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equilibrium with multiple
deviators**

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Abstract

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Keywords

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Non-cooperative equilibrium with multiple deviators *

Dmitry V. Levando[†]

July 5, 2016

Abstract

The paper defines a non-cooperative simultaneous game with a number of potential deviators being a parameter of the game. A definition of the game embeds a mechanism design. The game has an equilibrium in mixed strategies. The equilibrium encompasses intra and inter group externalities and an individual payoff allocation that make it different from a strong Nash, coalition-proof equilibrium and some other equilibrium concepts. We offer a non-cooperative stability criterion to describe a robustness of an equilibrium strategy profile to an increase in a number of deviators. The criterion may serve as a way to measure self-enforcement or a trust for the equilibrium in terms of a number of potential deviators and the mechanism design.

Keywords: Noncooperative Games

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1 Introduction

This paper studies the question coming from the paper "Equilibrium points in n-person games" by John Nash (1950). His paper established equilibrium condition for an assumption that every player considers himself to be a unique deviator. The natural questions are: if a player assumes that beside her/him there are also some other deviators, then which game is played, and what is an equilibrium concept for this game?

Consider an example with three players $N = \{1, 2, 3\}$. Strong Nash equilibrium of Aumann (1959) is based on a deviation of a coalition of any size, *without specification of internal coalition structure*. Are the coalition structures of all three players: $\{\{1\}, \{2\}, \{3\}\}$, $\{\{12\}, \{3\}\}$, $\{\{13\}, \{2\}\}$, $\{\{1\}, \{23\}\}$ are different or not for the strong Nash equilibrium. We can not have an unambiguous answer.¹

Traditional way to study the problem is through cooperative game theory. However Maskin (2011) pointed out: "I believe that there are three (related) reasons for the historical shift from cooperative to noncooperative theory: (a) most cooperative theory ignores externalities, the possibility that a coalition can be affected by the actions of those not in the coalition; (b) it assumes that a Pareto efficient outcome will be reached; and (c) it supposes that the grand coalition (the coalition of all players) will form."

Further: "These features of cooperative theory are problematic because

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¹This example also does not fit k -equilibrium approach as deviating sets have an ambiguous structure.

most applications of game theory to economics involve settings in which externalities are important, Pareto inefficiency arises, and the grand coalition does not form. ”

Consider herding behavior in financial markets. Herding occurs an individual action of many. But does herding takes place as a deviation of a player, a group of many agents or as a parallel deviation of many independent agents and/or smaller groups, totally of a size of one big group? How big should be the herding group(s) to become contagious for the whole market? What means an equilibrium market liquidity in terms of deviating sets of players to support a mixed strategies equilibrium? The same issue appears in studying international (im)migration, in labor market trade-union conflicts, etc. ²

The suggested approach changes the role of a central planner in comparison to cooperative approach of Nash. Nash suggested that a pre-play agreement can be enforced by a central planner. This paper offers another role for the central planner - to offer a mechanism design, what includes a maximum number of deviators, a list of partitions, a list of non-contradictory

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²The question of studying multiple deviators is far from a purely theoretical one. Take European currency system, which was designed as an equilibrium system for several heterogeneous interactive economies. A membership requires meeting some requirements, debt/GNP ratio, for simplicity. If there is only one deviator, then all other members are able to make a decision to implement a punishing regulation, and the initially desirable equilibrium presumable can be reinstalled. However if there are many deviators, do the deviating countries make a decision to penalize themselves? So comes the question - what is a maximum number of deviators, under which European Currency union remains stable? One may expect the same problem in international emission agreements.³ The Shapley value approach is inappropriate here as besides other limitations, it does not deal with deviations after a coalition is formed.

rules for group formation ⁴ and an enforcement. This approach is simpler in comparison to other more complicated and not yet developed alternatives for studying deviations of any number of players - an uncertainty in a number of potential deviators or an evolutionary selection of group formation rules.

