\theta_1 + \theta_2 - y_0}{2}$, best choice of S_1 is $y_1 = \theta_1$ if $\theta_1 \leq y_0$ and $y_1 = y_0$ if $\theta_1 \geq y_0$ and best choice of S_2 is $y_2 = \theta_2$ if $\theta_2 \leq y_0$ and $y_2 = y_0$ if $\theta_2 \geq y_0$.

Case 4 ($\theta_1 + \theta_2 \geq y_0$ and $\theta_1 - \theta_2 \geq y_0$):

We are in the region CFH , the best choice of R is $y_1 = y_0$ and $y_2 = 0$, best choice of S_1 is $y_1 = y_0$, best choice of S_2 is $y_2 = \theta_2$ if $\theta_2 \leq y_0$ and $y_2 = y_0$ if $\theta_2 \geq y_0$.

Notice that as we increase y_0 from 0 to 1, the regions AEH and CFH decreases and the region EDF increases. As $y_0 \geq 1$, there are only two regions where the region

Figure 1: Best choices of the Players

EDF expands to a pentagon and the region $GEFHB$ condenses to a triangle and both the regions AEH and CFH vanishes.

If we write the above cases in compact form for all players, the Receiver's optimal actions choice in the states (θ_1, θ_2) with the budget constraint y_0 is,

$(\gamma_1^R(\theta_1, \theta_2, y_0), \gamma_2^R(\theta_1, \theta_2, y_0))$ where

$$\begin{aligned} \gamma_1^R(\theta_1, \theta_2, y_0) &= \min \left[\max \left(0, \theta_1 - \max \left(0, \frac{\theta_1 + \theta_2 - y_0}{2} \right) \right), y_0 \right] \\ \gamma_2^R(\theta_1, \theta_2, y_0) &= \min \left[\max \left(0, \theta_2 - \max \left(0, \frac{\theta_1 + \theta_2 - y_0}{2} \right) \right), y_0 \right] \end{aligned} \quad (1)$$

Optimal action choice $(\gamma^{S_1}(\theta_1, \theta_2, y_0))$ of S_1 in the states (θ_1, θ_2) with the budget constraint y_0 is,

$$\gamma^{S_1}(\theta_1, \theta_2, y_0) = \min(\theta_1, y_0)$$

Optimal action choice of $\gamma^{S_2}(\theta_1, \theta_2, y_0)$ of S_2 in the states (θ_1, θ_2) with the budget constraint y_0 is,

$$\gamma^{S_2}(\theta_1, \theta_2, y_0) = \min(\theta_2, y_0)$$

In Figure 1, the best choice for R is O whereas the best choice for S_1 is the θ_1 -coordinate of E and the best choice for S_2 is the θ_2 -coordinate of F . There is a difference in the θ_1 -coordinate of O and E and hence there is a bias between S_1 and R which depends upon (θ_1, θ_2) . So the bias here is state-dependent as discussed in Gordon (2010)[7]. Similarly, there is a state-dependent bias between S_2 and R . For a given state (θ_1, θ_2) , the biases change if we change y_0 . When $y_0 \geq 2$, the biases disappear for all the states.

Let the strategy of S_i (here i denotes both 1, 2) be to choose a signaling rule (a probability distribution) $q_i(m_i|\theta_i, y_0)$ for a given $\theta_i \in \Theta_i$ such that

$$\int_{m_i \in M} q_i(m_i|\theta_i, y_0) dm_i = 1$$

where $q_i(m_i|\theta_i, y_0)$ gives the probability of sending message m_i given θ_i . The PBNE for this game is defined following Crawford and Sobel(1982)[3],

Definition 2.1 *The PBNE for simultaneous game at budget y_0 is defined as, S_1 chooses a signaling rule $q_1(m_1|\theta_1, y_0)$, S_2 chooses a signaling rule $q_2(m_2|\theta_2, y_0)$ and hearing the messages m_1 and m_2 , the Receiver takes actions $y_1(m_1, m_2, y_0)$ and $y_2(m_1, m_2, y_0)$ such that,*

1. *The message $m_1 \in M$ in the support of q_1 maximizes*

$$\int_0^1 \left[\int_{m_2 \in M} (y_1(m_1, m_2, y_0) - \theta_1)^2 q_2(m_2|\theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2$$

2. *The message $m_2 \in M$ in the support of q_2 maximizes*

$$\int_0^1 \left[\int_{m_1 \in M} (y_1(m_1, m_2, y_0) - \theta_2)^2 q_1(m_1|\theta_1, y_0) dm_1 \right] f_1(\theta_1) d\theta_1$$

3. *The Receiver's actions pairs $(y_1(m_1, m_2), y_2(m_1, m_2))$ satisfies,*

$$(y_1(m_1, m_2, y_0), y_2(m_1, m_2, y_0)) = \max_{y_1, y_2 \text{ s.t. } y_1 + y_2 \leq y_0} \int_0^1 \int_0^1 [-(y_1 - \theta_1)^2 - (y_2 - \theta_2)^2] P(\theta_1, \theta_2|m_1, m_2, y_0) d\theta_1 d\theta_2$$

$$\text{where } P(\theta_1, \theta_2 | m_1, m_2, y_0) = \frac{q_1(m_1 | \theta_1, y_0) q_2(m_2 | \theta_2, y_0) f_1(\theta_1) f_2(\theta_2)}{\int_0^1 \int_0^1 q_1(m_1 | \theta_1, y_0) q_2(m_2 | \theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2}$$

4. The off equilibrium path beliefs of R should be such that neither S_1 nor S_2 finds it profitable to deviate from the equilibrium path.

We can observe that, when there is no budget constraint or $y_0 \geq 2$ and the realized states are θ_1 and θ_2 , there exists a PBNE in which the Senders report the true states i.e. S_1 reports m_1 which indicates the true state θ_1 , S_2 reports m_2 which indicates the true state θ_2 and the Receiver believes them and take the actions $y_1 = \theta_1$ and $y_2 = \theta_2$ so that all players attain their maximum utility zero. But the above PBNE may not be possible as we restrict y_0 below 2. We study the equilibrium when we introduce the budget constraint $y_1 + y_2 \leq y_0 < 2$.

Remark 2.2 *A PBNE always exists. There always exists a babbling equilibrium of this game like the classical Cheap Talk games where both S_1 and S_2 blabber and R does not believe the Senders and take actions according to his prior belief.*

Consider S_1 sending a message with a signaling rule $q_1(m_1 | \theta_1, y_0)$ and S_2 sending a message with a signaling rule $q_2(m_2 | \theta_2, y_0)$. The Receiver's actions will be then,

$$\begin{aligned} & y_1(m_1, m_2, y_0), y_2(m_1, m_2, y_0) \\ &= \max_{y_1, y_2 \text{ s.t. } y_1 + y_2 \leq y_0} \int_0^1 \int_0^1 [-(y_1 - \theta_1)^2 - (y_2 - \theta_2)^2] P(\theta_1, \theta_2 | m_1, m_2, y_0) d\theta_1 d\theta_2 \\ &= \max_{y_1, y_2 \text{ s.t. } y_1 + y_2 \leq y_0} \int_0^1 \int_0^1 [-(y_1 - \theta_1)^2 - (y_2 - \theta_2)^2] \\ & \quad \frac{q_1(m_1 | \theta_1, y_0) q_2(m_2 | \theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2}{\int_0^1 \int_0^1 q_1(m_1 | \theta_1, y_0) q_2(m_2 | \theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2} \end{aligned} \quad (2)$$

The following lemma describes the optimal actions after hearing the messages and the proof is given in the appendix.

Lemma 2.3

$$y_1(m_1, m_2, y_0) = \min \left[\max \left(0, \Psi_1(m_1) - \max \left(0, \frac{\Psi_1(m_1) + \Psi_2(m_2) - y_0}{2} \right) \right), y_0 \right] \quad (3)$$

$$y_2(m_1, m_2, y_0) = \min \left[\max \left(0, \Psi_2(m_2) - \max \left(0, \frac{\Psi_1(m_1) + \Psi_2(m_2) - y_0}{2} \right) \right), y_0 \right] \quad (4)$$

$$\text{where } \Psi_1(m_1) = \frac{\int_0^1 \theta_1 q_1(m_1|\theta_1, y_0) f_1(\theta_1) d\theta_1}{\int_0^1 q_1(m_1|\theta_1, y_0) f_1(\theta_1) d\theta_1}, \Psi_2(m_2) = \frac{\int_0^1 \theta_2 q_2(m_2|\theta_2, y_0) f_2(\theta_2) d\theta_2}{\int_0^1 q_2(m_2|\theta_2, y_0) f_2(\theta_2) d\theta_2}$$

■

3 Equilibria

3.1 No Fully Revealing Equilibrium

First we check whether with the constraint $y_1 + y_2 \leq y_0 < 2$, a fully revealing equilibrium exists. A fully revealing equilibrium as defined in Battaglini (2002)[2] is an equilibrium in which for each Sender, for each of its state of the world, the information is perfectly transmitted, that is both the Senders reveal their states truthfully. Let S_1 report the true state $m_1 = \theta_1$, S_2 report the true state $m_2 = \theta_2$ and R believes them. Optimal actions $(y_1(\theta_1, \theta_2, y_0), y_2(\theta_1, \theta_2, y_0))$ of R are given in (3) and (4).

