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Empirical properties of group preference aggregation methods employed in AHP: Theory and evidence.

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Abstract
We study various methods of aggregating individual judgments and individual priorities in group decision making with the AHP. The focus is on the empirical properties of the various methods, mainly on the extent to which the various aggregation methods represent an accurate approximation of the priority vector of interest. We identify five main classes of aggregation procedures which provide identical or very similar empirical expressions for the vectors of interest. We also propose a method to decompose in the AHP response matrix distortions due to random errors and perturbations caused by cognitive bias predicted by the mathematical psychology literature. We test the decomposition with experimental data and find that perturbations in group decision making caused by cognitive distortions are more important than those caused by random errors. We propose methods to correct systematic distortions.

Keywords
Group decisions, matrix differentials, separable representations, cognitive distortions.

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1 Introduction

The Analytic Hierarchy Process (AHP) of Saaty (1977, 1980, 1986) is a technique for establishing priorities in multi-criteria decision making. It can be applied to both individual and group decisions. At the individual level, the procedure starts with the decision maker measuring on a ratio scale the relative dominance between any pair of items relevant in a decision problem: namely, for any pair of items i and j from a set of size n, the decision maker elicits the ratio w_i/w_j, in terms of underlying priority weights $\mathbf{w} = (w_1, ..., w_n)^\mathsf{T}$, with $w_1 > 0, ..., w_n > 0$. The procedure gives rise to $n(n-1)/2$ subjective ratio assessments, which the AHP conveniently stores in a subjective comparison matrix $\mathbf{A} = [\alpha_{ij}]$. Various prioritization procedures can then be used to extract the priority vector from $\mathbf{A}$. However, due to inconsistencies in the elicitation, the theoretical vector $\mathbf{w}$ cannot be known exactly, but only some perturbed vector $\mathbf{u}$ can be obtained. Therefore, a fundamental question in the AHP concerns the extent to which the empirical priority vector $\mathbf{u}$ provided by different methods represents an accurate approximation of the vector $\mathbf{w}$ of interest. Standard prioritization methods used in the AHP include: the classical maximum-eigenvalue (introduced by Saaty and defended since; Saaty 1977, 2003); the logarithmic least squares method (see Crawford and Williams 1985, de Jong 1984, Genest and Rivest 1994); the conceptually close row geometric mean method (Crawford and Williams 1985), which can be further distinguished in some different forms depending on the type of normalizations applied (e.g., Barzilai and Golany 1994, Escobar, Aguarón and Moreno-Jiménez 2004).

When applied to group decision making, several additional normative and behavioral issues arise, including considerations about the nature of the group formation and the relations between its members. Starting from classical works by Aczél and Saaty (1983) and Saaty (1989), a large and important literature has developed several techniques to extend AHP to group decisions. Further fundamental aspects in the definition of the aggregating procedures concern the level of aggregation, which can apply to individual priorities or individual judgments;[1] and the averaging methods, which refer to the question whether the arithmetic mean or the geometric mean (weighted or not) is used in the aggregation. Many possible combinations of techniques can be applied to AHP group decision making (discussions in Dyer and Forman 1992, Ramanathan and Ganesh 1994, Forman and Peniwati 1998), which can be supported by different normative models (analyses in Bryson 1996, Van Den Honert and Lootsma 1996, Van Den Honert 2001, Escobar and Moreno-Jiménez 2007, Dong et al. 2010, and references therein).

In Section 2 of the paper we will review the main issues arising in the above literature and will survey several methods of aggregation. One feature of the aggregating procedures which in our view still needs further scrutiny concerns the empirical performance of the various methods and their similarities on empirical grounds. In this paper we are concerned with the distance that, given the aggregation techniques employed, there is between the theoretical vector $\mathbf{w}$ of interest and its approximation $\mathbf{u}$. Often, in the AHP, the quality of the approximation $\mathbf{u}$ is assessed on the basis of the consistency of the response matrix $\mathbf{A}$. Important contributions in such a line for the AHP in group decision making have been obtained by several authors, including Xu (2000), Escobar, Aguarón and Moreno-Jiménez (2004), Moreno-Jiménez, Aguarón and Escobar (2007), Aull-Hyde, Erdogan and Duke (2006), Lin et al. (2008). Our approach complements those analyses. While near consistency is obviously a desirable normative property holding in response matrices in which perturbations are small, the converse is not true and a pairwise comparison matrix may be perfectly consistent, but "irrelevant and far off the mark of the true priority vector" (Saaty 2003, p. 86). We discuss in more detail this issue in Section 2.

We develop our approach in Sections 3 and 4. In Section 3 the algebraic expressions of vectors $\mathbf{u}$'s and $\mathbf{w}$'s obtained by different prioritization methods and aggregation rules are derived and compared theoretically and empirically. We approach the problem using the theory of matrix differentials (Magnus and Neudecker

[1] It should be clear that aggregating decision makers' preferences, whether in the form of judgments or priorities, is not the same as aggregating criteria.

1999), taking the first-order expansion of vector $\mathbf{u}$ around $\mathbf{w}$. We identify five main classes of aggregation methods which provide identical expressions for the first-order expansions of vectors $\mathbf{u}$'s around $\mathbf{w}$'s. We show the validity of the approximations by applying the theoretical expressions to comparison matrices obtained in three experiments conducted with human subjects in the domain of tangibles.

In Section 4, we analyse more closely the nature of the perturbations affecting the group's comparisons. In classical AHP the error terms of the comparison matrix $\mathbf{A}$ have been typically interpreted as caused by factors like trembling, rounding, computational mistakes, lapses of concentration. As a result, perturbation terms have generally been treated as stochastically unpredictable white noise errors (see, e.g., Genest and Rivest 1994). More recent studies in mathematical psychology have analysed theories of subjective ratio judgments belonging to a class of so-called separable representations (see Narens 1996, 2002, and Luce 2002, 2004). In these models people's ratio judgments, in addition to random errors, are affected by systematic distortions. These are due to a subjective weighting function which transforms numerical mathematical ratios into subjective perceived ratios. Various recent experimental evidence has given support to the predictions of models of separable representations (Ellermeier and Faulhammer 2000, Zimmer 2005, Steingrimsson and Luce 2005a, 2005b, 2006, 2007, Bernasconi, Choirat and Seri 2008, Augustin and Maier 2008). In previous works we have shown how the transformation function of separable representations can be fruitfully applied in the analysis of the AHP response matrices used for individual decision making (Bernasconi, Choirat and Seri 2010, 2011). Here we study the implications of separable representations for group decision making. We first provide a general method to decompose, in the first-order approximation of the difference between $\mathbf{u}$ and $\mathbf{w}$, a stochastic component due to random errors, and a deterministic component due to the individual subjective transformation functions. The properties of the decomposition are discussed theoretically and then applied to the data of our three experiments for one aggregation method as an example. The decompositions clearly show that the deterministic components of the aggregated perturbation terms are substantially larger than the ones due to random noise. Partly, this is consistent with the evidence found in previous analyses of perturbation terms in individual decision making; partly, it follows from the fact that, while the individual random noises tend to cancel out by the group averaging procedures, the deterministic components are not exposed to the same effect. In fact, given the similarity of the individual subjective weighting functions estimated across subjects, we observe a tendency for the associated distortions to be reinforced by the aggregating procedure of the AHP. We discuss methods to correct systematic distortions. In the concluding Section 5, we summarise how our analysis can contribute to the implementation of AHP in group decision making.

2 Basic issues in AHP-group aggregation

In this section we review some fundamental issues in AHP-group aggregation. We start to introduce some notation. For a $n-$vector $\mathbf{a}$, let $\bar{\mathbf{a}}$ be the $n-$vector defined by $\bar{\mathbf{a}} = [\bar{a}_i] = \left[a_i^{-1}\right]$; $\mathbf{u}_n$ is a $n-$vector composed of ones; $\mathbf{I}_n$ is the $(n \times n)-$identity matrix; $\mathbf{U}_n$ is a $(n \times n)-$matrix composed of ones; $\mathbf{e}_i$ is a vector of zeros with a one in the $i-$th position. $\mathbf{A}^\mathsf{T}$ is the transpose of the matrix $\mathbf{A}$. $\mathbf{A}^\ell$ denotes the ordinary product of the matrix $\mathbf{A}$ by itself, repeated ℓ times. The notations $\overline{\ln}\mathbf{A}$, $\overline{\exp}\mathbf{A}$ and $\mathbf{A}^{\odot\ell}$ denote the element-wise application of natural logarithm, exponential and power function (of degree ℓ) to a matrix $\mathbf{A}$. $\sum_{j=1}^n \mathbf{A}_j$ and $\bigodot_{j=1}^n \mathbf{A}_j$ respectively denote the sum and the element-wise product of a series of matrices.

We consider K individuals. In the following we will use an apex (k) to indicate any quantity for the $k-$th individual. As an example, $\mathbf{A}^{(k)} = \left[\alpha_{ij}^{(k)}\right]$ is the pairwise comparison matrix of the $k-$th individual. We say that the individual comparison matrix $\mathbf{A}^{(k)} = \left[\alpha_{ij}^{(k)}\right]$ respects the reciprocal symmetry property if $\alpha_{ij}^{(k)} = 1/\alpha_{ji}^{(k)}$ for every i and j. This is an important requisite for the $\alpha_{ij}^{(k)}$ to be measured on a ratio scale. In theory, a reciprocally symmetric matrix $\mathbf{A}^{(k)} = \left[\alpha_{ij}^{(k)}\right]$ is said to satisfy the property of cardinal consistency when for any three ratio judgments $\alpha_{ij}^{(k)}$, $\alpha_{il}^{(k)}$, $\alpha_{lj}^{(k)}$, the following holds: $\alpha_{ij}^{(k)} = \alpha_{il}^{(k)} \cdot \alpha_{lj}^{(k)}$. In practice, cardinal consistency is violated by individuals due to errors: these may be due to trembling, rounding and other unpredictable events. Indeed, we do not know of any fully (cardinally) consistent matrix from practice of order 5 or higher. A weaker requirement less often violated in practice is ordinal consistency, implying that when $\alpha_{il}^{(k)} > 1$ and $\alpha_{lj}^{(k)} > 1$ then also $\alpha_{ij}^{(k)} > 1$.[2]

Starting with Saaty (1977), a large literature has proposed different consistency indexes to evaluate the quality of the ratio assessments both for individuals decisions and for group decisions (recent contributions in Escobar, Aguarón and Moreno-Jiménez 2004, Moreno-Jiménez, Aguarón and Escobar 2007, Aull-Hyde, Erdogan and Duke 2006, Lin et al. 2008). As however remarked in the Introduction, it should be clear that consistency is only a proxy for the quality of the assessments. This is because while the theoretical case of no errors always implies full consistency, the converse is not true. A simple example is a judgment matrix $\mathbf{A}^{(k)}$ in which a decision maker always elicits responses $\alpha_{ij}^{(k)} = 1$ in all pairwise comparisons (i, j) and in any context. The resulting judgment matrix is fully consistent, but unlikely to be without errors.