Studying a deviation of more than one player is impossible without structuring a deviating set of players into subsets. The suggested approach resolves this problem by embedding a mechanism design into a definition of a simultaneous game.

Let K be a fixed number of deviators. We construct a non-cooperative game $\Gamma(K)$ with a modified strategy set and a mechanism design. Different values of K induce different non-cooperative equilibrium for every game $\Gamma(K)$. Every $\Gamma(K)$ includes a set of eligible partitions $\mathcal{P}(K)$ and family of group formation rules $\mathcal{R}(K)$ for every partition. Every partition is a standard non-cooperative game of John Nash. These games are united by mixed strategies within the same framework of the game $\Gamma(K)$. A player constructs his strategies over all such games or partitions. The equilibrium in mixed strategies exists and depends on the number K . The game $\Gamma(K)$ has reacher descriptive power than a deviation of a coalition of any size in a strong Nash equilibrium or in a core approach in cooperative game theory.

A suggested non-cooperative stability criterion is constructed as an answer for the question - does a change in a number of potential deviators changes an equilibrium strategy profile given mechanism design? This ques-

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⁴or coalition structures in terminology of Aumann, Dreze (1974)

tion is another side of opportunistic behavior in groups - what a player knows about a number of potential deviators from an equilibrium. The introduced K -Nash equilibrium does not have an uncertainty about K , although it is just a step further.

The paper has the following structure - Section 2 introduces a basic notation, Section 3 presents a game and its properties, Section 4 suggests a stability criterion and Section 5 discusses the results.

2 Basic notation

Consider a non-cooperative game (Nash, 1950) of N players, $i \in N$, N is finite. Every $i \in N$ is assigned a strategy set $S_i = \{s_i\}$, a closed convex subset from $\mathbb{R}$, and a payoff $U_i: \times_{i \in N} S_i \mapsto \mathbb{R}$, real valued Lebegue integrable function over $S = \times_{i \in N} S_i$. We assume that for all $i \in N$ there is a finite $M < \infty$ such that $\int_{s \in S} U_i(s) ds \leq M$. This guarantees compactness of a kernel of an expected utility integral operator for every player further. $\Gamma = \langle N, (S_i)_{i \in N}, (U_i)_{i \in N} \rangle$ is a standard non-cooperative game of Nash.

A mixed strategy σ_i (a probability measure S_i) of i is an element of an (infinitely-dimensional) probability space $\Delta(S_i)$ over a measurable set S_i . This set includes also probability measures concentrated in single points.

Expected utility of i is defined as a mapping

$$EU_i(\sigma_i, \sigma_{-i}): \Delta(S_i) \times \left(\times_{j \in N \setminus \{i\}} \Delta(S_j) \right) \mapsto \mathbb{R}$$

or in a functional form

$$EU_i(\sigma_i, \sigma_{-i}) = \int_{S=(S_j)_{j \in N}} U_i(s_i, s_{-i}) d\sigma_i d\sigma_{-i}.$$

The notation $\Delta(S_i) \times \left(\times_{j \in N \setminus \{i\}} \Delta(S_j) \right)$ emphasizes that a player makes his choice of a mixed strategy given mixed strategy profile of all other players.

It is a well-defined bounded linear functional over $\Delta(S_i)$, what implies compactness of expected utility, and uniform continuity (Ljusternik, Sobolev, (1982), also Lax (2002)). Notations $-i = (j)_{j \in N \setminus \{i\}}$, $s_{-i} = (s_j)_{j \in N \setminus \{i\}}$, $\sigma_{-i} = (\sigma_j)_{j \in N \setminus \{i\}}$ are standard. The set EU_i is bounded, compact, complete and continuous.