Given S_2 reports the true state θ_2 , the expected utility of S_1 by reporting the true state θ_1 is given at budget y_0 by,

$$\begin{aligned} EU^{S_1} &= \int_0^1 U^{S_1}(y_1(\theta_1, \theta_2, y_0), \theta_1) f(\theta_2) d\theta_2 \\ &= \int_0^1 -(y_1(\theta_1, \theta_2, y_0) - \theta_1)^2 f(\theta_2) d\theta_2 \end{aligned}$$

Consider S_1 contemplating a deviation to increase his utility. Let S_1 inflate his message by ϵ which means in state θ_1 , he sends a message signaling $\theta_1 + \epsilon$. The expected utility with ϵ deviation is given at budget y_0 by

$$\begin{aligned} EU^{S_1}(\epsilon) &= \int_0^1 U^{S_1}(y_1(\theta_1 + \epsilon, \theta_2, y_0), \theta_1) f(\theta_2) d\theta_2 \\ &= \int_0^1 -(y_1(\theta_1 + \epsilon, \theta_2, y_0) - \theta_1)^2 f(\theta_2) d\theta_2 \end{aligned}$$

If $EU^{S_1}(\epsilon) - EU^{S_1} > 0$ for some $\epsilon > 0$ for a given θ_1 and y_0 such that $\epsilon + \theta_1 \leq 1$, we can say that S_1 will find it profitable to inflate ϵ amount. The restriction $\epsilon + \theta_1 \leq 1$ is kept to enable us to stay inside our message space $M = [0, 1]$. I show in the following lemma that there is no fully-revealing equilibrium for each $y_0 \in (0, 2)$ by demonstrating that for each S_i , there exists some state $\theta_i \in (0, 1)$, such that S_i finds it profitable to

inflate $\epsilon > 0$ (depends upon θ_i) amount, given the other Sender reports truth and R believes them. The proof is given in the appendix, in the proof I have shown it for S_1 and the same proof holds also for S_2 .

Lemma 3.1 *There is no fully revealing equilibrium for $0 < y_0 < 2$.*

The intuition for the above lemma is that, the quadratic utility function decreases at a faster rate as we move farther from the ideal point (the peak) because of concavity. For a Sender say S_1 , at a given state θ_1 , for higher states of S_2 such that $\theta_1 + \theta_2 > y_0$ the actions taken by R is far from the ideal point, here the ideal point is θ_1 . So S_1 prefers to send a message indicating a slightly higher state $\theta_1 + \epsilon$ where $\epsilon > 0$. Thus he will lose utility for lower states of θ_2 , but will gain substantially (due to concavity) for higher states of θ_2 even though the action moves closer to the ideal point less than ϵ amount. Also even if the cardinality of higher states is very small, still S_1 can choose very very small $\epsilon > 0$ to increase his utility.

3.2 Interval Partition

Here I study whether we have interval partition, like the general Cheap Talk literature, of the state space of the Senders if they do not reveal fully in the equilibrium. In a one dimensional action space, the interval partition occurs if for messages m and m' , the actions are $y(m)$ and $y(m')$ respectively and $y(m) < y(m')$, then all the elements of the set $A = \{\theta : q(m|\theta) > 0\}$ are less than any element of the set $B = \{\theta : q(m'|\theta) > 0\}$ and conversely if $m(\theta^L)$ and $m'(\theta^H)$ are two messages from θ^L and θ^H respectively with $\theta^L < \theta^H$, then $y(m(\theta^L)) \leq y(m'(\theta^H))$. We can see that this makes the state space partitioned into intervals like Crawford and Sobel (1982) [3].

Since the action space of the Receiver is multidimensional in our analysis, we can not use the above rule to see if we have interval partition of the state space of a Sender. Instead of one action from a message, I'll use the average value of the actions (where the average is taken over the messages and the states of the other Sender) of one Sender to check for interval partition.

Definition 3.2 *The function $v(m_1)$ (the average value of actions of S_1 with message m_1) of S_1 by sending a message m_1 given the messaging rule $q_2(m_2|\theta_2, y_0)$ of S_2 is*

defined as

$$v(m_1) = \int_0^1 \left[\int_{M_2} y_1(m_1, m_2, y_0) q_2(m_2 | \theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2$$

Similarly, the function $v(m_2)$ of S_2 by sending a message m_2 given the messaging rule $q_1(m_1 | \theta_1, y_0)$ of S_1 is defined as

$$v(m_2) = \int_0^1 \left[\int_{M_1} y_2(m_1, m_2, y_0) q_1(m_1 | \theta_1, y_0) dm_1 \right] f_1(\theta_1) d\theta_1$$

So in our model, interval partition occurs if, m_1 and m'_1 are two messages with $v(m_1)$ and $v(m'_1)$ being the respective average actions and $v(m_1) < v(m'_1)$, then all the elements of the set $A = \{\theta_1 | q_1(m_1 | \theta_1) > 0\}$ are less than any element of the set $B = \{\theta_1 | q_1(m'_1 | \theta_1) > 0\}$. If it can be proved that this rule is satisfied in our model, then we can say that the state space of S_1 is partitioned into intervals in the equilibrium. The same way we can prove also for S_2 and I just focus the proof for S_1 in the following analysis.

Let $M_1 = \{m_1 : q_1(m_1 | \theta_1) > 0 \text{ for some } \theta_1 \in \Theta_1\}$ and $M_2 = \{m_2 : q_2(m_2 | \theta_2) > 0 \text{ for some } \theta_2 \in \Theta_2\}$. The set M_1 (similarly M_2) contain messages such that each message have been sent with positive probability for some θ_1 (respectively θ_2) in the equilibrium.

As defined above $\Psi_1(m_1) = \int_0^1 \theta_1 P(\theta_1 | m_1) f_1(\theta_1) d\theta_1$ denotes the expected value of the state θ_1 after hearing a message m_1 where $P(\theta_1 | m_1) = \frac{q_1(m_1 | \theta_1, y_0) f_1(\theta_1)}{\int_0^1 q_1(m_1 | \theta_1, y_0) f_1(\theta_1) d\theta_1}$. Let m_1^1 and m_1^2 be two different messages of S_1 .

Lemma 3.3 *If in a PBNE, $\Psi_1(m_1^1) \geq \Psi_1(m_1^2)$, then $y_1(m_1^1, m_2, y_0) \geq y_1(m_1^2, m_2, y_0)$ for any $m_2 \in M_2$ implying $v(m_1^1) \geq v(m_1^2)$. Conversely, if $v(m_1^1) > v(m_1^2)$, then $\Psi_1(m_1^1) > \Psi_1(m_1^2)$.*

The above lemma is quite straightforward because since $\Psi_1(m_1)$ is the expected value of the states from which the message m_1 has been sent, the message which signals a higher expected state induces a higher action from R for a given message of S_2 because of the utility function of R . Similarly, if $v(m_1^1) > v(m_1^2)$, then for some $m_2 \in M_2$, we have $y_1(m_1^1, m_2, y_0) > y_1(m_1^2, m_2, y_0)$ from definition of $v(m)$ which can happen only if $\Psi_1(m_1^1) > \Psi_1(m_1^2)$. We shall use this lemma to prove the following lemma which states

that in our model we have interval partition. The proof is in the appendix and I want to remind again that all the proofs for S_1 holds for S_2 .

Lemma 3.4 *If the messages $m_1(\theta_1^L)$ and $m_1(\theta_1^H)$ are from the states θ_1^L and θ_1^H respectively with $\theta_1^L < \theta_1^H$, then $v(m_1(\theta_1^L)) \leq v(m_1(\theta_1^H))$. Conversely, if for two messages m_1^L and m_1^H , we have $v(m_1^L) < v(m_1^H)$, then all the elements of the set $A = \{\theta_1 | q_1(m_1^L | \theta_1) > 0\}$ are less than any element of the set $B = \{\theta_1 | q_1(m_1^H | \theta_1) > 0\}$.*

The above lemma holds because R updates his belief using Bayes rule after hearing a message and the continuity of the utility function of S_1 in θ_1 . We can conclude from this lemma that $v(m(\theta))$ is monotonically increasing in θ where $m(\theta)$ comes from the equilibrium signaling rule.

4 Effect of Budget on Information Transmission

Here I show the effect of budget on information transmission with two types of equilibria given by: (1) Only one of the Senders reveals his state completely (2) Each Sender has two intervals. The information transmission is measured by the ex-ante expected utility of a player because with a finer partition of the state space, the ex-ante expected utility is higher as explained in Crawford and Sobel (1982) [3]. In our model, also for a given partition of one Sender, if finer is the partition of the other Sender, higher is the ex-ante expected utility. The ex-ante expected utility (EU) of players for our model is given by following Crawford and Sobel (1982) [3],

$$EU^{S_1} = \int_{\theta_1 \in \Theta_1} \int_{m_1 \in M} \left[\int_{\theta_2 \in \Theta_2} \int_{m_2 \in M} (y_1(m_1, m_2, y_0) - \theta_1)^2 q_2(m_2 | \theta_2, y_0) dm_2 f_2(\theta_2) d\theta_2 \right] q_1(m_1 | \theta_1, y_0) dm_1 f_1(\theta_1) d\theta_1$$

Similarly, the ex-ante expected utility of S_2 (we denote it as EU^{S_2}) is defined and the ex-ante expected utility of R (we denote it as EU^R) is the sum of the ex-ante expected utilities of S_1 and S_2 i.e. $EU^R = EU^{S_1} + EU^{S_2}$.

4.1 Equilibrium where One Sender Reveals Fully

We have seen before in the Lemma (3.1) that both the senders can not reveal their states completely. Consider S_2 sending messages with a signaling rule $q_2(m_2|\theta_1, y_0)$ where $m_2 \in M = [0, 1]$ and S_1 reveals his state completely (any bijection from $\Theta_1 \rightarrow M$) and so we can assume $\theta_1 = m_1$. The Receiver's actions are given by equations (3) and (4).

Let a partition of state space θ_2 be denoted by $b_0 = 1, b_1, b_2, \dots, b_N = 0$ for a given y_0 . The following proposition describes the equilibrium where only one of the Senders fully reveals his state and the proof is provided in the appendix. The proof also holds for S_2 as we can interchange S_1 with S_2 as they are in the same strategic position.