We denote the priority weights in the theoretical case of no errors in the judgment matrix as $\mathbf{w}^{(k)}$. Then, if $\mathbf{A}^{(k)}$ is the comparison matrix of individual k, we define the matrix of deviations, called $\mathrm{d}\mathbf{E}^{(k)}$, through the equality $\mathbf{A}^{(k)} = \left(\mathbf{w}^{(k)}\overline{\mathbf{w}^{(k)}}^{\mathsf{T}}\right) \odot \overline{\exp}\left(\mathrm{d}\mathbf{E}^{(k)}\right)$ or $\mathrm{d}\mathbf{E}^{(k)} = \overline{\ln}\left[\mathbf{A}^{(k)} \odot \left(\overline{\mathbf{w}^{(k)}}\mathbf{w}^{(k),\mathsf{T}}\right)\right] = \overline{\ln}\mathbf{A}^{(k)} + \overline{\ln}\left(\overline{\mathbf{w}^{(k)}}\mathbf{w}^{(k),\mathsf{T}}\right)$, with $\mathrm{d}\mathbf{E}^{(k)} = \mathbf{0}$ corresponding to the case of no deviations. The vectors of priority weights obtained from $\mathbf{A}^{(k)}$ with different prioritization methods are denoted as $\mathbf{u}^{(k)}$. In Bernasconi et al. (2011) we discuss the algebraic properties of $\mathbf{u}^{(k)}$.

When applied to group decision making, techniques are used to obtain a vector of priority weights valid for the group as a whole. In this article we are interested in comparing the difference $(\mathbf{u} - \mathbf{w})$, where $\mathbf{u}$ is the group priority vector obtained by the application of various techniques starting from the individual comparison matrices $\mathbf{A}^{(k)}$, and $\mathbf{w}$ is the vector that would be obtained by the application of the

[2] For example, in a test of ordinal consistency conducted with 69 subjects performing three independent ratio estimation tasks, we observed only 7 violations of ordinal consistency in the sum of the three experiments, less than 3.5% (see Bernasconi et al. 2010). Similar positive results are reported in test of the monotonicity property, which is the equivalent of ordinal consistency in the context of ratio production tasks in psychophysics (see, e.g., Agustin and Maier 2008).

same techniques in the theoretical case in which $d\mathbf{E}^{(k)} = \mathbf{0}$ for all k.

2.1 Group formation and member weights

A particularly important issue that arises in the AHP when applied to group decision making concerns how the group is formed and whether the members of the group are of equal importance. In many situations it is natural to assume that agents who agree to act as a group also agree to have equal importance in the group. There are however also many contexts in which members may be assigned different importance, for example because the group is already a well-established hierarchy in an organization, or because in the group there are agents more experts than others. It is then possible to use weights $\beta_1, \dots, \beta_K$ to measure the importance of every member of the group (see Forman and Peniwati 1998, Ramanathan and Ganesh 1994, Saaty 1994). In particular, the weights satisfy $\beta_k \geq 0$ for every k and $\sum_{k=1}^{K} \beta_k = 1$. In the simplest case, when all individuals have equal importance, it is $\beta_k = K^{-1}$ for every k. Consensus on different weights may be more difficult to achieve. In some situations, there can be an external source determining the weights or, as is sometimes referred to, a 'supra decision maker' (Ramanathan and Ganesh 1994). When this does not exist, it is in principle possible to use the AHP to determine the priorities weights for the group members. The problem is then to decide who should give the judgments to obtain the member weights. If it is the group itself, the issue is to determine the member weights for this meta-problem (Forman and Peniwati 1998). One possibility is to assume equal member weights at this upper level problem. An alternative way proposed by Ramanathan and Ganesh (1994) adopts a methodology in which each member of the group evaluate the importance of all group members, including himself or herself.[3] A problem of this approach is that the decision maker can exaggerate her importance if she has an advantage from doing that. A limit to this tendency could come from the fact that individuals who are discovered overrating themselves, giving biased judgments, or

[3] The method is based on an eigenvector approach which assumes that the members priority weights for the original decision problem and for the meta-problem should be the same (see Ramanathan and Ganesh 1994, p. 257 for details).

not being really expert, will be penalized in subsequent decisions by lower weights presumably assigned by others.[4] Other scholars have proposed approaches in which members only provide evaluations to some of the other members, which typically do not include themselves, with specific techniques proposed to recover member weights from incomplete pairwise comparisons (Lootsma 1997, Van Den Honert 2001). Other techniques can be applied to assign weights to members of homogeneous subgroups (Bollojou 2001). In any case, we remark that the analyses which will be developed in this paper apply and the results hold regardless of the methods used to weight the different decision makers.

2.2 Levels of aggregation

Following a large literature (Aczél and Saaty 1983, Saaty 1989, Dyer and Forman 1992, Ramanathan and Ganesh 1994, Forman and Peniwati 1998, Van Den Honert and Lootsma 1996, Van Den Honert 2001, Dong et al. 2010), the aggregation can be performed at two levels: AIJ (aggregation of individual judgments) consists in the aggregation of the individual comparison matrices $\mathbf{A}^{(k)}$ into one judgment matrix $\mathbf{A}$ valid for the group as a whole, and then in the computation of the group decision vector $\mathbf{u}$ from this matrix; AIP (aggregation of individual priorities) consists in the computation of the individual weights $\mathbf{u}^{(k)}$ from each $\mathbf{A}^{(k)}$ first, and then in obtaining the aggregated vector $\mathbf{u}$ from these. According to Forman and Peniwati (1998), the two methods may be seen to correspond to two different ways of considering the group: in the first, the group is taken as a sort of new individual, different from the simple collection of all its members; whereas in the second, the group is seen as a collection of independent agents maintaining their own identities. Others neglect this interpretation and compare the two methods simply on the basis of the axiomatic justifications of the procedures of aggregation.

[4]Obviously, such a system of sanctions can work the better the more possibilities there are to actually monitor the biases in the decision makers' judgments in various contexts. In a subsequent part of the paper we will further comment on the possibility to evaluate the extent to which a decision maker provides accurate judgments in the various contexts.

2.3 Procedures of aggregation

Indeed, depending on the level chosen for aggregation, different averaging procedures can be used for aggregation. In the context of the AIJ, the main aggregation method in the literature is the WGM (weighted geometric mean method) that is based on the computation of the element-wise weighted geometric mean of the comparison matrices, i.e. of the aggregated matrix $\mathbf{A} = [\alpha_{ij}]$ whose generic element is $\alpha_{ij} = \prod_{k=1}^{K} \left(\alpha_{ij}^{(k)}\right)^{\beta_k}$, where β_k is the weight for individual k. The use of this method in the context of the AIJ has sometimes been criticized because it violates the Pareto Principle with respect to individual priorities (e.g., Ramanathan and Ganesh 1994). However, according to Forman and Peniwati (1998), the Pareto Principle with respect to priorities is inapplicable in the context of AIJ precisely because aggregation concerns judgments, not priorities. On the other hand, when aggregating judgments, it has been demonstrated that WGM is indeed the only method which preserves the reciprocally symmetric structure of the judgment matrices[5] and satisfies the Pareto Principle over judgments and the so-called homogeneity condition,[6] whereas other procedures like the arithmetic mean do not (see Aczél and Saaty 1983, Aczél and Alsina 1986, Forman and Peniwati 1998). Sometimes in the AIJ in particular, assignment of different weights β_k's among agents k's may reflect different expertise with the purpose of assigning greater weights to judgments of more expert agents.

In the context of the AIP, the vectors of priorities $\mathbf{u}^{(k)} = \left[u_i^{(k)}\right]$ are first computed and then aggregated. For the AIP both the geometric and the arithmetic average satisfy the Pareto Principle over priorities and can therefore be used. In particular, for the AIP, the methods based on the weighted geometric mean considered in the literature are of two kinds: the normalized weighted geometric mean method (NWGM) is based on the computation of the geometric mean of the eigenvectors and on the normalization of the vector, and yields a vector $\mathbf{u}$ whose $i-$th element is given by $u_i = \Pi_{k=1}^{K}\left(u_i^{(k)}\right)^{\beta_k} / \sum_{h=1}^{n} \prod_{k=1}^{K}\left(u_h^{(k)}\right)^{\beta_k}$; the unnormalized weighted

[5]An aggregation procedure preserves the reciprocally symmetric structure if the aggregated matrix $\mathbf{A} = [\alpha_{ij}]$ is reciprocally symmetric when the individual matrices $\mathbf{A}^{(k)}$ are.

[6]The homogeneity condition requires that if all individuals judge a ratio λ times as large as another ratio, then the aggregated judgments should be λ times as large.

geometric mean method (UWGM) is based on the computation of the geometric mean of the eigenvectors without normalization (see, e.g., Forman and Peniwati 1998), and yields a vector $\mathbf{u}$ whose $i-$th element is given by $u_i = \prod_{k=1}^{K} \left(u_i^{(k)}\right)^{\beta_k}$. The weighted arithmetic mean method (WAM) is based on the arithmetic mean of the vectors, and yields a vector $\mathbf{u}$ whose $i-$th element is given by $u_i = \sum_{k=1}^{K} \beta_k u_i^{(k)}$ and is guaranteed to be normalized.

It is also worth noticing that a preference for geometric mean methods over arithmetic mean methods as aggregation procedures is sometimes justified on the ground that arithmetic methods are typically relevant when measurements possess only an interval scale meaning. However, in the context of AHP, in which measurements occurs on ratio scale and have precisely the meaning of representing how many more times an alternative dominates (in terms of preference or judgment) another alternative, the geometric mean is more suitable for aggregation since it directly implements the homogeneity condition (recent discussion in, e.g., Escobar and Moreno-Jiménez 2007).

2.4 Prioritization

Prioritization is the process of computing the priority vector from the judgments matrix. In the AIP, prioritization applies to the individual matrices $\mathbf{A}^{(k)}$, whereas in the AIJ it applies to the group matrix $\mathbf{A}$ directly.