Definition 1. *Nash equilibrium is a profile of mixed strategies $(\sigma_i^*)_{i \in N}$ (probability measures) such that for every $\sigma_i \neq \sigma_i^*$ there is $EU(\sigma_i^*, \sigma_{-i}^*) \geq EU(\sigma_i, \sigma_{-i}^*)$, $\forall i \in N$.*

The space $\Delta(S_i)$ is a bounded, complete, uniformly bounded and uniformly convex set (Balakrishnan, 1971). This set includes also probability measures concentrated in single points. The same properties of mixed strategies will reappear in the main model.

Existence of an equilibrium follows from Brower or Schauder fixed point theorems, John Nash used Kakutani theorem.

3 Arbitrary number of deviators

3.1 Partitions and partition dependent strategies

Let an integer K be a maximum number of potential deviators, $1 \leq K \leq N$. Further every fixed value of K , $K \in \{1, \dots, K\}$ means that a size of any deviating group in any partition *does not exceed* K . K is the same for all players, and is common knowledge.

Let $\mathcal{P}(K) = \{P^\beta : \beta\}$ be a family of all partitions of players from N with a partition index β . A size of any group g^β from a partition P^β can not accede the exogenously fixed parameter K . Let $G(K) = \#\mathcal{P}(K)$ is the size of the family $\mathcal{P}(K)$, i.e. $G(K)$ is a number of partitions in the game.⁵ The family $\mathcal{P}(K)$ (or coalition structures in terminology of Aumann and Dreze, 1974) is offered by a central planner.

For example, a set of players $N = \{1, 2, 3\}$, and let $K = 2$. Then possible partitions can be $P^{\beta=1} = \{\{1\}, \{2\}, \{3\}\}$, $P^{\beta=2} = \{\{1, 2\}, \{3\}\}$, $P^{\beta=3} = \{\{1, 3\}, \{2\}\}$, $P^{\beta=4} = \{\{1\}, \{2, 3\}\}$. The partition $P^{\beta=5} = \{1, 2, 3\}$ can not be reached as a size of the only group, a grand coalition, is greater than $K = 2$. These partition examples are used further in developing of the example.

Parameter K defines a maximum number of deviators: size of a group for any i is no grater that K . A bigger group can not be formed for the fixed K . For example, if $K = 1$ there is a final unique partition $\{i\} \cup \{N \setminus \{i\}\}$. If

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⁵Examples can be sizes of firms or social restrictions on a maximum group size.

$K = 2$, then player i can imagine two types of partitions after a deviation: $\{i\} \cup \{j\} \cup \{N \setminus \{i, j\} : j \neq i\}$ and $\{i, j\} \cup \{N \setminus \{i, j\} : j \neq i\}$.

Let β_i be an index for a partitions chosen by i , what unambiguously identifies a **chosen** group for i in partition P^{β_i} .⁶ Realization of a partition depends on strategies of all players. We leave for future research partition-specific externalities, when a final partition depends on a partition choice of *all* players.

The difference between indices β and β_i is the following: β_i is an index of a partition, which player i chooses, and β is an index of a realized partition P^β from $\mathcal{P}$, after all strategies of all players are announced. In the general case $\beta \neq \beta_i$ as a final partition depends on strategies of all players, and does not need to match all individual partition choices.

For every i a set of strategies for a desirable partition β_i is S_{β_i} , a subset from $\mathbb{R}$. A player i chooses a partition β_i and a corresponding action s_{β_i} from S_{β_i} . As an example for every i and β_i we take $S_{\beta_i} = [0, 1]_{\beta_i}$, where $[0, 1]$ is a domain of actions of i , who chooses a partition β_i .

Let $S_i(K) = \left\{ s_{\beta_i} \in S_{\beta_i} : \beta_i = 1, \dots, G(K) \right\}$ be a set of strategies of i with a general element s_{β_i} . Let $S(K) = \times_{i \in N} S_i(K)$ be a set of strategies of all players for a fixed K .

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⁶with the assumptions of participation in only one group for every i .