Proposition 4.1 *For $1.5 \leq y_0 < 2$, there exists a class of equilibria where S_1 tells truth. The class of equilibria is given by,*

1. *The strategy of S_1 is to send $m_1 = \theta_1$*
2. *There exists a positive integer $N(y_0)$ such that for every N with $1 \leq N \leq N(y_0)$, there exists a partition $b_0 = 1, b_1, b_2, \dots, b_N = 0$ of state space θ_2 where the strategy of S_2 is to send a message with signaling rule, $q_2(m_2|\theta_2, y_0)$ such that $q_2(m_2|\theta_2, y_0)$ is uniform, supported on $[b_j, b_{j+1}]$, if $\theta_2 \in (b_j, b_{j+1})$.*
3. *$N(y_0) = \left\lfloor \frac{1}{1-(2y_0-3)} \right\rfloor$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to x .*
4. *The partition satisfies the condition $\frac{b_1+b_0}{2} \leq y_0 - 1$.*
5. *$b_j = \frac{N-j}{N}$ and actions of R are given by $y_1(\theta_1, [b_j, b_{j+1}], y_0) = \theta_1$ and $y_2(\theta_1, [b_j, b_{j+1}], y_0) = \frac{b_j+b_{j+1}}{2}$ when $m_2 \in [b_j, b_{j+1}]$*

The above lemma is quite simple to understand because given S_1 sends the true message θ_1 , the condition $\frac{b_1+b_0}{2} \leq y_0 - 1$ ensures that we are always inside the budget because, $\frac{b_1+b_0}{2}$ is the maximum amount S_2 receives and the maximum value of $\theta_1 = 1$ and so we should have $1 + \frac{b_1+b_0}{2} \leq y_0$. If this condition is not satisfied, then S_1 can not reveal truthfully as he prefers to deviate for high value of θ_1 . Since we are always within the budget limit, the intervals are equally spaced for S_2 and there is a maximum value of the number of intervals of S_2 because within a budget limit we can have certain

number of equally spaced intervals. The following corollary describes the effect of y_0 on information transmission and the proof is provided in the appendix.

Corollary 4.2 *Consider the class of equilibria described in Proposition (4.1). As y_0 increases, $N(y_0)$ which is the maximum number of partitions possible, increases. The ex-ante expected utilities of players are $EU^{S_1} = 0$, $EU^{S_2} = -\frac{1}{12(N(y_0))^2}$, $EU^R = -\frac{1}{12(N(y_0))^2}$.*

This means a higher y_0 allows more information transmission in terms of ex-ante expected utility of the Receiver that represents the society because with a higher y_0 , the number of equally spaced intervals $N(y_0)$ of the state space of S_2 increases. The plot for ex-ante expected utility for R is provided later in Figure (4) which shows the effect of y_0 .

4.2 $\{2\} \times \{2\}$ Symmetric Equilibria

Here I analyze the PBNE where each Sender partitions his state space into two intervals. I consider the symmetric equilibria and limit my analysis for $y_0 \geq 1$ to keep the calculations simple. As the end points of any interval partition is always 0 and 1, so in the $\{2\} \times \{2\}$ symmetric equilibrium, $a_0 = b_0 = 1$ and $a_2 = b_2 = 0$ (in our notations a_0 always denotes the higher end of the interval which is 1). Hence, we just need to find the point $a_1 = b_1$ which is given in the following lemma and the proof is given in the appendix.

Lemma 4.3 *For $y_0 \geq 1.5$, the $\{2\} \times \{2\}$ symmetric equilibrium is given by $a_1 = b_1 = \frac{1}{2}$. For $1 \leq y_0 \leq 1.5$, $a_1 = b_1$ is given by the real solution of the cubic equation $3a_1^3 - a_1^2(4y_0 + 1) + a_1(y_0^2 + 4y_0 - 1) - y_0^2 = 0$.*

The ex-ante expected utility for $y_0 \geq 1.5$ is given by,

$$EU^{S_1} = EU^{S_2} = -\int_0^{\frac{1}{2}} (0.25 - \theta_1)^2 d\theta_1 - \int_{\frac{1}{2}}^1 (0.75 - \theta_1)^2 d\theta_1 = -\frac{1}{48}$$

For the Receiver $EU^R = EU^{S_1} + EU^{S_2} = -\frac{1}{24}$.

Consider Figure (2). For $1 \leq y_0 \leq 1.5$, as y_0 moves from 1 to 1.5, a_1 increases from 0.405 to 0.5 which means the ex-ante expected utility increases because if a_1 lies closer

Figure 2: Plot of a_1 with y_0

to 0.5, the ex-ante expected utility is higher. I have plotted the ex-ante expected utility in Figure (3) and (4). So in the $\{2\} \times \{2\}$ symmetric equilibrium, ex-ante expected utility increases with increase in y_0 from $1 \leq y_0 \leq 1.5$ and then remains constant for $1.5 \leq y_0 \leq 2$.

4.3 With Commitment

In the previous two types of equilibria I discussed the effect of budget and showed that a higher budget increases information transmission. Here I discuss a commitment strategy of R and compare the ex-ante expected utility with those of previous equilibria.

Consider again $1 \leq y_0 \leq 2$ and the Receiver with the commitment that she gives to one Sender say S_1 always what S_1 wants and then the rest she gives to S_2 based upon S_2 's need. I calculate the ex-ante expected utility for this case and they are given in the following lemma and the proof is given in the appendix. Here also, with a higher budget, there is more information transmission which can be seen in the Figure (3) and Figure (4).

Lemma 4.4 *With commitment where S_1 gets his desired amount, $EU^{S_1} = 0$, $EU^{S_2} = -\frac{(y_0-2)^4}{12}$, $EU^R = -\frac{(y_0-2)^4}{12}$.*

4.4 Comparing the Ex-ante Expected Utilities

Consider Figures (3) and (4) where we have plotted the ex-ante expected utilities of the above three cases. The $\{2\} \times \{2\}$ symmetric equilibrium does not do better than the equilibrium where one Sender fully reveals for $1.5 \leq y_0 < 2$. The $\{2\} \times \{2\}$ symmetric equilibrium does better than with commitment strategy for $1 \leq y_0 \leq 1.05$ (1.05 is the approximate value). But the commitment strategy does better than the two types of equilibria for $1.05 \leq y_0 < 2$. This comparison tells us to look for a PBNE for $1.05 \leq y_0 < 2$ which does better than commitment.

Figure 3: Ex-ante expected utilities with commitment and with $\{2\} \times \{2\}$ symmetric equilibrium for $1 \leq y_0 \leq 1.5$

Figure 4: Ex-ante expected utilities with commitment, with $\{2\} \times \{2\}$ symmetric equilibrium and with the equilibrium where one Sender tells truth for $1.5 \leq y_0 < 2$

5 Conclusion and Future Research

I discussed a model of distribution of a limited resource among multiple Senders by a Receiver where the Receiver represents the society (sum of the Senders) whereas each Sender cares for himself only. I showed that with a budget constraint, there is no fully revealing PBNE. I showed the effect of budget with a $\{2\} \times \{2\}$ symmetric equilibrium, with an equilibrium where only one Sender reveals truthfully and with a commitment strategy where R gives to one Sender his desired amount. I compared the ex-ante expected utilities of all these three cases and I showed that the commitment strategy is doing better than the other two cases for $1.05 \leq y_0 < 2$.

Here, I discuss if there is an equilibrium which does better than the commitment. I conjecture that there exists equilibria with large number of intervals (depends upon the budget) of the state space for both the Senders which does better than commitment. In fact we may take different number of symmetric intervals and for lower y_0 , small number of symmetric intervals may do better than commitment strategy, but as y_0 increases, we may need to take large number of symmetric intervals to do better than commitment. I have not yet proved the existence of an equilibrium with infinite intervals nor even of equilibria with more than 2 intervals which is left for future research. It may be shown that a symmetric equilibrium with $N + 1$ intervals has ex-ante expected utility

greater than the symmetric equilibrium with N intervals, so an equilibrium with infinite symmetric intervals is better than other symmetric equilibria. Therefore, I'll discuss certain results now that point to the existence of an infinite equilibrium and I discuss some properties of the equilibrium assuming it exists.

First I start with a lemma that states that the difference between the expected value of the states and the average value of the actions is higher for higher expected value of the states in any PBNE. This lemma I'll use to show that there is a converging sequence of the length of partition intervals if there exists a PBNE with infinite partition.

Lemma 5.1 $\Psi_1(m_1^1) - v(m_1^1) \geq \Psi_1(m_1^2) - v(m_1^2)$ if $\Psi_1(m_1^1) > \Psi_1(m_1^2)$

This lemma is intuitive because for a given message $m_2 \in M_2$ of S_2 , if $\Psi_1(m_1^1) > \Psi_1(m_1^2)$, then the distance between the θ_1 coordinate of the projection of the point $(\Psi_1(m_1^1), \Psi_1(m_2))$ and $\Psi_1(m_1^1)$ is higher than the the distance between the θ_1 coordinate of the projection of the point $(\Psi_1(m_1^2), \Psi_1(m_2))$ and $\Psi_1(m_1^2)$ and we can use the Figure (1) to see it graphically.

Now I describe the point, call it $\bar{\theta}_1$ where the sequence converges if an infinite partition equilibrium exists.

$$\text{Let } \bar{\theta}_1 = \arg \max_{\theta_1} [v(m_1(\theta_1)) = \Psi_1(m_1(\theta_1))]$$

The following lemma describes that the length of partition intervals decreases from right side (from 1 on the θ_1 axis) and converges to the point $\bar{\theta}_1$ in the infinite equilibrium. Since there are intervals of the state space, we have two messages m_1^1 and m_1^2 such that $v(m_1^1) \neq v(m_1^2)$ from Lemma (3.4).