In either case, the computation of the vectors can be performed using different methods. The classical one is the maximum eigenvalue (ME). It has been proposed by Saaty in his classical writings (1977, 1980) and confirmed since then (Saaty 1990, 2003). In the AIP, for a generic matrix $\mathbf{A}^{(k)} = \left[\alpha_{ij}^{(k)}\right]$, the maximum eigenvalue (ME) method yields the vector $\mathbf{u}^{(k)}$ defined as $\mathbf{A}^{(k)}\mathbf{u}^{(k)} = \lambda^{(k)}\mathbf{u}^{(k)}$ where $\lambda^{(k)}$ denotes the Perron root (maximum eigenvalue) of $\mathbf{A}^{(k)}$ and $\sum_{i=1}^{n} u_i^{(k)} = 1$. For the theoretical case of a judgment matrix with no errors, $\mathbf{w}^{(k)}$ is the vector containing the underlying priority weights. For this case, the ME is known to deliver $\mathbf{w}^{(k)}$ directly with the maximum eigenvalue being at its minimum $\lambda^{(k)} = n$. The classical AHP argument is to use the ME method even for the practical cases in which the matrices $\mathbf{A}^{(k)}$'s contain errors and are therefore not fully consistent, provided inconsistencies fall within given bounds. Therefore, the main normative justifications to use the ME lies in its algebraic properties (Saaty 2003).

A different prioritization method is the logarithmic least squares (LLS) method. It yields a vector $\mathbf{u}^{(k)}$ whose $i-$th element is given by $u_i^{(k)} = \left(\prod_{j=1}^{n} \alpha_{ij}\right)^{1/n} / \sum_{h=1}^{n} \left(\prod_{j=1}^{n} \alpha_{hj}\right)^{1/n}$. The main characteristic of this method is that it can be justified on the basis of statistical properties (classical references in de Jong 1984, and Genest and Rivest 1994). A variant of LLS studied by Crawford and Williams (1985) and applied to AHP in group decision making by Escobar, Aguarón and Moreno-Jiménez (2004) is the row geometric mean (RGM). It yields a vector $\mathbf{u}^{(k)}$ such that its $i-$th element is given by $u_i^{(k)} = \left(\prod_{j=1}^{n} \alpha_{ij}^{(k)}\right)^{1/n}$.

In the AIJ the same prioritization methods can be used to obtain the group priority vector $\mathbf{u}$ from the aggregated matrix $\mathbf{A}$.

3 Comparisons of $(\mathbf{u} - \mathbf{w})$ in AHP-group aggregations

One important question regarding the above aggregation procedures which we believe the previous literature has not fully addressed concerns the difference between the various methods on empirical grounds. In particular, how much empirical difference can we expect from the various techniques when applied to actual data, given the mathematical properties on which the techniques are based?

3.1 Classes of aggregation methods

To answer the above question we now analyze the algebraic characteristics of the priority weights $\mathbf{u}$ computed according to the different methods and compare the differences $(\mathbf{u} - \mathbf{w})$ obtained under the various methods. The results of the analysis are summarised in Table 1. In the table, we use the following notation. We introduce the matrices $\mathbf{W} \triangleq \left(\mathrm{diag}\left[\mathbf{w}\right] - \mathbf{w} \cdot \mathbf{w}^{\mathsf{T}}\right)$ and $\mathbf{W}^{(k)} \triangleq \left(\mathrm{diag}\left[\mathbf{w}^{(k)}\right] - \mathbf{w}^{(k)} \cdot \mathbf{w}^{(k),\mathsf{T}}\right)$. We recall that the matrix of errors of individual k, with respect to the case of consistency is $\mathrm{d}\mathbf{E}^{(k)} = \overline{\ln}\mathbf{A}^{(k)} + \overline{\ln}\left(\overline{\mathbf{w}^{(k)}}\mathbf{w}^{(k),\mathsf{T}}\right)$. We also notice that, whatever the method of aggregation, there is no guarantee that $\overline{\ln}\left[\mathbf{A} \odot \left(\overline{\mathbf{w}}\mathbf{w}^{\mathsf{T}}\right)\right]$ is a well-defined matrix of deviations from consistency (see Lin et al. 2008, p. 675, equation 16).

The first three columns of the table report the three dimensions of aggregations discussed above, namely the levels of the aggregation, the averaging procedures for the aggregation, the prioritization methods. The formulas for $\mathbf{u}$ and $\mathbf{w}$ obtained by the various techniques are shown in columns 3 and 5, respectively. To compare the formulas it is necessary to consider the different normalizations employed by the techniques. In particular, most prioritization methods (ME, LLS) require that the vector is such that $\sum_{j=1}^{n} w_j^{(k)} = 1$; nevertheless, other methods (RGM) require a vector such that $\prod_{j=1}^{n} w_j^{(k)} = 1$. This introduces a small difficulty, since $\mathbf{w}^{(k)}$ is normalized in different ways according to the method. In order to avoid complications, the table indicates the vector with the same symbol, specifying in each case the kind of normalization. The same fact happens with $\mathbf{w}$ that can be given by a weighted arithmetic mean of the individual $\mathbf{w}^{(k)}$'s or by a weighted geometric mean of the same vectors; in the latter case, it can be normalized through the alternative constraints $\sum_{j=1}^{n} w_j = 1$ or $\prod_{j=1}^{n} w_j = 1$. Also in this case, we use the same symbol, leaving the specification of the kind of vector to the context. A further problem is that, even if the individual vectors are normalized as $\sum_{j=1}^{n} w_j^{(k)} = 1$ or $\prod_{j=1}^{n} w_j^{(k)} = 1$, some aggregation methods yield a resulting aggregated vector with no normalization (see below for details).

The analytic derivations of all the expressions in Table 1 are obtained using the theory of matrix differentials (Magnus and Neudecker 1999) and are given in the Appendix. We remark that, up to the first order, all methods have the same kind of expansion, since they are given by:

$$\mathbf{u} \simeq \mathbf{w} + \frac{1}{n} \sum_{k=1}^{K} \beta_k \cdot \mathbf{B}^{(k)} \mathrm{d}\mathbf{E}^{(k)} \mathbf{u}_n \tag{1}$$

where the matrix $\mathbf{B}^{(k)}$, that is different across methods and can vary across individuals, is given in Table 1.

Comparing the formulas for $\mathbf{u}$'s and $\mathbf{w}$'s, the table identifies 5 different classes

Table 1: Characteristics of priority weights

			Priorities						
			Formula of $\mathbf{u}$	$\mathbf{B}^{(k)}$	Norm. of $\mathbf{u}$	Formula of $\mathbf{w}$	Norm. of $\mathbf{w}$	Norm. of $\mathbf{w}^{(k)}$	Class
AIJ	WGM	ME	—	$\mathbf{W}$	$\mathbf{u}_n^\intercal \mathbf{u} = 1$	$\frac{\bigodot_{k=1}^K \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k}}{\left[\mathbf{u}_n^\intercal \cdot \bigodot_{k=1}^K \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k}\right]}$	$\mathbf{u}_n^\intercal \mathbf{w} = 1$	$\mathbf{u}_n^\intercal \mathbf{w}^{(k)} = 1$	1
		LLS	—	$\mathbf{W}$	$\mathbf{u}_n^\intercal \mathbf{u} = 1$	$\frac{\bigodot_{k=1}^K \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k}}{\left[\mathbf{u}_n^\intercal \cdot \bigodot_{k=1}^K \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k}\right]}$	$\mathbf{u}_n^\intercal \mathbf{w} = 1$	$\mathbf{u}_n^\intercal \mathbf{w}^{(k)} = 1$	1
		RGM	$\bigodot_{k=1}^K \left(\mathbf{u}^{(k)}\right)^{\odot \beta_k}$	$\operatorname{diag}(\mathbf{w})$	$\prod_{j=1}^n u_j = 1$	$\bigodot_{k=1}^K \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k}$	$\prod_{j=1}^n w_j = 1$	$\prod_{j=1}^n w_j^{(k)} = 1$	2
AIP	NWGM	ME	$\frac{\bigodot_{k=1}^K \left(\mathbf{u}^{(k)}\right)^{\odot \beta_k}}{\mathbf{u}_n^\intercal \bigodot_{k=1}^K \left(\mathbf{u}^{(k)}\right)^{\odot \beta_k}}$	$\mathbf{W}$	$\mathbf{u}_n^\intercal \mathbf{u} = 1$	$\frac{\bigodot_{k=1}^K \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k}}{\left[\mathbf{u}_n^\intercal \cdot \bigodot_{k=1}^K \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k}\right]}$	$\mathbf{u}_n^\intercal \mathbf{w} = 1$	$\mathbf{u}_n^\intercal \mathbf{w}^{(k)} = 1$	1
		LLS	$\frac{\bigodot_{k=1}^K \left(\mathbf{u}^{(k)}\right)^{\odot \beta_k}}{\mathbf{u}_n^\intercal \bigodot_{k=1}^K \left(\mathbf{u}^{(k)}\right)^{\odot \beta_k}}$	$\mathbf{W}$	$\mathbf{u}_n^\intercal \mathbf{u} = 1$	$\frac{\bigodot_{k=1}^K \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k}}{\left[\mathbf{u}_n^\intercal \cdot \bigodot_{k=1}^K \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k}\right]}$	$\mathbf{u}_n^\intercal \mathbf{w} = 1$	$\mathbf{u}_n^\intercal \mathbf{w}^{(k)} = 1$	1
		RGM	$\frac{\bigodot_{k=1}^K \left(\mathbf{u}^{(k)}\right)^{\odot \beta_k}}{\mathbf{u}_n^\intercal \bigodot_{k=1}^K \left(\mathbf{u}^{(k)}\right)^{\odot \beta_k}}$	$\mathbf{W}$	$\mathbf{u}_n^\intercal \mathbf{u} = 1$	$\frac{\bigodot_{k=1}^K \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k}}{\left[\mathbf{u}_n^\intercal \cdot \bigodot_{k=1}^K \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k}\right]}$	$\mathbf{u}_n^\intercal \mathbf{w} = 1$	$\prod_{j=1}^n w_j^{(k)} = 1$	1
	UWGM	ME	$\bigodot_{k=1}^K \left(\mathbf{u}^{(k)}\right)^{\odot \beta_k}$	$\operatorname{diag}(\mathbf{w}) - \mathbf{w} \cdot \mathbf{w}^{(k),\intercal}$	none	$\bigodot_{k=1}^K \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k}$	none	$\mathbf{u}_n^\intercal \mathbf{w}^{(k)} = 1$	3
		LLS	$\bigodot_{k=1}^K \left(\mathbf{u}^{(k)}\right)^{\odot \beta_k}$	$\operatorname{diag}(\mathbf{w}) - \mathbf{w} \cdot \mathbf{w}^{(k),\intercal}$	none	$\bigodot_{k=1}^K \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k}$	none	$\mathbf{u}_n^\intercal \mathbf{w}^{(k)} = 1$	3
		RGM	$\bigodot_{k=1}^K \left(\mathbf{u}^{(k)}\right)^{\odot \beta_k}$	$\operatorname{diag}(\mathbf{w})$	$\prod_{j=1}^n u_j = 1$	$\bigodot_{k=1}^K \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k}$	$\prod_{j=1}^n w_j = 1$	$\prod_{j=1}^n w_j^{(k)} = 1$	2
	WAM	ME	$\sum_{k=1}^K \beta_k \mathbf{u}^{(k)}$	$\mathbf{W}^{(k)}$	$\mathbf{u}_n^\intercal \mathbf{u} = 1$	$\sum_{k=1}^K \beta_k \mathbf{w}^{(k)}$	$\mathbf{u}_n^\intercal \mathbf{w} = 1$	$\mathbf{u}_n^\intercal \mathbf{w}^{(k)} = 1$	4
		LLS	$\sum_{k=1}^K \beta_k \mathbf{u}^{(k)}$	$\mathbf{W}^{(k)}$	$\mathbf{u}_n^\intercal \mathbf{u} = 1$	$\sum_{k=1}^K \beta_k \mathbf{w}^{(k)}$	$\mathbf{u}_n^\intercal \mathbf{w} = 1$	$\mathbf{u}_n^\intercal \mathbf{w}^{(k)} = 1$	4
		RGM	$\sum_{k=1}^K \beta_k \mathbf{u}^{(k)}$	$\operatorname{diag}\left(\mathbf{w}^{(k)}\right)$	none	$\sum_{k=1}^K \beta_k \mathbf{w}^{(k)}$	none	$\prod_{j=1}^n w_j^{(k)} = 1$	5

of methods according to the first-order expansion of the vector $\mathbf{u}$ in terms of the matrix $\mathrm{d}\mathbf{E}^{(k)}$.