3.1.1 An example of strategy sets for different K

A set of strategies of i for $K = 1$ is $S_i(K = 1) = [0, 1]_{\beta_i=1}$ as for $K = 1$ any partition with group size more than 1 can not be formed. A set of strategies of i for $K = 2$ is $S_i(2) = \{[0, 1]_{\beta_i=1} \sqcup [0, 1]_{\beta_i=2} \sqcup [0, 1]_{\beta_i=3} \sqcup [0, 1]_{\beta_i=4}\}$. A set of strategies of i for $K = 3$ is

$$S_i(3) = \{[0, 1]_{\beta_i=1} \sqcup [0, 1]_{\beta_i=2} \sqcup [0, 1]_{\beta_i=3} \sqcup [0, 1]_{\beta_i=4} \sqcup [0, 1]_{\beta_i=5}\},$$

where $\sqcup$ is a disjoint union symbol.

A set of all strategies of all players for $K = 1$ is

$$S(K = 1) = [0, 1]_{\beta_1=1} \times [0, 1]_{\beta_2=1} \times [0, 1]_{\beta_3=1}.$$

A set of all strategies of all players for $K = 2$ is

$$S(K = 2) = S_1(K = 2) \times S_2(K = 2) \times S_3(K = 2) =$$

$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=4}$
$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=4}$
$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=4}$
$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=1}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=4}$
$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=4}$
$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=4}$
$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=4}$
$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=2}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=4}$
$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=4}$
$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=4}$
$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=4}$
$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=3}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=4}$
$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=1}, [0, 1]_{\beta_3=4}$
$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=2}, [0, 1]_{\beta_3=4}$
$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=3}, [0, 1]_{\beta_3=4}$
$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=1}$	$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=2}$	$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=3}$	$[0, 1]_{\beta_1=4}, [0, 1]_{\beta_2=4}, [0, 1]_{\beta_3=4}$

Every cell in Table 3.1.1 is a three-dimensional unit cube, which corresponds to the choices of partitions by all three players. The set $S(K = 1)$ is in a frame as a top-left cell of Table 3.1.1). We can easily see that $S(K = 1) \subset S(K = 2)$ (The same stays true as a general rule, which further we address as a nesting property for strategies. We do not present $S(K = 3)$ as it demands much space.

3.2 Group formation rules

A definition of a game embeds mechanism design, i.e. a number of deviators, a collection of eligible partitions and a family of group formation rules for these partitions. Partitions were described above, here we present group formation rules and some their properties.

A family of group formation rules $\mathcal{R}(K)$ prescribes how partitions from $\mathcal{P}(K)$ divide a set of all strategies S into non-overlapping domains with

complete coverage of S . In other words: the $\mathcal{R}(P)$ assigns a unique final partition P^β for every element s from S . Every partition P^β is a separate non-cooperative game of Nash. Another interpretation is that $\mathcal{R}(K)$ is a set of social rules or norms defined on a set of all strategies $S(K)$.

“The rules can be interpreted as different institutional settings in which treaty formations take place”, (Finus and Rundshagen, 2003). An example further demonstrates that different chosen strategies may lead to the same partition. A triple $\{K, \mathcal{P}(K), \mathcal{R}(P)\}$ we address as a mechanism design.

Formally the family of coalition formation rules $\mathcal{R}(K)$ is defined as

$$\mathcal{R}(K): S \mapsto \left\{ S_\beta: \beta = 1, G(K); S_\beta \subset R^N; S = \cup_\beta S_\beta, \begin{cases} \bar{\beta} \neq \hat{\beta} \\ S_{\bar{\beta}} \cap S_{\hat{\beta}} = \{\emptyset\} \end{cases} \right\}.$$

A set S_β is a set of all strategies of all players for a partition P^β . S_β may be open and consist of disjoint sets. Every S_β is a bounded measurable subset in R^N . The family has a property:

$$\forall (s_{\beta_i})_{i \in N} \in (S_i(K))_{i \in N} \quad \exists! \beta \mid s_\beta = (s_{\beta_i})_{i \in N} \in S_\beta,$$

every strategy profile s from S is assigned a unique partition P^β .