Lemma 5.2 *If $v(m_1^1) > v(m_1^2)$ and $v(m_1^2) < \Psi_1(m_1^2)$, then $v(m_1^2) < \Psi_1(m_1^2) < v(m_1^1) < \Psi_1(m_1^1)$. If there are infinite intervals of the state space Θ_1 in the equilibrium, then for $\theta_1 \geq \bar{\theta}_1$, interval points converge to $\bar{\theta}_1$ which implies there will be truth revelation for $\theta_1 \leq \bar{\theta}_1$.*

Let a grid of state space $\Theta = \Theta_1 \times \Theta_2$ be given by a partition of state space θ_1 by $a_0 = 1, a_1, a_2, \dots, a_i, \dots$ and a partition of state space θ_2 be denoted by $b_0 =$

Figure 5: Infinite intervals of both the state spaces

$1, b_1, b_2, \dots, b_j, \dots$. For all $0 < y_0 \leq 2$, the general class of equilibria is conjectured as follows² :

1. For each $M \in \mathbb{N}$ and $N \in \mathbb{N}$, there exists a grid given by interval points $a_0 = 1, a_1, a_2, \dots, a_M$ of state space Θ_1 and by $b_0 = 1, b_1, b_2, \dots, b_N$ of state space Θ_2 where the strategy of S_1 is to send a message with signaling rule $q_1(m_1|\theta_1, y_0)$ such that $m_1 \in (a_i, a_{i+1})$ if $\theta_1 \in [a_i, a_{i+1}]$, the strategy of S_2 is to send a message with signaling rule $q_2(m_2|\theta_2, y_0)$ such that $m_2 \in (b_j, b_{j+1})$ if $\theta_2 \in [b_j, b_{j+1}]$
2. For each $N \in \mathbb{N}$, there is a set of equilibria where there is infinite intervals of the state space Θ_1 . The partition points of state θ_1 converge to $\bar{a} = \min\{\max\{0, y_0 -$

²Before I have always kept my analysis limited to $1 \leq y_0 \leq 2$. But the conjecture about the general class of equilibria, I make for all $0 < y_0 \leq 2$.

$\frac{1+b_1}{2}\}, 1\}$ and for $\theta_1 \leq \bar{a}$, there is truth-revelation. Similarly for each $M \in \mathbb{N}$, we can state for θ_2 .

3. There is a set of equilibria where both the state spaces have infinite intervals. The interval points of state θ_1 converge to $\bar{a} = \min\{\max\{0, y_0 - \frac{1+b_1}{2}\}, 1\}$ and the interval points of state θ_2 converge to $\bar{b} = \min\{\max\{0, y_0 - \frac{1+a_1}{2}\}, 1\}$ and for $\theta_1 \leq \bar{a}$ and $\theta_2 \leq \bar{b}$ there is truth-revelation.

The graphical illustration is provided in the Figure (5). The existence of the above class of equilibria may be proved using the lattice theory approach adopted in Gordon (2010)[7]. Regarding whether these type of equilibria does better than commitment, I think equilibrium with both the state space having large number of intervals (depends on y_0) may do better than commitment. However the computation of the interval points are particularly difficult. Because first the indifference conditions are cubic equations in a_i and b_j . Second how to choose in which regions of the state space the point $y_1([a_i, a_{i+1}], [b_j, b_{j+1}], y_0)$ belongs i.e. whether $y_1([a_i, a_{i+1}], [b_j, b_{j+1}], y_0) = \frac{a_i+a_{i+1}}{2}$ or $\frac{a_i+a_{i-1}}{2} - \frac{\frac{a_i+a_{i-1}}{2} + \frac{b_j+b_{j+1}}{2} - y_0}{2}$ or 0 or y_0 as there are so many possibilities, but all may not give feasible solutions. Otherwise, we may use some numerical techniques to consider for different number of symmetric intervals, calculate the ex-ante expected utility and compare it with the commitment strategy and in this way it may be that for higher y_0 , we may need to take larger number of intervals to do better than commitment.

A Appendix

Proof of Lemma (2.3)

Proof To find the optimal solutions of the optimization problem (2) that precedes the lemma, consider the following optimization problem,

$$\max_{y_1, y_2} \int_0^1 \int_0^1 [-(y_1 - \theta_1)^2 - (y_2 - \theta_2)^2] \frac{q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2}{\int_0^1 \int_0^1 q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2} d\theta_1 d\theta_2 \quad (5)$$

To maximize, we take derivative with respect to y_1 and y_2 and equaling to zero,

$$\begin{aligned} \int_0^1 \int_0^1 [-2(y_1 - \theta_1)] \frac{q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2}{\int_0^1 \int_0^1 q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2} &= 0 \\ \Rightarrow y_1(m_1, m_2, y_0) &= \frac{\int_0^1 \int_0^1 \theta_1 q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2}{\int_0^1 \int_0^1 q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2} \\ &= \frac{\int_0^1 \theta_1 q_1(m_1|\theta_1, y_0) f_1(\theta_1) d\theta_1}{\int_0^1 q_1(m_1|\theta_1, y_0) f_1(\theta_1) d\theta_1} \end{aligned} \quad (6)$$

The last equality is because $f_1(\theta_1)$ and $f_2(\theta_2)$ are independent and $q_1(m_1|\theta_1, y_0)$ and $q_2(m_2|\theta_2, y_0)$ are independent.

$$\begin{aligned} \int_0^1 \int_0^1 [-2(y_2 - \theta_2)] \frac{q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2}{\int_0^1 \int_0^1 q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2} &= 0 \\ \Rightarrow y_2(m_1, m_2, y_0) &= \frac{\int_0^1 \int_0^1 \theta_2 q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2}{\int_0^1 \int_0^1 q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2} \\ &= \frac{\int_0^1 \theta_2 q_2(m_2|\theta_2, y_0) f_2(\theta_2) d\theta_2}{\int_0^1 q_2(m_2|\theta_2, y_0) f_2(\theta_2) d\theta_2}. \end{aligned} \quad (7)$$

The last equality is again due to the fact that $f_1(\theta_1)$ and $f_2(\theta_2)$ are independent and $q_1(m_1|\theta_1, y_0)$ and $q_2(m_2|\theta_2, y_0)$ are independent.

If the above optimal solutions satisfy $y_1 + y_2 \leq y_0$ then it is the solution to the Receiver's optimization problem, otherwise we consider the following optimization problem which will give the solutions to the optimization problem.

$$\begin{aligned} \max_{0 \leq y_1 \leq y_0} \int_0^1 \int_0^1 [-(y_1 - \theta_1)^2 - (y_0 - y_1 - \theta_2)^2] \\ \frac{q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2}{\int_0^1 \int_0^1 q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2} \end{aligned} \quad (8)$$

Taking derivative with respect to y_1 and equaling to zero and using the fact that $f_1(\theta_1)$ and $f_2(\theta_2)$ are independent and $q_1(m_1|\theta_1, y_0)$ and $q_2(m_2|\theta_2, y_0)$ are independent,

$$\begin{aligned} \int_0^1 \int_0^1 [-2(y_1 - \theta_1) + 2(y_0 - y_1 - \theta_2)] \\ \frac{q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2}{\int_0^1 \int_0^1 q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2} \\ \Rightarrow y_1 = \frac{y_0}{2} + \int_0^1 \int_0^1 \frac{(\theta_1 - \theta_2) q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2}{2 \int_0^1 \int_0^1 q_1(m_1|\theta_1, y_0) q_2(m_2|\theta_2, y_0) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2} \\ = \frac{y_0}{2} + \int_0^1 \frac{\theta_1 q_1(m_1|\theta_1, y_0) f_1(\theta_1) d\theta_1}{2 \int_0^1 q_1(m_1|\theta_1, y_0) f_1(\theta_1) d\theta_1} - \int_0^1 \frac{\theta_2 q_2(m_2|\theta_2, y_0) f_2(\theta_2) d\theta_2}{2 \int_0^1 q_2(m_2|\theta_2, y_0) f_2(\theta_2) d\theta_2} \end{aligned}$$

But as $0 \leq y_1 \leq y_0$, the optimal solutions are,

$$y_1(m_1, m_2, y_0) = \min \left[\max \left(0, \frac{y_0}{2} + \int_0^1 \frac{\theta_1 q_1(m_1|\theta_1, y_0) f_1(\theta_1) d\theta_1}{2 \int_0^1 q_1(m_1|\theta_1, y_0) f_1(\theta_1) d\theta_1} - \int_0^1 \frac{\theta_2 q_2(m_2|\theta_2, y_0) f_2(\theta_2) d\theta_2}{2 \int_0^1 q_2(m_2|\theta_2, y_0) f_2(\theta_2) d\theta_2} \right), y_0 \right] \quad (9)$$

$$y_2(m_1, m_2, y_0) = y_0 - y_1(m_1, m_2, y_0) = \min \left[\max \left(0, \frac{y_0}{2} + \int_0^1 \frac{\theta_2 q_2(m_2|\theta_2, y_0) f_2(\theta_2) d\theta_2}{2 \int_0^1 q_2(m_2|\theta_2, y_0) f_2(\theta_2) d\theta_2} - \int_0^1 \frac{\theta_1 q_1(m_1|\theta_1, y_0) f_1(\theta_1) d\theta_1}{2 \int_0^1 q_1(m_1|\theta_1, y_0) f_1(\theta_1) d\theta_1} \right), y_0 \right] \quad (10)$$

To find equation (10), use different cases of equation (9). The above solutions can be seen as the optimal actions of the Receiver if the true states were the solutions to optimization problem (5). If we write in compact form of both the cases of optimal solutions $y_1 + y_2 \leq y_0$ and $y_1 + y_2 \geq y_0$, the solution is given by similar to equation (1),

$$y_1(m_1, m_2, y_0) = \min \left[\max \left(0, \Psi_1(m_1) - \max \left(0, \frac{\Psi_1(m_1) + \Psi_2(m_2) - y_0}{2} \right) \right), y_0 \right] \quad (11)$$

$$y_2(m_1, m_2, y_0) = \min \left[\max \left(0, \Psi_2(m_2) - \max \left(0, \frac{\Psi_1(m_1) + \Psi_2(m_2) - y_0}{2} \right) \right), y_0 \right] \quad (12)$$

where $\Psi_1(m_1) = \int_0^1 \frac{\theta_1 q_1(m_1|\theta_1, y_0) f_1(\theta_1) d\theta_1}{\int_0^1 q_1(m_1|\theta_1, y_0) f_1(\theta_1) d\theta_1}$, $\Psi_2(m_2) = \int_0^1 \frac{\theta_2 q_2(m_2|\theta_2, y_0) f_2(\theta_2) d\theta_2}{\int_0^1 q_2(m_2|\theta_2, y_0) f_2(\theta_2) d\theta_2}$