Methods of class 1 (AIJ-WGM-ME/LLS, AIP-NGWM-ME/LLS/RGM) yield a vector $\mathbf{u}$ such that $\mathbf{u}_n^{\mathsf{T}}\mathbf{u} = 1$, and $\mathbf{u} \simeq \mathbf{w} + \frac{1}{n}\sum_{k=1}^{K}\beta_k \cdot \mathbf{W}\mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n$. These methods are insensitive to the normalization adopted for $\mathbf{u}^{(k)}$: this is due to the fact that they are homogeneous of degree 1 in each vector, so that different eigenvectors can even be normalized in different ways. As remarked above, this is a very appealing property of WGM. Moreover, AIJ-WGM-LLS and AIP-NWGM-LLS/RGM yield exactly the same priority vector. Clearly, the remarkable result for the models of this class is that they obtain the same priority vectors regardless whether the aggregation procedures are applied at the level of judgments (AIJ) or of priorities (AIP).

The two methods of class 2 (AIJ-WGM-RGM and AIP-UWGM-RGM) yield exactly the same priorities (see Escobar, Aguarón and Moreno-Jiménez 2004). In this case, $\mathbf{u}$ is normalized so that $\prod_{j=1}^{n} u_j = 1$ and $\mathbf{u} \simeq \mathbf{w} + \frac{1}{n}\sum_{k=1}^{K}\beta_k \cdot \mathrm{diag}\left(\mathbf{w}\right) \cdot \mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n$.

Methods of class 3 (AIP-UWGM-ME/LLS) yield the formula $\mathbf{u} \simeq \mathbf{w} + \frac{1}{n}\sum_{k=1}^{K}\beta_k \cdot \left(\mathrm{diag}\left(\mathbf{w}\right) - \mathbf{w} \cdot \mathbf{w}^{(k),\mathsf{T}}\right) \cdot \mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n$. Unfortunately, the priority vector so obtained is not normalized.

Methods of class 4 (AIP-WAM-ME/LLS) yield a vector $\mathbf{u}$ normalized as $\mathbf{u}_n^{\mathsf{T}}\mathbf{u} = 1$, with expansion $\mathbf{u} \simeq \mathbf{w} + \frac{1}{n}\sum_{k=1}^{K}\beta_k \cdot \mathbf{W}^{(k)} \cdot \mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n$.

The only method of class 5 (AIP-WAM-RGM) yields $\mathbf{u} \simeq \mathbf{w} + \frac{1}{n}\sum_{k=1}^{K}\beta_k \cdot \mathrm{diag}\left(\mathbf{w}^{(k)}\right) \cdot \mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n$. The problem of this aggregation method is that the eigenvector is not normalized in any way.

3.2 Empirical computations

In Table 2, we compute the vectors of priority weights aggregated according to the different methods for three experiments described in Bernasconi, Choirat and Seri (2010). In the experiments, 69 individuals were asked to elicit individual comparison matrices in three domains respectively concerning 5 probabilities from games of chances, 5 distances of Italian cities from Milan, and the rainfalls in 5 European cities

TABLE 2: Aggregations of priority weights by different methods

a) Chances experiment

Methods			Weights					Class
AIJ	WGM	ME	0.13579354	0.04955144	0.44610785	0.08617203	0.28237514	1
		LLS	0.13553307	0.04975188	0.44674197	0.08740280	0.28057028	1
		RGM	0.9085515	0.3335138	2.9947533	0.5859083	1.8808145	2
AIP	NWGM	ME	0.13542866	0.04855110	0.44577453	0.08675934	0.28348636	1
		LLS	0.13553307	0.04975188	0.44674197	0.08740280	0.28057028	1
		RGM	0.13553307	0.04975188	0.44674197	0.08740280	0.28057028	1
	UWGM	ME	0.12688924	0.04548973	0.41766634	0.08128876	0.26561121	3
		LLS	0.12703102	0.04663093	0.41871767	0.08191999	0.26296999	3
		RGM	0.9085515	0.3335138	2.9947533	0.5859083	1.8808145	2
	WAM	ME	0.14088845	0.05057028	0.43650269	0.09161485	0.28042373	4
		LLS	0.14072004	0.05222758	0.43764566	0.09114555	0.27826117	4
		RGM	0.9871902	0.3640906	3.1903676	0.6399945	2.0140510	5

b) Distances experiment

Methods			Weights					Class
AIJ	WGM	ME	0.25623855	0.45062333	0.16563406	0.05025878	0.07724528	1
		LLS	0.25798675	0.44838493	0.16582196	0.05014743	0.07765893	1
		RGM	1.7255649	2.9990583	1.1091133	0.3354151	0.5194280	2
AIP	NWGM	ME	0.25859236	0.44989872	0.16458325	0.05010357	0.07682210	1
		LLS	0.25798675	0.44838493	0.16582196	0.05014743	0.07765893	1
		RGM	0.25798675	0.44838493	0.16582196	0.05014743	0.07765893	1
	UWGM	ME	0.25132578	0.43725633	0.15995837	0.04869563	0.07466336	3
		LLS	0.25101157	0.43626196	0.16133863	0.04879159	0.07555927	3
		RGM	1.7255649	2.9990583	1.1091133	0.3354151	0.5194280	2
	WAM	ME	0.25371327	0.44514466	0.16635863	0.05295901	0.08182443	4
		LLS	0.25379045	0.44391684	0.16718922	0.05277836	0.08232513	4
		RGM	1.7616022	3.1411006	1.1353818	0.3501750	0.5459828	5

c) Rainfalls experiment

Methods			Weights					Class
AIJ	WGM	ME	0.05764504	0.22923164	0.44891332	0.17619264	0.08801735	1
		LLS	0.05756653	0.22998892	0.44788431	0.17623212	0.08832813	1
		RGM	0.3690836	1.4745569	2.8715770	1.1298991	0.5663092	2
AIP	NWGM	ME	0.05758913	0.22966776	0.44501392	0.17920920	0.08851999	1
		LLS	0.05756653	0.22998892	0.44788431	0.17623212	0.08832813	1
		RGM	0.05756653	0.22998892	0.44788431	0.17623212	0.08832813	1
	UWGM	ME	0.05341867	0.21303581	0.41278716	0.16623133	0.08210960	3
		LLS	0.05334509	0.21312346	0.41504023	0.16330873	0.08185089	3
		RGM	0.3690836	1.4745569	2.8715770	1.1298991	0.5663092	2
	WAM	ME	0.05973668	0.23167645	0.42525063	0.18546677	0.09786947	4
		LLS	0.05950081	0.23215688	0.42752466	0.18299506	0.09782259	4
		RGM	0.3982207	1.6170153	3.0168890	1.2651230	0.6623847	5

in November 2001.[7] The computations in the Table aggregate individual judgments or individual priorities, depending on the method, over the 69 individuals using equal weights. The results of the computations confirm the validity of the approach and of the classification discussed above. In particular, consistently with the algebraic expressions derived in Table 1, we find that the differences between the methods belonging to each of the 5 classes are very small, namely 0 up to the third decimal in all the three experiments; while they are larger between methods of different classes. It is just worth remarking that this finding is virtually independent of the size of the group, since the group size n is not an asymptotic parameter for any class.

We also observe that since the methods differ only in the second and higher orders, the results indirectly support the validity of the first order approximation, as previously indicated for individual decision making.[8]

3.3 Discussion and implications

The classification in Table 1 and the empirical computations in Table 2 put also some order on the issues discussed in Section 2. First of all, although the analysis identifies five main classes of methods, two of them (classes 3 and 5) imply that the priority vectors obtained are not normalized and are therefore unsuitable for actual implementation. Methods in class 2 are characterized by the use of the RGM as prioritization procedure (when applied in combination with AIJ/WGM and AIP/UWGM). It is quite interesting that, even if this method is often considered similar to LLS, the two can produce results that are more different than those obtained under LLS and ME. On the contrary, notwithstanding the disputes between the latter two methods often encountered in the literature, it is remarkable that on the empirical ground ME and LLS generate virtually *identical* priorities when applied in *all* aggregating procedures. Overall, the analysis of this Section can be

[7] The instructions and the full data set of the experiments are available from authors.

[8] For individual decision making (in Bernasconi, Choirat and Seri 2011), we have actually computed the second-order term and shown that it is much smaller that the first-order one. Similar computations are available here for the interested reader.

viewed to speak moderately in favour of models of class 1. In addition to the interesting properties listed above, including the equivalence of the priority vectors obtained under AIP and AIJ, the use of the WGM as averaging procedure for the models of this class can be supported over (for example) models of class 4, on the basis of the normative argument outlined in Section 2, namely that the geometric mean fits better than the arithmetic mean the notion of ratio scale measures underlying both judgments and priorities.

4 Decomposition of $(\mathbf{u} - \mathbf{w})$ in group aggregation methods: the effect of systematic distortions

What does it cause the departure of $\mathbf{u}$ from its true value $\mathbf{w}$? Classical AHP has not generally spent much attention to discuss the nature of the perturbations occurring in ratio estimation tasks and has generally assumed that they are due to random errors.[9] In recent years, studies in mathematical psychology have focussed on systematic distortions occurring in subjective ratio estimations, which can be formalised in so-called separable forms (Narens 1996, 2002, and Luce 2002, 2004).