Domain of $\mathcal{R}(K)$ is a set of all strategies of all players $S(K)$. The range of $\mathcal{R}(K)$ is a collection of sets $\{S_\beta \mid \beta = 1, \dots, G(K)\}$, where every component S_β contains all strategy profiles for a partition P^β from $\mathcal{P}(K)$.

The construction implies that there are two expansions for S : one from in-

put strategies $S(K) = \times_{i \in N} S_i(K)$, and one from partition-specific strategies $S = \cup_{\beta} S_{\beta}$.

All mismatches between individual partition choices must be anticipated in $\mathcal{R}(K)$. Thus a choice of a partition becomes also a choice of a structure for partition-specific externalities. Studying properties of $\mathcal{R}$ we leave for a future paper as well as partition-specific externalities.

It is clear that the family $\mathcal{R}(K)$ is more complicated than a simple coalition function (for example, as in Finus and Rundshagen, 2003). The suggested approach is also different from partition approach.

3.2.1 An example: a unanimous group formation rule

Let $\mathcal{R}(K)$ be a unanimous group formation rule, i.e. when any group g^{β} in any P^{β} can be formed only from a unanimous agreement (choices) of players from g^{β} . However the same group can be present in different partitions, $g^{\bar{\beta}} \in P^{\bar{\beta}}$, $g^{\bar{\bar{\beta}}} \in P^{\bar{\bar{\beta}}}$, but $P^{\bar{\beta}} \neq P^{\bar{\bar{\beta}}}$.

Let all players from a group g^{β_0} choose the same partition P^{β_0} , then formation of the P^{β_0} still depends on partitions chosen by all other players. But any final partition will contain the unanimously chosen group g^{β_0} .

Let there is a mismatch in choices of players i_1, i_2 , who choose different partitions $P^{\bar{\beta}}$ and $P^{\bar{\bar{\beta}}_2}$, while $i_1 \in g^{\bar{\beta}}$, $i_2 \in g^{\bar{\bar{\beta}}}$, $i_1, i_2 \in g^{\bar{\bar{\beta}}}$ but $i_1 \notin g^{\bar{\beta}}$. Then partition $P^{\bar{\bar{\beta}}}$ can not be formed from a unanimous rule, and in any final partition a player i_2 obtains a singleton. We can not say anything definite about final allocation of player i_1 as there is no enough information.

Table Table 3.2.1 presents transformation of chosen strategy profiles into partition specific strategy profiles. The first column contains values for a number of deviators. In this column we used nested property of strategy sets. We can also see that for every K and every β a set $S_\beta(K) \subset R^3$.

Table 1: Unanimous rule formation of partition-specific strategy sets $\{S_\beta\}$ from individual choices of players, $N = 3$.