■

Proof of Lemma (3.1)

Proof : Case 1 : $0 < \theta_1 < y_0$, $0 < y_0 < 0.5$

$$\begin{aligned}
EU^{S_1}(\theta_1, y_0) &= \int_0^1 U^{S_1}(y_1(\theta_1, \theta_2, y_0), \theta_1) f(\theta_2) d\theta_2 \\
&= \int_0^{y_0-\theta_1} -(y_1(\theta_1, \theta_2, y_0) - \theta_1)^2 f(\theta_2) d\theta_2 + \int_{y_0-\theta_1}^{y_0+\theta_1} -(y_1(\theta_1, \theta_2, y_0) - \theta_1)^2 f(\theta_2) d\theta_2 \\
&\quad + \int_{y_0+\theta_1}^1 -(y_1(\theta_1, \theta_2, y_0) - \theta_1)^2 f(\theta_2) d\theta_2 \\
&= \int_0^{y_0-\theta_1} -(\theta_1 - \theta_1)^2 d\theta_2 + \int_{y_0-\theta_1}^{y_0+\theta_1} -(\theta_1 - \frac{\theta_1 + \theta_2 - y_0}{2} - \theta_1)^2 d\theta_2 \\
&\quad + \int_{y_0+\theta_1}^1 -(0 - \theta_1)^2 d\theta_2 = -\frac{1}{3}(\theta_1)^2(-3 + \theta_1 + 3y_0)
\end{aligned}$$

Let's choose an ϵ very small such that $\theta_1 + \epsilon < y_0$.

$$\begin{aligned}
EU^{S_1}(\theta_1 + \epsilon, y_0) &= \int_0^1 U^{S_1}(y_1(\theta_1 + \epsilon, \theta_2, y_0), \theta_1) f(\theta_2) d\theta_2 \\
&= \int_0^{y_0-(\theta_1+\epsilon)} -(y_1(\theta_1 + \epsilon, \theta_2, y_0) - \theta_1)^2 f(\theta_2) d\theta_2 \\
&\quad + \int_{y_0-(\theta_1+\epsilon)}^{y_0+(\theta_1+\epsilon)} -(y_1(\theta_1 + \epsilon, \theta_2, y_0) - \theta_1)^2 f(\theta_2) d\theta_2 + \int_{y_0+(\theta_1+\epsilon)}^1 -(y_1(\theta_1, \theta_2, y_0) - \theta_1)^2 f(\theta_2) d\theta_2 \\
&= \int_0^{y_0-(\theta_1+\epsilon)} -(\theta_1 + \epsilon - \theta_1)^2 d\theta_2 + \int_{y_0-(\theta_1+\epsilon)}^{y_0+(\theta_1+\epsilon)} -(\theta_1 + \epsilon - \frac{\theta_1 + \epsilon + \theta_2 - y_0}{2} - \theta_1)^2 d\theta_2 \\
&\quad + \int_{y_0+(\theta_1+\epsilon)}^1 -(0 - \theta_1)^2 d\theta_2 = \frac{1}{3}(\epsilon^3 + 3\epsilon\theta_1^2 + 3\epsilon^2(\theta_1 - y_0) + \theta_1^2(-3 + \theta_1 + 3y_0))
\end{aligned}$$

$$EU^{S_1}(\theta_1 + \epsilon, y_0) - EU^{S_1}(\theta_1, y_0) = \frac{1}{3}\epsilon(\epsilon^2 + 3\theta_1^2 - 3\epsilon(y_0 - \theta_1))$$

If we choose $\epsilon < \frac{3\theta_1^2}{3(y_0-\theta_1)}$, the above term is always positive and hence deviation is profitable.

Case 2 : $1 - y_0 > \theta_1 \geq y_0$, $0 < y_0 < 0.5$

$$\begin{aligned}
EU^{S_1} &= \int_0^{\theta_1-y_0} -(y_0 - \theta_1)^2 d\theta_2 \\
&\quad + \int_{\theta_1-y_0}^{\theta_1+y_0} -(\theta_1 - \frac{\theta_1 + \theta_2 - y_0}{2} - \theta_1)^2 d\theta_2 + \int_{\theta_1+y_0}^1 -(0 - \theta_1)^2 d\theta_2 \\
&= \frac{1}{3}(-3\theta_1 y_0^2 + y_0^3 + \theta_1^2(-3 + 6y_0))
\end{aligned}$$

Let's choose an ϵ very small such that $\theta_1 + \epsilon < 1 - y_0$.

$$\begin{aligned}
EU^{S_1}(\epsilon) &= \int_0^{\theta_1 + \epsilon - y_0} -(y_0 - \theta_1)^2 d\theta_2 \\
&+ \int_{\theta_1 + \epsilon - y_0}^{\theta_1 + \epsilon + y_0} -(\theta_1 + \epsilon - \frac{\theta_1 + \epsilon + \theta_2 - y_0}{2} - \theta_1)^2 d\theta_2 + \int_{\theta_1 + \epsilon + y_0}^1 -(0 - \theta_1)^2 d\theta_2 \\
&= \frac{1}{3}(3\theta_1(2\epsilon - y_0)y_0 + y_0^2(-3\epsilon + y_0) + \theta_1^2(-3 + 6y_0))
\end{aligned}$$

$$EU^{S_1}(\epsilon) - EU^{S_1} = \epsilon(2\theta_1 - y_0)y_0$$

The above term is always positive and hence deviation is profitable.

Case 3 : $1 - y_0 \leq \theta_1 < 1$, $y_0 \leq 0.5$

$$\begin{aligned}
EU^{S_1} &= \int_0^{\theta_1 - y_0} -(y_0 - \theta_1)^2 d\theta_2 \\
&+ \int_{\theta_1 - y_0}^1 -(\theta_1 - \frac{\theta_1 + \theta_2 - y_0}{2} - \theta_1)^2 d\theta_2 = \frac{1}{12}(-4(\theta_1 - y_0)^3 - (1 + \theta_1 - y_0)^3)
\end{aligned}$$

Let's choose an ϵ very small such that $\theta_1 + \epsilon < 1$.

$$\begin{aligned}
EU^{S_1}(\epsilon) &= \int_0^{\theta_1 + \epsilon - y_0} -(y_0 - \theta_1)^2 d\theta_2 \\
&+ \int_{\theta_1 + \epsilon - y_0}^1 -(\theta_1 + \epsilon - \frac{\theta_1 + \epsilon + \theta_2 - y_0}{2} - \theta_1)^2 d\theta_2 \\
&= -(\theta_1 - y_0)^2(\epsilon + \theta_1 - y_0) + \frac{1}{12}(8(\theta_1 - y_0)^3 + (-1 + \epsilon - \theta_1 + y_0)^3)
\end{aligned}$$

$$EU^{S_1}(\epsilon) - EU^{S_1} = \frac{1}{12}\epsilon(3 + 6(\theta_1 - y_0) - 9(\theta_1 - y_0)^2 + \epsilon^2 - 3\epsilon(1 + \theta_1 - y_0))$$

The above term is positive as if we take $\epsilon < \frac{3 + 6(\theta_1 - y_0) - 9(\theta_1 - y_0)^2}{3(1 + \theta_1 - y_0)}$ and so a deviation is profitable.

Case 4 : $0 < \theta_1 < 1 - y_0$, $0.5 < y_0 < 1$

$$\begin{aligned}
EU^{S_1}(\theta_1, y_0) &= \int_0^{y_0 - \theta_1} -(\theta_1 - \theta_1)^2 d\theta_2 \\
&+ \int_{y_0 - \theta_1}^{y_0 + \theta_1} -(\theta_1 - \frac{\theta_1 + \theta_2 - y_0}{2} - \theta_1)^2 d\theta_2 + \int_{y_0 + \theta_1}^1 -(0 - \theta_1)^2 d\theta_2 \\
&= -\frac{1}{3}(\theta_1)^2(-3 + \theta_1 + 3y_0)
\end{aligned}$$

Let's choose an ϵ very small such that $\theta_1 + \epsilon < y_0$.

$$\begin{aligned}
EU^{S_1}(\theta_1 + \epsilon, y_0) &= \int_0^{y_0 - (\theta_1 + \epsilon)} -(\theta_1 + \epsilon - \theta_1)^2 d\theta_2 \\
&+ \int_{y_0 - (\theta_1 + \epsilon)}^{y_0 + (\theta_1 + \epsilon)} -(\theta_1 + \epsilon - \frac{\theta_1 + \epsilon + \theta_2 - y_0}{2} - \theta_1)^2 d\theta_2 + \int_{y_0 + (\theta_1 + \epsilon)}^1 -(0 - \theta_1)^2 d\theta_2 \\
&= \frac{2\epsilon^3}{3} + \frac{2\theta_1^3}{3} + \epsilon^2(-\theta_1 + y_0) - \theta_1^2(-1 + \epsilon + \theta_1 + y_0)
\end{aligned}$$

$$EU^{S_1}(\theta_1 + \epsilon, y_0) - EU^{S_1}(\theta_1, y_0) = -\frac{2\epsilon^3}{3} + \epsilon\theta_1^2 + \epsilon^2(\theta_1 - y_0)$$