4.1 Separable representations

Following the literature on separable forms, from now on we assume that the elicited ratios in the AHP are generated according to the following model:[10]

$$\alpha_{ij} = W^{-1}\left(\frac{\psi(x_i)}{\psi(x_j)}\right) \cdot e_{ij} \qquad (2)$$

where the functions ψ and W are respectively called *psychophysical* and *subjective weighting* function, $W^{-1}(\cdot)$ is the inverse of $W(\cdot)$, $\psi(x_1), ..., \psi(x_n)$ are the psychological perceptions of the stimuli intensities corresponding to the priority weights $w_i = \frac{\psi(x_i)}{\sum_j \psi(x_j)}$ (for $i = 1, ..., n$); and where e_{ij} are the more classical multiplicative

[9]See de Jong (1984) and Genest and Rivest (1994) for classical works on the stochastic structure of the error terms in the AHP.

[10]For a more general discussion on the relation between the AHP and the modern theory of separable representations we refer to Bernasconi, Choirat and Seri (2010).

random error terms introduced by the AHP. The functions ψ and W indicate that two independent transformations may occur in a ratio estimation: one of the stimuli intensities (embodied in ψ), and the other of numbers (entailed in W). Support for separable forms has been found in a series of recent experiments which have been conducted to test some of their underlying properties and which have estimated different functional specifications of ψ and W (see, among others, Ellermeier and Faulhammer 2000, Zimmer 2005, Steingrimsson and Luce 2005a, 2005b, 2006, 2007, Bernasconi, Choirat and Seri 2008). Specifically relevant are the distortions caused by W, which are sometimes interpreted as cognitive ones. In fact, findings have shown that systematic distortions of numbers (actually, ratios) due to W follow a very intuitive pattern: namely, people tend to overestimate low ratios and underestimate high ratios, with the tendency to underestimate increasing as the ratios get increasingly larger than one. Such tendency is also consistent with the probability transformations observed to affect people decisions under conditions of risk and uncertainty and used to develop descriptive models of choice in that area (like the famous Cumulative Prospect Theory by Tversky and Kahneman 1992).

4.2 Deterministic distortions in $(\mathbf{u}-\mathbf{w})$

Here we study the effect of deterministic distortions due to the subjective weighting function W in group aggregation methods. We set $e_{ij}=\exp(\nu_{ij})$, where $\nu_{ij}=-\nu_{ji}$ and the ν_{ij}'s are independent and identically distributed random variables with $\mathbb{E}(\nu_{ij})=0$ and $\mathbb{V}(\nu_{ij})=\sigma^2$. Then, the previous formula (2) can be rewritten as:

$$\alpha_{ij}=\frac{w_i}{w_j}\cdot\exp\left\{\ln\left[\frac{w_j}{w_i}\cdot W^{-1}\left(\frac{w_i}{w_j}\right)\right]+\nu_{ij}\right\}=\frac{w_i}{w_j}\cdot e^{\mathrm{d}\varepsilon_{ij}} \tag{3}$$

where:

$$\mathrm{d}\varepsilon_{ij}=\ln\left[\frac{w_j}{w_i}\cdot W^{-1}\left(\frac{w_i}{w_j}\right)\right]+\nu_{ij}. \tag{4}$$

In order to respect the property of reciprocal symmetry, we need $\mathrm{d}\varepsilon_{ij}=-\mathrm{d}\varepsilon_{ji}$.

Taking a polynomial approximation, it is then possible to write $W^{-1}(\cdot)$ as (see, e.g., Bernasconi, Choirat and Seri 2011, p. 156):

$$\begin{aligned} W^{-1}(x) &= \exp\left\{\sum_{\ell=0}^{L}\phi_\ell\cdot[\ln(x)]^\ell\right\} \\ &= x\cdot\exp\left\{\sum_{\ell=2}^{L}\phi_\ell\cdot[\ln(x)]^\ell\right\} \end{aligned} \tag{5}$$

so that, when $\|\phi_\ell\|_\infty = \max_{2\le\ell\le L}|\phi_\ell| \downarrow 0$, $W^{-1}(x)\to x$. Coefficients ϕ_ℓ's therefore approximate the effect of the systematic distortion. Substituting in equation (4) we obtain:

$$\mathrm{d}\varepsilon_{ij} = \ln\left[\frac{w_j}{w_i}\cdot W^{-1}\left(\frac{w_i}{w_j}\right)\right] + \nu_{ij} = \sum_{\ell=2}^{L}\phi_\ell\cdot[\ln(w_i/w_j)]^\ell + \nu_{ij}. \tag{6}$$

Thus, under the hypotheses that $\|\phi_\ell\|_\infty = \max_{2\le\ell\le L}|\phi_\ell| \downarrow 0$ and $\sigma\downarrow 0$, $\mathrm{d}\varepsilon_{ij}$ is asymptotically negligible. This expansion holds for all the individuals in the group, so that we put an apex (k) on the quantities appearing in the above formula; for ease of notation, we suppose that L is independent of k, since this can always be achieved introducing some zero coefficients $\phi_\ell^{(k)}$'s.

We define the vectors of weights $\mathbf{w}^{(k)} \triangleq \left[w_i^{(k)}\right]$, the matrices of distortions $\mathrm{d}\mathbf{E}^{(k)} \triangleq \left[\varepsilon_{ij}^{(k)}\right]$ and of random errors $\mathbf{N}^{(k)} \triangleq \left[\nu_{ij}^{(k)}\right]$. In matrix notation, it is possible to write:

$$\begin{aligned} \mathrm{d}\mathbf{E}^{(k)} &= \sum_{\ell=2}^{L}\phi_\ell^{(k)}\cdot\left[\overline{\ln}\left(\mathbf{w}^{(k)}\overline{\mathbf{w}^{(k)}}^{\mathsf{T}}\right)\right]^{\odot\ell} + \mathbf{N}^{(k)} + o\left(\|\phi_\ell\|_\infty\right) + o_{\mathbb{P}}(\sigma) \\ &= \sum_{\ell=2}^{L}\phi_\ell^{(k)}\cdot\left[\left(\overline{\ln}\mathbf{w}^{(k)}\right)\cdot\mathbf{u}_n^{\mathsf{T}} - \mathbf{u}_n\cdot\left(\overline{\ln}\mathbf{w}^{(k)}\right)^{\mathsf{T}}\right]^{\odot\ell} + \mathbf{N}^{(k)} + o\left(\|\phi_\ell\|_\infty\right) + o_{\mathbb{P}}(\sigma) \end{aligned} \tag{7}$$

While the order of the approximation can obviously be extended to any desired degree, we retain the approximation in the first non-zero term:[11]

$$\mathrm{d}\mathbf{E}^{(k)} \simeq \phi_3^{(k)}\cdot\left[\left(\overline{\ln}\mathbf{w}^{(k)}\right)\cdot\mathbf{u}_n^{\mathsf{T}} - \mathbf{u}_n\cdot\left(\overline{\ln}\mathbf{w}^{(k)}\right)^{\mathsf{T}}\right]^{\odot 3} + \mathbf{N}^{(k)}. \tag{8}$$

[11] Remark that here and in the following $\phi_2^{(k)}$ is zero since the function $\ln\left[W^{(k)}\right]^{-1}(\exp(\cdot))$ is supposed to be skew-symmetric.

This is indeed sufficient to characterize the regularities appearing in several empirical data of individual decision making (Bernasconi, Choirat and Seri 2008, 2010). All of the theoretical computations can evidently be repeated within the more general model introduced above.

Replacing now $\mathrm{d}\mathbf{E}^{(k)}$ in the general formula (1) for $\mathbf{u}$, we obtain:

$$\begin{aligned}\mathbf{u} &\simeq \mathbf{w}+\frac{1}{n}\sum_{k=1}^{K}\beta_k\cdot\mathbf{B}^{(k)}\mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n\\ &\simeq \mathbf{w}+\frac{1}{n}\sum_{k=1}^{K}\beta_k\cdot\phi_3^{(k)}\cdot\mathbf{B}^{(k)}\left[\left(\overline{\ln}\mathbf{w}^{(k)}\right)\cdot\mathbf{u}_n^{\mathsf{T}}-\mathbf{u}_n\cdot\left(\overline{\ln}\mathbf{w}^{(k)}\right)^{\mathsf{T}}\right]^{\odot 3}\mathbf{u}_n\\ &\quad+\frac{1}{n}\sum_{k=1}^{K}\beta_k\cdot\mathbf{B}^{(k)}\mathbf{N}^{(k)}\mathbf{u}_n. \end{aligned}\tag{9}$$

The variance of the stochastic part is given by:

$$\mathbb{V}\left(\frac{1}{n}\sum_{k=1}^{K}\beta_k\cdot\mathbf{B}^{(k)}\mathbf{N}^{(k)}\mathbf{u}_n\right)=\frac{1}{n}\sum_{k=1}^{K}\beta_k^2\sigma^{(k),2}\cdot\left\{\mathbf{B}^{(k)}\cdot\left(\mathbf{I}_n-\frac{1}{n}\mathbf{U}_n\right)\cdot\mathbf{B}^{(k),\mathsf{T}}\right\};\tag{10}$$

the "bias" due to the deterministic part is given by:

$$\mathbb{E}\left(\frac{1}{n}\sum_{k=1}^{K}\beta_k\cdot\mathbf{B}^{(k)}\mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n\right)=\frac{1}{n}\sum_{k=1}^{K}\beta_k\cdot\phi_3^{(k)}\cdot\mathbf{B}^{(k)}\left[\left(\overline{\ln}\mathbf{w}^{(k)}\right)\cdot\mathbf{u}_n^{\mathsf{T}}-\mathbf{u}_n\cdot\left(\overline{\ln}\mathbf{w}^{(k)}\right)^{\mathsf{T}}\right]^{\odot 3}\mathbf{u}_n.\tag{11}$$

Equations (10) and (11) provide the basis to assess the relative contributions on the difference $\mathrm{d}\mathbf{u}\simeq\mathbf{u}-\mathbf{w}$ of the stochastic components due to $\nu_{ij}^{(k)}$ and of the deterministic distortions due to the subjective weighting function $W^{(k)}(\cdot)$. In fact, for the case of individual decision making ($K=1$), it is shown in Bernasconi, Choirat and Seri (2011) that when $|\phi_3|$ is equal to the standard error of the noise σ and the elements of $\mathbf{w}$ range on a small interval, then the effects of the deterministic distortions and of the stochastic terms are comparable; while when the stimuli in $\mathbf{w}$ are very different, the effects of the deterministic distortions are much larger than those due to the stochastic errors. This result is in line with the so-called homogeneity axiom of the AHP (Saaty 1986), which requires that the stimuli used in the AHP must be in a range of comparability.