K	choices of $i = 1$	choices of $i = 2$	choices of $i = 3$	final $S_\beta(K)$
1, 2, 3	$[0, 1]_{\beta_1=\{\{1\},\{2\},\{3\}\}}$	$[0, 1]_{\beta_2=\{\{1\},\{2\},\{3\}\}}$	$[0, 1]_{\beta_3=\{\{1\},\{2\},\{3\}\}}$	$[0, 1]_{\beta=\{\{1\},\{2\},\{3\}\}}^3$
2, 3	$[0, 1]_{\beta_1=any}$	$[0, 1]_{\beta_2=\{\{1\},\{2\},\{3\}\}}$	$[0, 1]_{\beta_3=\{\{1\},\{2\},\{3\}\}}$	$[0, 1]_{\beta=\{\{1\},\{2\},\{3\}\}}^3$
2, 3	$[0, 1]_{\beta_1=\{\{1\},\{2\},\{3\}\}}$	$[0, 1]_{\beta_2=any}$	$[0, 1]_{\beta_3=\{\{1\},\{2\},\{3\}\}}$	$[0, 1]_{\beta=\{\{1\},\{2\},\{3\}\}}^3$
2, 3	$[0, 1]_{\beta_1=\{\{1\},\{2\},\{3\}\}}$	$[0, 1]_{\beta_2=\{\{1\},\{2\},\{3\}\}}$	$[0, 1]_{\beta_3=any}$	$[0, 1]_{\beta=\{\{1\},\{2\},\{3\}\}}^3$
2, 3	$[0, 1]_{\beta_1=\{\{1,3\},\{2\}\}}$	$[0, 1]_{\beta_2=\{\{1,2\},\{3\}\}}$	$[0, 1]_{\beta_3=\{\{1\},\{2,3\}\}}$	$[0, 1]_{\beta=\{\{1\},\{2\},\{3\}\}}^3$
2, 3	$[0, 1]_{\beta_1=\{\{1,2\},\{3\}\}}$	$[0, 1]_{\beta_2=\{\{1\},\{2,3\}\}}$	$[0, 1]_{\beta_3=\{\{1,3\},\{2\}\}}$	$[0, 1]_{\beta=\{\{1\},\{2\},\{3\}\}}^3$
2, 3	$[0, 1]_{\beta_1=\{\{1,2\},\{3\}\}}$	$[0, 1]_{\beta_2=\{\{1\},\{2,3\}\}}$	$[0, 1]_{\beta_3=any}$	$[0, 1]_{\beta=\{\{1,3\},\{2\}\}}^3$
2, 3	$[0, 1]_{\beta_1=\{\{1,3\},\{2\}\}}$	$[0, 1]_{\beta_2=\{\{1,2\},\{3\}\}}$	$[0, 1]_{\beta_3=any}$	$[0, 1]_{\beta=\{\{1\},\{2,3\}\}}^3$
2, 3	$[0, 1]_{\beta_1=\{\{1,2\},\{3\}\}}$	$[0, 1]_{\beta_2=\{\{1,2\},\{3\}\}}$	$[0, 1]_{\beta_3=any}$	$[0, 1]_{\beta=\{\{1,2\},\{3\}\}}^3$
3	$[0, 1]_{\beta_1=\{1,2,3\}}$	$[0, 1]_{\beta_2=\{1,2,3\}}$	$[0, 1]_{\beta_3=\{1,2,3\}}$	$[0, 1]_{\beta=\{1,2,3\}}^3$

We can see that the same partitions with three singletons appear in three different cases, thus family formation rule $\mathcal{R}(K)$ does not have an invertible transformation.

3.3 Payoffs and definition of a game

Let

$$U_i^\beta(s_1(K), \dots, s_N(K)) : S_\beta \mapsto \mathbb{R}$$

be a payoff function for i in a partition P^β , defined for every strategy profile $s_\beta = (s_1, \dots, s_N) \in S_\beta(K) \subset R^N$. All functions $U_i^\beta(s_1, \dots, s_N)$ are integrable, for every partition P^β there is $\int_{(s_1, \dots, s_N) \in S_\beta} U_i^\beta(s_1, \dots, s_N) ds_1 \dots ds_N < \infty$, as every $S_\beta(K) \subset R^N$.

Every partition is a separate non-cooperative game of Nash. The games are united by mechanism design and mixed strategy sets of the players defined over all possible strategy profiles, framed by partitions.

This payoff functions captures both intra and inter group externalities. Intra-group externalities are natural for any game. However the presence of inter-group externalities is not so well studied. Their importance for group formation is well discussed in Olson (p.17, 1971).