Case 5 : $y_0 > \theta_1 \geq 1 - y_0$, $0.5 < y_0 < 1$

$$\begin{aligned}
EU^{S_1} &= \int_0^{y_0 - \theta_1} -(\theta_1 - \theta_1)^2 d\theta_2 \\
&+ \int_{y_0 - \theta_1}^1 -(\theta_1 - \frac{\theta_1 + \theta_2 - y_0}{2} - \theta_1)^2 d\theta_2 = \frac{1}{12}(8(\theta_1 - y_0)^3 - (1 + \theta_1 - y_0)^3)
\end{aligned}$$

Let's choose an ϵ very small such that $\theta_1 + \epsilon < y_0$.

$$\begin{aligned}
EU^{S_1}(\epsilon) &= \int_0^{y_0 - \theta_1 - \epsilon} -(\theta_1 + \epsilon - \theta_1)^2 d\theta_2 \\
&+ \int_{y_0 - \theta_1 - \epsilon}^1 -(\theta_1 + \epsilon - \frac{\theta_1 + \epsilon + \theta_2 - y_0}{2} - \theta_1)^2 d\theta_2 \\
&= \epsilon^2(\epsilon + \theta_1 - y_0) + \frac{1}{12}(-8\epsilon^3 + (-1 + \epsilon - \theta_1 + y_0)^3)
\end{aligned}$$

$$EU^{S_1}(\epsilon) - EU^{S_1} = \frac{1}{12}\epsilon(5\epsilon^2 + \epsilon(-3 - 9(y_0 - \theta_1)) + 3(1 + \theta_1 - y_0)^2)$$

The above term is positive for $\epsilon < \frac{3(1 + \theta_1 - y_0)^2}{3 + 9(y_0 - \theta_1)}$ and hence deviation is profitable.

Case 6 : $y_0 \leq \theta_1 < 1$, $0.5 \leq y_0 < 1$

$$\begin{aligned}
EU^{S_1} &= \int_0^{\theta_1 - y_0} -(y_0 - \theta_1)^2 d\theta_2 \\
&+ \int_{\theta_1 - y_0}^1 -(\theta_1 - \frac{\theta_1 + \theta_2 - y_0}{2} - \theta_1)^2 d\theta_2 = \frac{1}{12}(-4(\theta_1 - y_0)^3 - (1 + \theta_1 - y_0)^3)
\end{aligned}$$

Let's choose an ϵ very small such that $\theta_1 + \epsilon < 1$.

$$\begin{aligned}
EU^{S_1}(\epsilon) &= \int_0^{\theta_1 + \epsilon - y_0} -(y_0 - \theta_1)^2 d\theta_2 \\
&+ \int_{\theta_1 + \epsilon - y_0}^1 -(\theta_1 + \epsilon - \frac{\theta_1 + \epsilon + \theta_2 - y_0}{2} - \theta_1)^2 d\theta_2 \\
&= -(\theta_1 - y_0)^2(\epsilon + \theta_1 - y_0) + \frac{1}{12}(8(\theta_1 - y_0)^3 + (-1 + \epsilon - \theta_1 + y_0)^3)
\end{aligned}$$

$$EU^{S_1}(\epsilon) - EU^{S_1} = \frac{1}{12}\epsilon(3 + 6(\theta_1 - y_0) - 9(\theta_1 - y_0)^2 + \epsilon^2 - 3\epsilon(1 + \theta_1 - y_0))$$

The above term is positive as if we take $\epsilon < \frac{3+6(\theta_1-y_0)-9(\theta_1-y_0)^2}{3(1+\theta_1-y_0)}$ and so a deviation is profitable.

Case 7 : $y_0 - 1 \geq \theta_1 > 0, 2 > y_0 > 1$

$$EU^{S_1} = \int_0^1 -(\theta_1 - \theta_1)^2 d\theta_2 = 0$$

This is the maximum utility that can be obtained and hence deviation is not profitable.

Case 8 : $y_0 - 1 < \theta_1 < 1, 2 > y_0 \geq 1$

$$\begin{aligned} EU^{S_1} &= \int_0^{y_0-\theta_1} -(\theta_1 - \theta_1)^2 d\theta_2 \\ &= \int_{y_0-\theta_1}^1 -(\theta_1 - \frac{\theta_1 + \theta_2 - y_0}{2} - \theta_1)^2 d\theta_2 = -\frac{1}{12}(1 + \theta_1 - y_0)^3 \end{aligned}$$

Let's choose an ϵ very small such that $\theta_1 + \epsilon < 1$.

$$\begin{aligned} EU^{S_1}(\epsilon) &= \int_0^{y_0-\theta_1-\epsilon} -(\theta_1 + \epsilon - \theta_1)^2 d\theta_2 \\ &+ \int_{y_0-\theta_1-\epsilon}^1 -(\theta_1 + \epsilon - \frac{\theta_1 + \epsilon + \theta_2 - y_0}{2} - \theta_1)^2 d\theta_2 \\ &= \epsilon^2(\epsilon + \theta_1 - y_0) + \frac{1}{12}(-8\epsilon^3 + (-1 + \epsilon - \theta_1 + y_0)^3) \end{aligned}$$

$$EU^{S_1}(\epsilon) - EU^{S_1} = \frac{1}{12}\epsilon(5\epsilon^2 + \epsilon(-3 - 9(y_0 - \theta_1)) + 3(1 + \theta_1 - y_0)^2)$$

The above term is positive for $\epsilon < \frac{3(1+\theta_1-y_0)^2}{3+9(y_0-\theta_1)}$ and hence deviation is profitable.

So we have analyzed all cases and proved the stated lemma. ■

Proof of Lemma (3.3)

Proof : It can be easily proved using the formula for optimal action given in equation (3) for all possible cases.

If $\Psi_1(m_1^1) = \Psi_1(m_1^2)$, then $y_1(m_1^1, m_2, y_0) = y_1(m_1^2, m_2, y_0)$. If $\Psi_1(m_1^1) > \Psi_1(m_1^2)$, $\Psi_1(m_1^1) + \Psi_2(m_2) - y_0 \leq 0$, then $\Psi_1(m_1^1) + \Psi_2(m_2) - y_0 \leq 0$ and so we have $y_1(m_1^1, m_2, y_0) = \Psi_1(m_1^1) > y_1(m_1^2, m_2, y_0) = \Psi_1(m_1^2)$. Similarly we can consider other cases and prove it. But the result can be seen conveniently graphically because $y_1(m_1, m_2, y_0)$ is the orthogonal projection of the point $(\Psi_1(m_1), \Psi_2(m_2))$ on to the budget line. For the converse,

if $v(m_1^1) > v(m_1^2)$, then for some $m_2 \in M_2$, we have $y_1(m_1^1, m_2, y_0) > y_1(m_1^2, m_2, y_0)$ from the definition of $v(m)$. If we refer the equation (3) for $y_1(m_1, m_2, y_0)$, we can immediately derive that $\Psi_1(m_1^1) > \Psi_1(m_1^2)$.

Proof of Lemma (3.4)

Proof If $\Psi_1(m_1(\theta_1^L)) \leq \Psi_1(m_1(\theta_1^H))$, then from Lemma (3.3), $v(m_1(\theta_1^L)) \leq v(m_1(\theta_1^H))$. So we assume $\Psi_1(m_1(\theta_1^L)) > \Psi_1(m_1(\theta_1^H))$. Let's prove by contradiction and assume that $v(m_1(\theta_1^L)) > v(m_1(\theta_1^H))$ which implies for some $m_2 \in M_2$, $y_1(m_1(\theta_1^L), m_2, y_0) > y_1(m_1(\theta_1^H), m_2, y_0)$. Since S_1 prefers $m_1(\theta_1^H)$ at θ_1^H we have,,

$$\begin{aligned} & - \int_0^1 \left[\int_{M_2} (y_1(m_1(\theta_1^H), m_2, y_0) - \theta_1^H)^2 q_2(m_2 | \theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2 \\ & > - \int_0^1 \left[\int_{M_2} (y_1(m_1(\theta_1^L), m_2, y_0) - \theta_1^H)^2 q_2(m_2 | \theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2 \\ \Rightarrow \theta_1^H & < \frac{\int_0^1 \left[\int_{M_2} ((y_1(m_1(\theta_1^H), m_2, y_0))^2 - (y_1(m_1(\theta_1^L), m_2, y_0))^2) q_2(m_2 | \theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2}{2 \int_0^1 \left[\int_{M_2} (y_1(m_1(\theta_1^H), m_2, y_0) - y_1(m_1(\theta_1^L), m_2, y_0)) q_2(m_2 | \theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2} \end{aligned}$$

We have taken the strict relation because for some $m_2 \in M_2$, $y_1(m_1(\theta_1^L), m_2, y_0) > y_1(m_1(\theta_1^H), m_2, y_0)$. Since S_1 prefers $m_1(\theta_1^L)$ at θ_1^L we also have,

$$\begin{aligned} & - \int_0^1 \left[\int_{M_2} (y_1(m_1(\theta_1^L), m_2, y_0) - \theta_1^L)^2 q_2(m_2 | \theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2 \\ & > - \int_0^1 \left[\int_{M_2} (y_1(m_1(\theta_1^H), m_2, y_0) - \theta_1^L)^2 q_2(m_2 | \theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2 \\ \Rightarrow \theta_1^L & > \frac{\int_0^1 \left[\int_{M_2} ((y_1(m_1(\theta_1^H), m_2, y_0))^2 - (y_1(m_1(\theta_1^L), m_2, y_0))^2) q_2(m_2 | \theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2}{2 \int_0^1 \left[\int_{M_2} (y_1(m_1(\theta_1^H), m_2, y_0) - y_1(m_1(\theta_1^L), m_2, y_0)) q_2(m_2 | \theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2} \end{aligned}$$

The above relations imply $\theta_1^L > \theta_1^H$ which is a contradiction.