The same result holds here for $\left|\phi_3^{(k)}\right| = \sigma^{(k)}$, provided that the $\mathbf{w}^{(k)}$'s (and the $\mathbf{B}^{(k)}$'s as a consequence) are not too dissimilar between individuals.[12] Moreover, it is important to emphasize that, in the latter case, the summands for the individuals in equation (11) are also comparable and do not tend to cancel out across individuals. On the other hand, equation (10) shows that when K is large enough and the β_k's are far away from the extreme case in which one β is 1 and the other ones are 0, the variance of the stochastic terms (due to $\beta_k^2\sigma^{(k),2}$) tends to get smaller with K. Therefore, this means that, whereas the stochastic component tends to be averaged out in group decision making, the same does not necessarily happen to the deterministic part.

In order to illustrate how the decomposition of equation (9) works in practice, the previous formulas (10) and (11) are applied to the data of the experiments described above with methods of class 1. In that case, $\mathbf{B}^{(k)} = \mathbf{W} = \left(\operatorname{diag}[\mathbf{w}] - \mathbf{w}\cdot\mathbf{w}^{\mathsf{T}}\right)$, so that the variance of the stochastic part and the bias due to the deterministic part are respectively given by:

$$
\begin{aligned}
\mathbb{V}\left(\frac{1}{n}\sum_{k=1}^{K}\beta_k\cdot\mathbf{B}^{(k)}\mathbf{N}^{(k)}\mathbf{u}_n\right) &= \frac{1}{n}\cdot\left(\sum_{k=1}^{K}\beta_k^2\sigma^{(k),2}\right)\cdot\mathbf{W}^2 \\
\mathbb{E}\left(\frac{1}{n}\sum_{k=1}^{K}\beta_k\cdot\mathbf{B}^{(k)}\mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n\right) &= \frac{1}{n}\cdot\mathbf{W}\cdot\sum_{k=1}^{K}\beta_k\cdot\phi_3^{(k)}\cdot\left[\left(\overline{\ln}\mathbf{w}^{(k)}\right)\cdot\mathbf{u}_n^{\mathsf{T}} - \mathbf{u}_n\cdot\left(\overline{\ln}\mathbf{w}^{(k)}\right)^{\mathsf{T}}\right]^{\odot 3}\mathbf{u}_n.
\end{aligned}
$$

In Table 3, we provide a comparison of the different contributions to $\mathbf{u}$.[13] In the computations, the parameters $\mathbf{w}^{(k)}$'s and $\phi_3^{(k)}$'s appearing in the above

[12] In this respect we also remark that evidence available in Bernasconi, Choirat and Seri (2010) shows that the coefficients $\phi_3^{(k)}$'s assume in fact quite similar values across individuals.

[13] With the above expressions, a limiting case that well illustrates the averaging effect can be obtained when all individuals are equally weighted ($\beta_k = K^{-1}$), have the same variances ($\sigma^{(k),2} = \sigma^2$), the same deterministic distortions ($\phi_3^{(k)} = \phi_3$), and the $\mathbf{w}^{(k)}$'s are equal so that $\mathbf{w}^{(k)} = \mathbf{w}$. In such a case:

$$
\begin{aligned}
\mathbb{V}\left(\frac{1}{n}\sum_{k=1}^{K}\beta_k\cdot\mathbf{B}^{(k)}\mathbf{N}^{(k)}\mathbf{u}_n\right) &= \frac{1}{nK}\cdot\sigma^2\cdot\mathbf{W}^2 \\
\mathbb{E}\left(\frac{1}{n}\sum_{k=1}^{K}\beta_k\cdot\mathbf{B}^{(k)}\mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n\right) &= \frac{1}{n}\cdot\mathbf{W}\cdot\phi_3\cdot\left[\left(\overline{\ln}\mathbf{w}\right)\cdot\mathbf{u}_n^{\mathsf{T}} - \mathbf{u}_n\cdot\left(\overline{\ln}\mathbf{w}\right)^{\mathsf{T}}\right]^{\odot 3}\mathbf{u}_n.
\end{aligned}
$$

Therefore, the deterministic distortion is equal to the one of a single individual. On the other hand, the variance of the stochastic part is equal to the same quantity for a single individual divided by K.

Table 3: Decomposition of factors contributing to aggregated priority weights - methods of class 1

a) Chances experiment

w	0.10394883	0.02773954	0.52207564	0.05891376	0.28732224
Effect of W	0.031213212	0.016974320	-0.078534838	0.023084544	0.007262762
Effect of noise	-0.001433746	-0.000182062	0.008228877	0.001401058	-0.008014128
remainder: AIJ-WGM-ME	0.002065247	0.005019648	-0.005661833	0.002772673	-0.004195734
remainder: AIJ-WGM-LLS	0.001804774	0.005220089	-0.005027708	0.004003439	-0.006000594
remainder: AIP-NWGM-ME	0.001700367	0.004019311	-0.005995150	0.003359983	-0.003084512
remainder: AIP-NWGM-LLS	0.001804774	0.005220089	-0.005027708	0.004003439	-0.006000594
remainder: AIP-NWGM-RGM	0.001804774	0.005220089	-0.005027708	0.004003439	-0.006000594

b) Distances experiment

w	0.24313450	0.54210640	0.13481816	0.02807432	0.05186662
Effect of W	0.02535673	-0.09595689	0.03184458	0.01723847	0.02151710
Effect of noise	-0.005262513	0.005719404	-0.000790453	-0.000296605	0.000630167
remainder: AIJ-WGM-ME	-0.006990169	-0.001245584	-0.000238222	0.005242587	0.003231388
remainder: AIJ-WGM-LLS	-0.005241967	-0.003483988	-0.00005032866	0.005131237	0.003645046
remainder: AIP-NWGM-ME	-0.004636354	-0.001970196	-0.001289040	0.005087379	0.002808211
remainder: AIP-NWGM-LLS	-0.005241967	-0.003483988	-0.00005032866	0.005131237	0.003645046
remainder: AIP-NWGM-RGM	-0.005241967	-0.003483988	-0.00005032866	0.005131237	0.003645046

c) Rainfall experiment

w	0.03335836	0.21287038	0.54011748	0.15225254	0.06140124
Effect of W	0.01955399	0.02501103	-0.09549464	0.02746318	0.02346643
Effect of noise	-0.000622832	-0.003889342	0.006172079	-0.001863904	0.000203999
remainder: AIJ-WGM-ME	0.005355531	-0.004760431	-0.001881604	-0.001659171	0.002945675
remainder: AIJ-WGM-LLS	0.005277020	-0.004003153	-0.002910622	-0.001619700	0.003256455
remainder: AIP-NWGM-ME	0.005299617	-0.004324307	-0.005781009	0.001357388	0.003448311
remainder: AIP-NWGM-LLS	0.005277020	-0.004003153	-0.002910622	-0.001619700	0.003256455
remainder: AIP-NWGM-RGM	0.005277020	-0.004003153	-0.002910622	-0.001619700	0.003256455

formulas are replaced by their estimates (see below on how to obtain the estimates). Here vectors $\mathbf{w}$'s in the various experiments are computed according to the methods of class 1 (see Table 1). The "effect of W" is given by $\frac{1}{n} \cdot \mathbf{W} \cdot \left(\frac{1}{K} \sum_{k=1}^{K} \cdot \phi_3^{(k)} \cdot \left[\left(\overline{\ln} \mathbf{w}^{(k)} \right) \cdot \mathbf{u}_n^{\mathsf{T}} - \mathbf{u}_n \cdot \left(\overline{\ln} \mathbf{w}^{(k)} \right)^{\mathsf{T}} \right]^{\odot 3} \right) \cdot \mathbf{u}_n$ and the "effect of noise" is given by $\frac{1}{n} \mathbf{W} \cdot \left(\frac{1}{K} \sum_{k=1}^{K} \mathbf{N}^{(k)} \right) \cdot \mathbf{u}_n$. The "remainder" is $\mathbf{u} - \mathbf{w} - \mathrm{d}\mathbf{u}$ and may be different according to the various aggregation methods of the class. It is evident that both the effect of the noise and of the remainders computed according to any method are much smaller than the effect of the deterministic distortions due to W.

4.3 Corrections of deterministic distortions

The results of the previous subsection shows that it is important to correct for systematic distortions in group aggregation. A direct way to make the corrections is provided by the computations underlying Table 3, which decompose the various terms affecting $\mathbf{u}$. The method is based on individual-specific estimates of parameters $\phi_3^{(k)}$. Obviously, the closer are the individual estimates to the true parameters $\phi_3^{(1)}, ..., \phi_3^{(K)}$, the better are the corrections.

A method to obtain consistent estimates, denoted with $\overline{\phi}_3^{(1)}, ..., \overline{\phi}_3^{(K)}$, from *any* judgment matrix $\mathbf{A}^{(k)}$ is developed in Bernasconi et al. (2010). The method is in fact a generalization of the LLS approach to obtain the priority vector $\mathbf{w}^{(k)}$, according to the analysis of Genest and Rivest (1994). The procedure has been used in the three experiments eliciting probabilities in games of chances, distances between cities, and rainfalls in European cities. Estimates have found values of $\overline{\phi}_3^{(k)}$ between -1 and 0 for the large majority of the 69 subjects participating in the study, with medians of the individual estimates very close to -0.03 in all the three experiments. The findings are consistent with the tendency of people to overestimate low ratios and underestimate high ratios, as predicted by cognitive arguments.

Subject-specific estimates of $\overline{\phi}_3^{(k)}$'s are useful to rank the extent of systematic biases of different individuals. Indeed, if one believes that cognitive biases in the perception of numbers are related to a more general attitude of agents to produce inconsistent reasonings,[14] subject-specific estimates $\overline{\phi}_3^{(k)}$'s might also be useful to determine the weights of the subjects in the group. For example, greater weights $\beta_1, \ldots, \beta_K$ could be assigned to members with lower $\overline{\phi}_3^{(k)}$'s because considered more generally capable to provide coherent judgments.

In some cases, it may be too costly to obtain a full set of subject-specific estimates of $\overline{\phi}_3^{(k)}$. A possibility in these cases is to use a "representative" model of $\overline{\phi}_3$ to correct all individual judgment matrices $\mathbf{A}^{(k)}$'s.[15] Representative models are computationally simpler and statistically less demanding to estimate than individual-specific models. Approaches based on "representative" agents are adopted in many theories of decision making. For example, in the contexts of theories for decision under risk there are many studies which provide estimates to be used generally for the probability subjective weighting function. This is precisely possible because the individual probability transformation function is stable across contexts and similar between individuals.

A "representative" model based on the median values of the subject-specific estimates of $\overline{\phi}_3^{(k)}$ to obtain corrected individual $\overline{w}_j^{(k)}$ is discussed in Bernasconi et al. (2010). The results of that paper shows that corrections based on individual-specific estimates are similar to those obtained by the median representative model. The analysis conducted here indicates that corrections of the median model may be even more performing in group decision making: when averaging over individuals, the use of a representative agent may be able to remove (almost) all of the effect of systematic distortions, while the law of large numbers remove the effect of the noise.