Let

$$\Gamma(K) = \left\langle N, \left\{ K, \mathcal{P}(K), \mathcal{R}(K) \right\}, \left(S_i(K), U_i^\beta(K) \right)_{i \in N} \right\rangle$$

be a non-cooperative game of N players, where mechanism design consists of no more than K deviators, final partitions $\mathcal{P}(K)$, a family of group formation rules $\mathcal{R}(K)$, and individual properties of the players are: strategy sets $S_i(K)$ and partition contingent payoffs $\mathcal{U} = \{U^\beta \mid \beta\}$. It is clear that for every P^β individual payoffs can be defined independently.

As far as with an increase of K sets of strategies are nested than payoffs functions for a game with less K are included into a game with a greater K , $\Gamma(K_1) \subset \Gamma(K_2)$, if $K_1 < K_2$. This property will be used further for

construction of a non-cooperative stability criterion.

Timing of the game is traditional for simultaneous games. Before the game all players obtain all fundamentals of the game from $\Gamma(K)$ and construct their beliefs about what may other players do. Then all players move simultaneously. Construction of an equilibrium does not need axioms besides individual rationality of a non-cooperative game of Nash.

3.3.1 An example of payoffs

We continue the example above. Let individual payoffs be defined in the simplest way.⁷ Let s_i be a strategy of i and $s = (s_i)_{i \in N}$ be a strategy profile which leads to a realization of a partition P^β with a partition-specific strategy profile $s_\beta \equiv s$. Table 2 presents the payoffs for $K = 1, 2, 3$:

Table 2: Payoffs in a unanimous group formation game for different K . Individual partition-contingent payoff depends only on i 's strategy, however a realization of a partition depends on a whole strategy profile s .

K	partition	U_1^β	U_2^β	U_3^β
1, 2, 3	$\{\{1\}, \{2\}, \{3\}\}$	$s_{1,\beta=1} + 2$	$s_{2,\beta=1} + 2$	$s_{3,\beta=1} + 2$
2, 3	$\{\{1, 2\}, \{3\}\}$	$s_{1,\beta=2} + 5$	$s_{2,\beta=2} + 7$	$s_{3,\beta=2} + 6$
2, 3	$\{\{1, 3\}, \{2\}\}$	$s_{1,\beta=3} + 6$	$s_{2,\beta=3} + 5$	$s_{3,\beta=3} + 7$
2, 3	$\{\{1\}, \{2, 3\}\}$	$s_{1,\beta=4} + 7$	$s_{2,\beta=4} + 6$	$s_{3,\beta=4} + 5$
3	$\{1, 2, 3\}$	$s_{1,\beta=5} - 1$	$s_{2,\beta=5} - 1$	$s_{3,\beta=5} - 1$

An increase in K adds new subsets due to nested property of strategy

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⁷The payoffs do not have explicit intra and inter group externalities mentioned above, but have another type of externalities, which for the moment we leave without a special term, something like partition-specific or social norm specific externalities.

sets, $S_i(K = 1) \subset S_i(K = 2) \subset S_i(K = 3)$, thus after an increase in K we need to define payoffs only for the newly added areas. An increase in K means a consideration of three games: $\Gamma(K = 1)$, $\Gamma(K = 2)$ and $\Gamma(K = 3)$. Payoffs in the game are constructed in such a way that even $K = 3$, i.e. in the game $\Gamma(K = 3)$, neither player is interested to choose a grand coalition $\{1, 2, 3\}$ in comparison to a case of the game $\Gamma(K = 2)$. This means that an increase in a number of deviators from $K = 2$ to $K = 3$ does not result in better off for players.

Table 3: Payoff maximizing strategies of players for a unanimous group formation rule with different number of deviators K

players	$s_i^*(K = 1)$	$s_i^*(K = 2)$	$s_i^*(K = 3)$
$i = 1$	$1_{\beta_1=1}$	$1_{\beta_1=2}$ $1_{\beta_1=3}$	the same
$i = 2$	$1_{\beta_2=1}$	$1_{\beta_2=2}$ $1_{\beta_2=4}$	the same
$i = 3$	$1_{\beta_3=1}$	$1_{\beta_3=1}$ $1_{\beta_3=4}$	the same

Table 3 contains payoff maximizing pure strategy profiles, but they are not unique for $K > 1$. The next section presents a mixed strategies equilibrium.