Conversely, let $v(m_1^L) < v(m_1^H)$, then for some $m_2 \in M_2$, $y_1(m_1^L, m_2, y_0) < y_1(m_1^H, m_2, y_0)$. Let θ_1^L be any state from which m_1^L is sent and θ_1^H be any state from which m_1^H is sent. Then we can calculate like the above relations (just the signs reversed) to conclude that $\theta_1^L < \theta_1^H$ which completes the proof. ■

Proof of Proposition (4.1)

Proof : Notice that for $y_0 \geq 2$, both the players will tell truth as I have said before. So we focus on $1.5 \leq y_0 < 2$ and the condition why the limit 1.5 has been set will be clear later. Given the Senders' strategies, the best actions of the Receiver are $y_1(\theta_1, m_2, y_0) = \theta_1$ and $y_2(\theta_1, m_2, y_0) = \frac{b_j + b_{j+1}}{2}$ if $m_2 \in [b_j, b_{j+1}]$ from equations (3) and (4). The condition $\frac{b_1 + b_0}{2} \leq y_0 - 1$ makes sure that the optimal actions satisfy the budget constraint as $y_1(\theta_1, m_2, y_0) = \theta_1 \leq 1$ and the maximum value of $y_2(\theta_1, m_2, y_0) = \frac{b_1 + b_0}{2}$ when $m_2 \in [b_1, b_0]$.

If we do not impose the condition $\frac{b_1 + b_0}{2} \leq y_0 - 1$, then the optimal solutions for $1 > \theta_1 \geq y_0 - \frac{b_1 + b_0}{2}$ are given by $y_1 = \frac{\theta_1 + y_0}{2} - \frac{b_{j+1} + b_j}{4}$ and $y_2 = \frac{y_0 - \theta_1}{2} + \frac{b_{j+1} + b_j}{4}$ from equations (3) and (4). But these actions by the Receiver can not be part of equilibrium as S_1 will find it profitable to inflate the message to get closer to his optimal action $y_1 = \theta_1$ as here the Receiver's optimal action $y_1 = \frac{\theta_1 + y_0}{2} - \frac{b_{j+1} + b_j}{4} = \theta_1 - \frac{\theta_1 + \frac{b_{j+1} + b_j}{2}}{2} < \theta_1$.

Now to make S_2 incentive compatible at the partition points, his utility at b_j , between sending a message in $[b_{j-1}, b_j]$ and $[b_j, b_{j+1}]$ should be same and following Crawford and Sobel (1982)[3], the indifference condition for S_2 is given for $j = 0, 1, \dots, N - 1$ by,

$$\begin{aligned} U^{S_2}(y_2(\theta_1, [b_j, b_{j+1}], y_0), b_{j+1}) &= U^{S_2}(y_2(\theta_1, [b_{j+1}, b_{j+2}], y_0), b_{j+1}) \\ \Rightarrow -(\frac{b_j + b_{j+1}}{2} - b_{j+1})^2 &= -(\frac{b_{j+1} + b_{j+2}}{2} - b_{j+1})^2 \\ \Rightarrow \frac{b_j + b_{j+1}}{2} - b_j + 1 &= \frac{b_{j+1} + b_{j+2}}{2} - b_{j+1} \\ \Rightarrow b_{j+1} &= \frac{b_j + b_{j+2}}{2} \end{aligned}$$

Since $b_0 = 1$ then from $b_{j+1} = \frac{b_j + b_{j+2}}{2}$ we get, $b_1 = \frac{1 + b_2}{2}$, ..., $b_j = 1 - j + b_1 j$ and $b_N = 1 - N + b_1 N$. As $b_N = 0$, we get, $b_1 = \frac{N-1}{N}$. Substituting backwards, we have, $b_j = 1 - j + \frac{N-1}{N}j = \frac{N-j}{N}$. Consider the condition $\frac{b_1 + b_0}{2} \leq y_0 - 1$. Putting the value of b_1 and b_0 , we get, $\frac{\frac{N-1}{N} + 1}{2} \leq y_0 - 1$ which gives, $N \leq \frac{1}{1 - (2y_0 - 3)}$. So the $N(y_0)$ stated in the proposition is $N(y_0) = \left\lfloor \frac{1}{1 - (2y_0 - 3)} \right\rfloor$ where $\lfloor x \rfloor$ denotes the greatest integer lower or equal to x . For each $1 \leq N \leq N(y_0)$, we can describe a partition and the partition points can be calculated as done above. Notice that as $y_0 < 1.5$, then $N(y_0) = 0$, but it is impossible as the minimum value of N is 1 which means there is no partition. So we have considered $y_0 \geq 1.5$. The other way of looking at it is if S_1 tells truth and if S_2 does not send any information (he blabbers), then the best action for R is to take $y_2 = 0.5$ which will require at least $y_0 = 1.5$ when $\theta_1 = 1$. ■

Proof of Corollary (4.2)

Proof : I have established during the proof of the proposition that $N(y_0) = \left\lfloor \frac{1}{1-(2y_0-3)} \right\rfloor$. As y_0 increases, it is clear that $N(y_0)$ increases. For $y_0 = 1.5$, $N(y_0) = 1$. If we set $N(y_0) = 2$, we get $y_0 = 1.75$ and at $y_0 \approx 1.833$, $N(y_0) = 3$. As $y_0 \rightarrow 2$, we get that $N(y_0) \rightarrow \infty$ which satisfies the observation that there will be fully revelation when $y_0 = 2$. Since S_1 reveals truthfully, $EU^{S_1} = 0$.

$$\begin{aligned} EU^{S_2} &= - \sum_{j=0}^{N(y_0)} \int_{\frac{N(y_0)-(j+1)}{N(y_0)}}^{\frac{N(y_0)-j}{N(y_0)}} \left(\frac{2N(y_0)-(2j+1)}{2N(y_0)} - \theta_1 \right)^2 d\theta_1 \\ &= \sum_{j=0}^{N(y_0)} \left[\frac{\left(\frac{2N(y_0)-(2j+1)}{2N(y_0)} - \theta_1 \right)^3}{3} \right]_{\frac{N(y_0)-(j+1)}{N(y_0)}}^{\frac{N(y_0)-j}{N(y_0)}} = \frac{1}{3} \sum_{j=0}^{N(y_0)} \left(\frac{-1}{2N(y_0)} \right)^3 - \left(\frac{1}{2N(y_0)} \right)^3 \\ &= \frac{1}{3} \frac{-2}{(N(y_0))^3} N(y_0) = -\frac{1}{12(N(y_0))^2} \end{aligned}$$

■

Proof of Lemma (4.3)

Proof : With $\{2\} \times \{2\}$ symmetric equilibrium, the intervals are $[a_2 = 0, a_1]$, $[a_1, a_0 = 1]$ for S_1 and the mid points are $\frac{a_1}{2}$ and $\frac{1+a_1}{2}$ respectively. Similarly for S_2 , the intervals are $[b_2 = 0, b_1]$, $[b_1, b_0 = 1]$ and the mid points are $\frac{b_1}{2}$ and $\frac{1+b_1}{2}$ respectively. So the points to calculate the actions of R that we consider are, $(\frac{a_1}{2}, \frac{b_1}{2})$, $(\frac{a_1}{2}, \frac{1+b_1}{2})$, $(\frac{1+a_1}{2}, \frac{b_1}{2})$ and $(\frac{1+a_1}{2}, \frac{1+b_1}{2})$. Let (x, y) be any point. The optimal action of R in the direction of θ_1 which is y_1 can be any of $0, y_0, x - \frac{x+y}{2}$ and x and similarly the optimal action of R in the direction of θ_2 which is y_2 can be any of $0, y_0, y - \frac{x+y}{2}$ or y . The actions taken by R for our above four points can be any of these values, but all may not give meaningful solutions. I considered all the possibilities and the only feasible solution (y_1, y_2) for $1.5 \leq y_0 \leq 2$ is given by the actions same as the points which means from the indifference condition of S_1 that $a_1 = b_1 = \frac{1}{2}$. For $1 \leq y_0 \leq 1.5$, the feasible solution is given by the actions as follows: for $(\frac{a_1}{2}, \frac{b_1}{2})$ is same $((\frac{a_1}{2}, \frac{b_1}{2}))$, for $(\frac{a_1}{2}, \frac{1+b_1}{2})$ is same $(\frac{a_1}{2}, \frac{1+b_1}{2})$, for $(\frac{1+a_1}{2}, \frac{b_1}{2})$ is same $(\frac{1+a_1}{2}, \frac{b_1}{2})$ and for $(\frac{1+a_1}{2}, \frac{1+b_1}{2})$ is $(\frac{a_1+1}{2} - \frac{\frac{a_1+1}{2} + \frac{b_1+1}{2} - y_0}{2}, \frac{b_1+1}{2} - \frac{\frac{a_1+1}{2} + \frac{b_1+1}{2} - y_0}{2})$ and so the indifference condition for S_1 is given by (for S_2 , we have the same equation when we consider symmetric equilibrium where $a_1 = b_1$),

$$-\left(\frac{a_1+1}{2} - a_1\right)^2 b_1 - \left(\frac{a_1+1}{2} - \frac{\frac{a_1+1}{2} + \frac{b_1+1}{2} - y_0}{2} - a_1\right)^2 (1 - b_1) = -\left(\frac{a_1}{2} - a_1\right)^2$$

Setting $a_1 = b_1$, we get

$$3a_1^3 - a_1^2(4y_0 + 1) + a_1(y_0^2 + 4y_0 - 1) - y_0^2 = 0$$

For $1 \leq y_0 \leq 1.5$, we have a value of a_1 which satisfies the above equation. $\blacksquare$

Proof of Lemma (4.4)

Proof : With commitment, the ex-ante $EU^{S_1} = 0$ as S_1 gets its desired amount always.