[14] For example, in the classical expected utility theory of choice under risk, outcome-probabilities correspond to decision weights and agents who subjectively transform probabilities are often considered exposed to a form of irrational behaviour (on this issue see, e.g., discussion in Neilson 2003).

[15] For example, a simple model that can be used to correct the entries of a judgment matrix can be obtained replacing $\alpha_{ij}^{(k)}$ with $\overline{\alpha}_{ij}^{(k)} = \exp\left\{\ln \alpha_{ij}^{(k)} - \overline{\phi}_3^{(k)} \cdot \left[\ln \alpha_{ij}^{(k)}\right]^3\right\}$, where $\overline{\phi}_3^{(k)}$ is an estimate of $\phi_3^{(k)}$. Substituting the expression in equations (5) and (6) (and using the fact that the errors are asymptotically negligible), one obtains:

$$\ln \overline{\alpha}_{ij}^{(k)} \simeq \ln\left(w_i^{(k)}/w_j^{(k)}\right) + \left(\phi_3^{(k)} - \overline{\phi}_3^{(k)}\right) \cdot \left[\ln\left(w_i^{(k)}/w_j^{(k)}\right)\right]^3.$$

Clearly, the expression confirms that the closer are the estimates $\overline{\phi}_3^{(k)}$'s to the true $\phi_3^{(k)}$'s, the better are the corrections. Using a "representative" estimate $\overline{\phi}_3$ for all the members of a group can in some cases be a simplifying procedure which reduces the cost of estimating subject-specific distortions.

5 Conclusion

Here we have developed a framework to compare the *theoretical* priority vector of interest $\mathbf{w}$ with the *empirical* priority vector $\mathbf{u}$, which can be obtained by various combinations of prioritization methods and aggregation rules in the AHP for group decision making. The analysis based on the first-order differential of $\mathbf{u}$ around $\mathbf{w}$ has shown how to identify five main classes of combinations of procedures which predict very similar values for vectors $\mathbf{u}$'s. The predictions are confirmed in experimental tests conducted with real human subjects in the domain of tangibles.

Our results are useful for implementation of group aggregation. As discussed in Section 2, a large literature has debated on different aspects of the aggregating techniques which can be employed in AHP group decision making. The theoretical analysis has shown, and the empirical applications have confirmed, that some aspects of the dispute may be less relevant than previously thought. On the one hand, the choice of the levels at which to conduct the aggregation, namely whether of individual judgments (AIJ) or individual priorities (AIP), has little relevance for the empirical results of the aggregation when the weighted geometric mean method is used as averaging procedure. In this respect, there seems to be little reason to dispute whether the group should be considered as a 'new agent' or a 'collection of independent individuals' as sometimes argued in connection with the choice of the level of aggregation.[16] On the other hand, the aggregated empirical results seem to be more sensitive to the averaging procedures chosen in the aggregation and the prioritization methods. Regarding the former issues, we have shown that several aggregating techniques discussed in the literature yield an aggregated priority vector that is not normalized. This implies a degree of indeterminacy in the vectors obtained which rules out the procedures. We have classified various aggregating methods which should not be considered for this reason. As concerns the methods of prioritization, our analysis has shown that, in practice, there are effectively no differences between the classical maximum eigenvalue (ME) and the logarithmic

[16] This issue can be more relevant in the choice of the weights β_k's.

least squares (LLS) when applied in the various aggregating procedures used in the AHP. Quite interestingly, we have shown that there is more difference between the previous two methods and the row geometric mean (RGM), despite the latter is often considered in the literature as similar to LLS.

The analysis has also shown how to decompose the first-order difference $d\mathbf{u} \simeq \mathbf{u} - \mathbf{w}$ in the components due to random errors and the components caused by systematic cognitive distortions in the perception of ratios consistent with the so-called separable representations of mathematical psychology. The importance to distinguish the two components has been previously documented, for individual decision making, both theoretically and empirically. In particular, deterministic distortions tend to be larger than those due to stochastic errors. The results are confirmed here for group decision making. Moreover, in group decision making, the effect of the deterministic distortions may become even more important because, while the stochastic errors tend to be averaged out with the size of the group, the deterministic distortions do not necessarily follow the same law. This implies that it is even more important to correct for deterministic distortions in group decision making than in individual decision making. The correction procedure can be based either on individual-specific estimates of the parameters of the distortions (parameters $\overline{\phi}_3^{(k)}$'s in Section 4), or on representative agent estimates, if the computational cost of estimation are too high.

The proposed analysis has focussed on the empirical properties of group preference aggregation methods in the AHP. It complements more standard approaches which look at consistency measures and judge the quality of $\mathbf{u}$ on the basis of those measures. Consistency is a very important requirement in order to use AHP techniques properly, both in individual and in group decision making; but, by itself, consistency doesn't say anything on the quality of $\mathbf{u}$ to represent the true priority vector $\mathbf{w}$ of interest (Saaty 2003). Future research must better integrate consistency considerations with the evaluation and decomposition of the (first-order approximation) difference $d\mathbf{u} \simeq \mathbf{u} - \mathbf{w}$ into a unified framework.

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A Proofs

In the proofs we introduce the notation $\alpha_{0,ij}^{(k)} = {}^{w_i^{(k)}}/_{w_j^{(k)}}$. $\mathbf{A}_0^{(k)} = \left[\alpha_{0,ij}^{(k)}\right]$ is the matrix filled with the elements $\alpha_{0,ij}^{(k)}$. Moreover, in general we will prove the results for the maximal eigenvector (ME) method and we will extend these results to the logarithmic least squares (LLS) method using the equivalence up to the first order proved in Genest and Rivest (1994, equation (6)).

A.1 Proofs - AIJ-WGM-ME/LLS

The generic element of matrix $\mathbf{A}$ is:

$$\alpha_{ij} \;=\; \prod_{k=1}^{K}\left(\alpha_{ij}^{(k)}\right)^{\beta_k} = \prod_{k=1}^{K}\left(\alpha_{0,ij}^{(k)}\right)^{\beta_k} \cdot e^{\sum_{k=1}^{K}\beta_k\cdot \mathrm{d}\varepsilon_{ij}^{(k)}}. \tag{12}$$

Therefore, it is possible to reason as if $\alpha_{0,ij} = \prod_{k=1}^{K}\left(\alpha_{0,ij}^{(k)}\right)^{\beta_k}$ and $\mathrm{d}\varepsilon_{ij} = \sum_{k=1}^{K}\beta_k \cdot \mathrm{d}\varepsilon_{ij}^{(k)}$. It is clear that, if all the matrices $\mathbf{A}_0^{(k)} = \left[\alpha_{0,ij}^{(k)}\right]$ for $k = 1,\ldots,K$ are consistent, then also the matrix $\mathbf{A}_0 = [\alpha_{0,ij}]$ is consistent. Indeed $\alpha_{0,ij}\alpha_{0,j\ell}\alpha_{0,\ell i} = \prod_{k=1}^{K}\left(\alpha_{0,ij}^{(k)}\alpha_{0,j\ell}^{(k)}\alpha_{0,\ell i}^{(k)}\right)^{\beta_k} = 1$. Moreover the Perron right eigenvector of the matrix $\mathbf{A}_0$ is proportional to $\bigodot_{k=1}^{K}\left(\mathbf{w}^{(k)}\right)^{\odot\beta_k}$ and is determined in such a way that its elements sum up to 1: therefore, it is given by $\mathbf{w} \triangleq \bigodot_{k=1}^{K}\left(\mathbf{w}^{(k)}\right)^{\odot\beta_k} / \left[\mathbf{u}_n^{\mathsf{T}} \cdot \bigodot_{k=1}^{K}\left(\mathbf{w}^{(k)}\right)^{\odot\beta_k}\right]$. Using results in Section 4 in Bernasconi, Choirat and Seri (2011), up to the first order, the maximal eigenvector is given by $\mathbf{u} \simeq \mathbf{w} + \frac{1}{n}\sum_{k=1}^{K} \beta_k \cdot \mathbf{W} \cdot \mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n$.

A.2 Proofs - AIJ-WGM-RGM

The generic element of matrix $\mathbf{A}$ is given in (12). The $i-$th element of $\mathbf{u}$ obtained through the RGM is:

$$\begin{aligned} u_i &= \left(\prod_{j=1}^{n} \prod_{k=1}^{K} \left(\alpha_{ij}^{(k)}\right)^{\beta_k} \right)^{1/n} = \left(\prod_{k=1}^{K} \frac{w_i^{(k)}}{\left(\prod_{j=1}^{n} w_j^{(k)}\right)^{1/n}} \right)^{\beta_k} \cdot e^{\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{K}\beta_k \cdot \mathrm{d}\varepsilon_{ij}^{(k)}} \\ &= \left(\prod_{k=1}^{K} w_i^{(k)} \right)^{\beta_k} \cdot e^{\frac{1}{n}\sum_{k=1}^{K}\beta_k \cdot \sum_{j=1}^{n} \mathrm{d}\varepsilon_{ij}^{(k)}}. \end{aligned}$$

The matrix formulation is obtained setting $\mathbf{w} = \bigodot_{k=1}^{K} \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k}$ and remarking that

$$\begin{aligned} \mathbf{u} &= \mathbf{w} \odot \overline{\exp}\left\{ \frac{1}{n}\sum_{k=1}^{K} \beta_k \cdot \mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n \right\} \simeq \mathbf{w} \odot \left(\mathbf{u}_n + \frac{1}{n}\sum_{k=1}^{K} \beta_k \cdot \mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n \right) \\ &= \mathbf{w} + \frac{1}{n}\sum_{k=1}^{K} \beta_k \cdot \mathrm{diag}\left(\mathbf{w}\right) \cdot \mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n. \end{aligned}$$