$$\begin{aligned} EU^{S_2} &= - \int_0^1 \left[\int_0^{\min\{y_0 - \theta_2, 1\}} (\theta_2 - \theta_1)^2 d\theta_1 + \int_{\min\{y_0 - \theta_2, 1\}}^1 (y_0 - \theta_1 - \theta_2)^2 d\theta_1 \right] d\theta_2 \\ &= - \int_0^1 \left[\int_{\min\{y_0 - \theta_2, 1\}}^1 (y_0 - \theta_1 - \theta_2)^2 d\theta_1 \right] d\theta_2 = \int_0^1 \left[\left(\frac{(y_0 - \theta_1 - \theta_2)^3}{3} \right)_{\min\{y_0 - \theta_2, 1\}}^1 \right] d\theta_2 \\ &= \int_{y_0 - 1}^1 \frac{(y_0 - \theta_2 - 1)^3}{3} d\theta_2 = - \frac{(y_0 - 2)^4}{12} \end{aligned}$$

$$EU^R = EU^{S_1} + EU^{S_2} = - \frac{(y_0 - 2)^4}{12}. \quad \blacksquare$$

Proof of Lemma (5.1)

Proof Consider two messages m_1^1 and m_1^2 in the equilibrium such that $\Psi_1(m_1^1) > \Psi_1(m_1^2)$. Now we have,

$$\begin{aligned} \Psi_1(m_1^1) - v(m_1^1) &\geq \Psi_1(m_1^2) - v(m_1^2) \\ \Rightarrow \Psi_1(m_1^1) - \int_0^1 \left[\int_{M_2} y_1(m_1^1, m_2, y_0) q_2(m_2 | \theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2 \\ &\geq \Psi_1(m_1^2) - \int_0^1 \left[\int_{M_2} y_1(m_1^2, m_2, y_0) q_2(m_2 | \theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2 \\ \Rightarrow \int_0^1 \left[\int_{M_2} (\Psi_1(m_1^1) - y_1(m_1^1, m_2, y_0)) q_2(m_2 | \theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2 \\ &\geq \int_0^1 \left[\int_{M_2} (\Psi_1(m_1^2) - y_1(m_1^2, m_2, y_0)) q_2(m_2 | \theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2 \end{aligned}$$

Now we can substitute the value of $y_1(m_1, m_2, y_0)$ from equation (3) into the above equation and see that the above equation always holds when $\Psi_1(m_1^1) > \Psi_1(m_1^2)$. $\blacksquare$

Proof of Lemma (5.2)

Proof : If $v(m_1^1) > v(m_1^2)$ and $v(m_1^2) < \Psi_1(m_1^2)$, then for some $m_2 \in M_2$, $y_1(m_1^2, m_2, y_0) < \Psi_1(m_1^2)$ which implies for the same $m_2 \in M_2$, we have $y_1(m_1^1, m_2, y_0) < \Psi_1(m_1^1)$ as

$\Psi_1(m_1^1) \geq \Psi_1(m_1^2)$ from equation (3). So we have $v(m_1^1) < \Psi_1(m_1^1)$. So we just need to show $\Psi_1(m_1^2) < v(m_1^1)$. It can be seen from above (during the proof of Lemma (3.4) in the appendix) that m_1^2 is sent below

$$\theta_1 = \frac{\int_0^1 \left[\int_{M_2} ((y_1(m_1^1, m_2, y_0))^2 - (y_1(m_1^2, m_2, y_0))^2) q_2(m_2|\theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2}{2 \int_0^1 \left[\int_{M_2} (y_1(m_1^1, m_2, y_0) - y_1(m_1^2, m_2, y_0)) q_2(m_2|\theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2}$$

The message m_1^2 is sent for states above and below $\theta_1 = \Psi_1(m_1^2)$ as $\Psi_1(m_1^2)$ represents the average of the states that the message is sent using Bayes rule and so we have,

$$\Psi_1(m_1^2) < \frac{\int_0^1 \left[\int_{M_2} ((y_1(m_1^1, m_2, y_0))^2 - (y_1(m_1^2, m_2, y_0))^2) q_2(m_2|\theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2}{2 \int_0^1 \left[\int_{M_2} (y_1(m_1^1, m_2, y_0) - y_1(m_1^2, m_2, y_0)) q_2(m_2|\theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2}$$

Now I aim to show that

$$\frac{\int_0^1 \left[\int_{M_2} ((y_1(m_1^1, m_2, y_0))^2 - (y_1(m_1^2, m_2, y_0))^2) q_2(m_2|\theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2}{2 \int_0^1 \left[\int_{M_2} (y_1(m_1^1, m_2, y_0) - y_1(m_1^2, m_2, y_0)) q_2(m_2|\theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2} < v(m_1^1)$$

. i.e.

$$\begin{aligned} & \frac{\int_0^1 \left[\int_{M_2} ((y_1(m_1^1, m_2, y_0))^2 - (y_1(m_1^2, m_2, y_0))^2) q_2(m_2|\theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2}{\int_0^1 \left[\int_{M_2} (y_1(m_1^1, m_2, y_0) - y_1(m_1^2, m_2, y_0)) q_2(m_2|\theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2} \\ & < 2 \int_0^1 \left[\int_{M_2} y_1(m_1^1, m_2, y_0) q_2(m_2|\theta_2, y_0) dm_2 \right] f_2(\theta_2) d\theta_2 \\ & \Rightarrow \int_0^1 \int_{M_2} (y_1(m_1^1, m_2, y_0) - y_1(m_1^2, m_2, y_0)) \\ & \quad (y_1(m_1^1, m_2, y_0) + y_1(m_1^2, m_2, y_0)) q_2(m_2|\theta_2, y_0) dm_2 f_2(\theta_2) d\theta_2 \\ & < \left(\int_0^1 \int_{M_2} (y_1(m_1^1, m_2, y_0) - y_1(m_1^2, m_2, y_0)) q_2(m_2|\theta_2, y_0) dm_2 f_2(\theta_2) d\theta_2 \right) \\ & \quad \left(2 \int_0^1 \int_{M_2} y_1(m_1^1, m_2, y_0) q_2(m_2|\theta_2, y_0) dm_2 f_2(\theta_2) d\theta_2 \right) \\ & \Rightarrow \int_0^1 \int_{M_2} (y_1(m_1^1, m_2, y_0) - y_1(m_1^2, m_2, y_0)) \\ & \quad (y_1(m_1^1, m_2, y_0) + y_1(m_1^2, m_2, y_0)) q_2(m_2|\theta_2, y_0) dm_2 f_2(\theta_2) d\theta_2 \\ & < \left(\int_0^1 \int_{M_2} (y_1(m_1^1, m_2, y_0) - y_1(m_1^2, m_2, y_0)) q_2(m_2|\theta_2, y_0) dm_2 f_2(\theta_2) d\theta_2 \right) \\ & \quad \left(\int_0^1 \int_{M_2} (y_1(m_1^1, m_2, y_0) + y_1(m_1^1, m_2, y_0)) q_2(m_2|\theta_2, y_0) dm_2 f_2(\theta_2) d\theta_2 \right) \end{aligned}$$

The above holds by triangle inequality for integration and the fact that $y_1(m_1^1, m_2, y_0) \geq y_1(m_1^2, m_2, y_0)$ for each $m_2 \in M_2$ and so we have $\Psi_1(m_1^2) < v(m_1^1)$. Now I want to show if there exists infinite intervals of the state space Θ_1 , then for $\theta_1 \geq \bar{\theta}_1$ whether interval points converge to $\bar{\theta}_1$ which implies there will be truth revelation for $\theta_1 \leq \bar{\theta}_1$. If interval points converge to a point, from Lemma (5.1) and the fact that $\Psi_1(m_1^2) < v(m_1^1) < \Psi_1(m_1^1)$, we can see that interval points can converge only to the left side direction (towards 0 on the θ_1 axis) and not to the right side direction. Assume the interval points do not converge to $\bar{\theta}_1$, then we can see from the indifference condition of S_1 that there will be only finite intervals and so we can not have truth revelation for $\theta_1 \leq \bar{\theta}_1$. ■

References

- [1] Attila Ambrus and Satoru Takahashi. Multi-sender Cheap Talk with Restricted State Spaces. *Theoretical Economics*, 3(1):1–27, March, 2008.
- [2] Marco Battaglini. Multiple Referrals and Multidimensional Cheap Talk. *Econometrica*, 70(4):1379–1401, July, 2002.
- [3] Vincent P. Crawford and Joel Sobel. Strategic Information Transmission. *Econometrica*, 50(6):1431–1451, 1982.
- [4] Joseph Farrell and Robert Gibbons. Cheap Talk with Two Audiences. *American Economic Review*, 79(5):1214–23, December, 1989.
- [5] Lars Frisell and Johan N. M. Lagerlf. A Model of Reputation in Cheap Talk. *Scandinavian Journal of Economics*, 109(1):49–70, March, 2007.
- [6] Thomas W. Gilligan and Keith Krehbiel. Asymmetric Information and Legislative Rules with a Heterogeneous Committee. *American Journal of Political Science*, 33(2):459–490, 1989.
- [7] Sidartha Gordon. On Infinte Cheap Talk Equilibria. revise and resubmit, *Journal of Economic Theory*, August 2010.

- [8] Vijay Krishna and John Morgan. A Model of Expertise. *The Quarterly Journal of Economics*, 116(2):747–775, May, 2001.
- [9] Ming Li. To Disclose or Not to Disclose: Cheap Talk with Uncertain Biases. Working Papers 04003, Concordia University, Department of Economics, August, 2003.
- [10] Andrew McGee and Huanxing Yang. Cheap Talk with Two Senders and Complementary Information. Unpublished, November, 2009.
- [11] Paul Newman and Richard Sansing. Disclosure Policies with Multiple Users. *Journal of Accounting Research*, 31(1):92–112, 1993.
- [12] Ronny Razin and Gilat Levy. Multidimensional Cheap Talk. 2004 Meeting Papers 184, Society for Economic Dynamics, 2004.