A.3 Proofs - AIP-UWGM-ME/LLS

In this case, the method is based on the aggregation of the individual vectors as $\mathbf{u} = \bigodot_{k=1}^{K} \left(\mathbf{u}^{(k)}\right)^{\odot \beta_k}$, i.e. without normalization. The vector of priorities is given by:

$$\begin{aligned} \mathbf{u} &\simeq \bigodot_{k=1}^{K} \left(\mathbf{w}^{(k)} + \mathrm{d}\mathbf{u}^{(k)}\right)^{\odot \beta_k} = \bigodot_{k=1}^{K} \left[\mathbf{w}^{(k)} \odot \left(\mathbf{u}_n + \mathrm{diag}\left(\overline{\mathbf{w}^{(k)}}\right) \cdot \mathrm{d}\mathbf{u}^{(k)}\right)\right]^{\odot \beta_k} \\ &= \bigodot_{k=1}^{K} \left(\mathbf{w}^{(k)}\right)^{\odot \beta_k} \odot \left(\mathbf{u}_n + \beta_k \cdot \mathrm{diag}\left(\overline{\mathbf{w}^{(k)}}\right) \cdot \mathrm{d}\mathbf{u}^{(k)}\right) \\ &\simeq \mathbf{w} + \mathbf{w} \odot \sum_{k=1}^{K} \beta_k \cdot \mathrm{diag}\left(\overline{\mathbf{w}^{(k)}}\right) \cdot \mathrm{d}\mathbf{u}^{(k)} = \mathbf{w} + \sum_{k=1}^{K} \beta_k \cdot \mathrm{diag}\left(\mathbf{w}\right) \mathrm{diag}\left(\overline{\mathbf{w}^{(k)}}\right) \cdot \mathrm{d}\mathbf{u}^{(k)} \end{aligned}$$

where $\mathbf{w} \triangleq \bigodot_{k=1}^{K}\left(\mathbf{w}^{(k)}\right)^{\odot\beta_k}$. Replacing the formula for $\mathrm{d}\mathbf{u}^{(k)}$ we get the desired result:

$$
\begin{aligned}
\mathbf{u} &\simeq \mathbf{w} + \frac{1}{n}\sum_{k=1}^{K}\beta_k \cdot \left[\operatorname{diag}(\mathbf{w})\operatorname{diag}\left(\overline{\mathbf{w}^{(k)}}\right)\mathbf{W}^{(k)}\right] \cdot \mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n \\
&= \mathbf{w} + \frac{1}{n}\sum_{k=1}^{K}\beta_k \cdot \left(\operatorname{diag}(\mathbf{w}) - \mathbf{w}\cdot\mathbf{w}^{(k),\mathsf{T}}\right) \cdot \mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n.
\end{aligned}
$$

A.4 Proofs - AIP-UWGM-RGM

See Escobar, Aguarón and Moreno-Jimenéz (2004) for a proof of the equality of AIP-UWGM-RGM and AIJ-WGM-RGM.

A.5 Proofs - AIP-NWGM-ME/LLS

In this case $\mathbf{u} = \frac{\bigodot_{k=1}^{K}\left(\mathbf{u}^{(k)}\right)^{\odot\beta_k}}{\mathbf{u}_n^{\mathsf{T}}\bigodot_{k=1}^{K}\left(\mathbf{u}^{(k)}\right)^{\odot\beta_k}}$. We start from AIP-UWGM-ME and we normalize it; remark that at present $\mathbf{w} = \bigodot_{k=1}^{K}\left(\mathbf{w}^{(k)}\right)^{\odot\beta_k}$ where $\mathbf{u}_n^{\mathsf{T}}\mathbf{w}^{(k)} = 1$. The vector of priorities is given by:

$$
\begin{aligned}
\mathbf{u} &\simeq \frac{\mathbf{w} + \frac{1}{n}\sum_{k=1}^{K}\beta_k \cdot \left(\operatorname{diag}(\mathbf{w}) - \mathbf{w}\cdot\mathbf{w}^{(k),\mathsf{T}}\right)\cdot \mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n}{\mathbf{u}_n^{\mathsf{T}}\mathbf{w}\cdot\left[1 + \frac{\frac{1}{n}\sum_{k=1}^{K}\beta_k\cdot\mathbf{u}_n^{\mathsf{T}}\left(\operatorname{diag}(\mathbf{w})-\mathbf{w}\cdot\mathbf{w}^{(k),\mathsf{T}}\right)\cdot\mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n}{\mathbf{u}_n^{\mathsf{T}}\mathbf{w}}\right]} \\
&\simeq \left(\frac{\mathbf{w}}{\mathbf{u}_n^{\mathsf{T}}\mathbf{w}} + \frac{\frac{1}{n}\sum_{k=1}^{K}\beta_k \cdot \left(\operatorname{diag}(\mathbf{w}) - \mathbf{w}\cdot\mathbf{w}^{(k),\mathsf{T}}\right)\cdot \mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n}{\mathbf{u}_n^{\mathsf{T}}\mathbf{w}}\right)\cdot \\
&\quad \cdot\left(1 - \frac{\frac{1}{n}\sum_{k=1}^{K}\beta_k \cdot \mathbf{u}_n^{\mathsf{T}}\left(\operatorname{diag}(\mathbf{w}) - \mathbf{w}\cdot\mathbf{w}^{(k),\mathsf{T}}\right)\cdot \mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n}{\mathbf{u}_n^{\mathsf{T}}\mathbf{w}}\right) \\
&\simeq \frac{\mathbf{w}}{\mathbf{u}_n^{\mathsf{T}}\mathbf{w}} + \left(\mathbf{I}_n - \frac{\mathbf{w}\mathbf{u}_n^{\mathsf{T}}}{\mathbf{u}_n^{\mathsf{T}}\mathbf{w}}\right)\cdot\frac{\frac{1}{n}\sum_{k=1}^{K}\beta_k \cdot \left(\operatorname{diag}(\mathbf{w}) - \mathbf{w}\cdot\mathbf{w}^{(k),\mathsf{T}}\right)\cdot \mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n}{\mathbf{u}_n^{\mathsf{T}}\mathbf{w}} \\
&= \frac{\mathbf{w}}{\mathbf{u}_n^{\mathsf{T}}\mathbf{w}} + \frac{1}{n}\sum_{k=1}^{K}\beta_k\cdot\left(\frac{\operatorname{diag}(\mathbf{w})}{\mathbf{u}_n^{\mathsf{T}}\mathbf{w}} - \frac{\mathbf{w}\mathbf{w}^{\mathsf{T}}}{\left(\mathbf{u}_n^{\mathsf{T}}\mathbf{w}\right)^2}\right)\cdot\mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n.
\end{aligned}
$$

If we redefine $\mathbf{w} \triangleq \bigodot_{k=1}^{K}\left(\mathbf{w}^{(k)}\right)^{\odot\beta_k} / \left[\mathbf{u}_n^{\mathsf{T}}\cdot\bigodot_{k=1}^{K}\left(\mathbf{w}^{(k)}\right)^{\odot\beta_k}\right]$, we have $\mathbf{u} \simeq \mathbf{w} + \frac{1}{n}\sum_{k=1}^{K}\beta_k\cdot\mathbf{W}\cdot\mathrm{d}\mathbf{E}^{(k)}\mathbf{u}_n$.

A.6 Proofs - AIP-NWGM-RGM

Starting from AIP-UWGM-RGM (or equivalently AIJ-WGM-RGM) and normalizing it, we get that the $i-$th weight is:

$$\frac{\left(\prod_{j=1}^{n}\prod_{k=1}^{K}\left(\alpha_{ij}^{(k)}\right)^{\beta_k}\right)^{1/n}}{\sum_{h=1}^{n}\left(\prod_{j=1}^{n}\prod_{k=1}^{K}\left(\alpha_{hj}^{(k)}\right)^{\beta_k}\right)^{1/n}}.$$

On the other hand, for AIP-UWGM-LLS, the $i-$th element is $\prod_{k=1}^{K}\left(\frac{\left(\prod_{j=1}^{n}\alpha_{ij}^{(k)}\right)^{1/n}}{\sum_{h=1}^{n}\left(\prod_{j=1}^{n}\alpha_{hj}^{(k)}\right)^{1/n}}\right)^{\beta_k}$. From this, the element for AIP-NWGM-LLS is easily seen to be:

$$\frac{\prod_{k=1}^{K}\left(\frac{\left(\prod_{j=1}^{n}\alpha_{ij}^{(k)}\right)^{1/n}}{\sum_{h=1}^{n}\left(\prod_{j=1}^{n}\alpha_{hj}^{(k)}\right)^{1/n}}\right)^{\beta_k}}{\sum_{\ell=1}^{n}\prod_{k=1}^{K}\left(\frac{\left(\prod_{j=1}^{n}\alpha_{\ell j}^{(k)}\right)^{1/n}}{\sum_{h=1}^{n}\left(\prod_{j=1}^{n}\alpha_{hj}^{(k)}\right)^{1/n}}\right)^{\beta_k}} = \frac{\prod_{k=1}^{K}\left(\left(\prod_{j=1}^{n}\alpha_{ij}^{(k)}\right)^{1/n}\right)^{\beta_k}}{\sum_{\ell=1}^{n}\prod_{k=1}^{K}\left(\left(\prod_{j=1}^{n}\alpha_{ij}^{(k)}\right)^{1/n}\right)^{\beta_k}}$$

that coincides with the above expression for AIP-NWGM-RGM.

A.7 Proofs - AIP-WAM-ME/LLS

Set $\mathbf{u} = \sum_{k=1}^{K}\beta_k\mathbf{u}^{(k)}$ and $\mathbf{w} = \sum_{k=1}^{K}\beta_k\mathbf{w}^{(k)}$. The vector of priorities is given by:

$$\mathbf{u} \simeq \sum_{k=1}^{K}\beta_k\left(\mathbf{w}^{(k)} + \mathrm{d}\mathbf{u}^{(k)}\right) = \mathbf{w} + \sum_{k=1}^{K}\beta_k\mathrm{d}\mathbf{u}^{(k)}$$

from which the result easily follows.

A.8 Proofs - AIP-WAM-RGM

The generic element of the vector is given by

$$\begin{aligned} u_i &= \sum_{k=1}^{K} \beta_k u_i^{(k)} = \sum_{k=1}^{K} \beta_k \left(\prod_{j=1}^{n} \alpha_{ij}^{(k)} \right)^{1/n} \\ &= \sum_{k=1}^{K} \beta_k \left(\prod_{j=1}^{n} \alpha_{0,ij}^{(k)} \cdot e^{\mathrm{d}\varepsilon_{ij}^{(k)}} \right)^{1/n} = \sum_{k=1}^{K} \beta_k \frac{w_i^{(k)}}{\left(\prod_{j=1}^{n} w_j^{(k)}\right)^{1/n}} \cdot e^{\frac{1}{n}\sum_{j=1}^{n} \mathrm{d}\varepsilon_{ij}^{(k)}}, \end{aligned}$$

from which:

$$\mathbf{u} = \sum_{k=1}^{K} \beta_k \frac{\mathbf{w}^{(k)}}{\left(\prod_{j=1}^{n} w_j^{(k)}\right)^{1/n}} \odot \overline{\exp} \left\{ \frac{1}{n} \mathrm{d}\mathbf{E}^{(k)} \mathbf{u}_n \right\} \simeq \mathbf{w} + \frac{1}{n} \sum_{k=1}^{K} \beta_k \cdot \mathrm{diag}\left(\mathbf{w}^{(k)}\right) \cdot \mathrm{d}\mathbf{E}^{(k)} \mathbf{u}_n$$

where $\mathbf{w} = \sum_{k=1}^{K} \beta_k \mathbf{w}^{(k)}$.