3.4 Mixed strategies, expected utility and definition of an equilibrium

Let $\Delta(S_i(K))$ be an infinitely-dimensional probability space over $S_i(K)$, i.e. a set of mixed strategies (probability measures) over $S_i(K)$.

Expected utility for i is defined as a sum of partition-contingent payoffs, like expected utility over contingent states in finance. For every i expected utility is a mapping

$$EU_i^{\mathcal{R}(K)}: \Delta(S_i(K)) \times \left(\times_{j \neq i} \Delta S_j(K) \right) \mapsto R$$

it is a well-defined bounded linear functional over $\Delta(S_i(K))$. Expected utility is written down in this way to emphasize that every i chooses his/her mixed strategy over $S_i(K)$ given mixed strategy profile of all other players.

Expected utility can be written in more details in terms of partition-specific strategies

$$EU_i^{\mathcal{R}(K)}(\sigma_i(K), \sigma_{-i}(K)) = \sum_{\beta=1, G(K)} \int_{S_\beta} U_i^\beta(s_i, s_{-i}) d\sigma_i(K) d\sigma_{-i}(K),$$

where every partition P^{β_j} is available for i from $\mathcal{P}$.

A mixed strategy of all other players $\sigma_{-i}(K)$ is defined as usual, $\sigma_{-i}(K) \in (\Delta(S_j(K)))_{j \neq i}$. By $U_i^\beta(s_i, s_{-i})$ we denote a payoff of i in partition $\beta = 1, \dots, G(K)$ from a strategy profile $(s_i, s_{-i}) \in S_\beta$.

Due to the property of S , which has two expansions, we can write down expected utility in terms of original strategies, but is it much more complicated expression. Relation between the two representations of mixed strategies is explained in a subsection on measure transportation.

Clearly the set $EU_i^{\mathcal{R}(K)}$ is bounded, complete, compact and continuous

over a set of all mixed strategies $\left(\Delta(S_i)\right)_{i \in N}$. Continuity for $EU_i^{\mathcal{R}(K)}$ is a uniform continuity, (Lax, 2005). All these properties are crucial for application of fixed point theorems.

3.4.1 An example of mixed strategies

A set of mixed strategies of i in the game above with $N = 3$ and $K = 1$ is

$$\Delta\left(S_i(K = 1)\right) = \Delta\left(\{[0, 1]_{\beta_i=}\}, \text{ such that}$$

$$\int_{S_i(K=1)} d\sigma_i = \int_{s_i \in [0,1]_{\beta_i=\{\{1\},\{2\},\{3\}\}}} d\sigma_i(s_i) = 1$$

.

A set of mixed strategies of i in the game above with $N = 3$ and $K = 2$ is

$$\Delta\left(S_i(K = 2)\right) = \Delta\left(\{[0, 1]_{\beta_i=\{\{1\},\{2\},\{3\}\}}, [0, 1]_{\beta_i=\{\{1,2\},\{3\}\}}, [0, 1]_{\beta_i=\{\{1,3\},\{2\}\}}, [0, 1]_{\beta_i=\{\{1\},\{2,3\}\}}\right),$$

and

$$\begin{aligned} \int_{S_i(K=2)} d\sigma_i = & \int_{s_i \in [0,1]_{\beta_i=\{\{1\},\{2\},\{3\}\}}} d\sigma_i(s_i) + \int_{s_i \in [0,1]_{\beta_i=\{\{1,2\},\{3\}\}}} d\sigma_i(s_i) \\ & + \int_{s_i \in [0,1]_{\beta_i=\{\{1,3\},\{2\}\}}} d\sigma_i(s_i) + \int_{s_i \in [0,1]_{\beta_i=\{\{1\},\{2,3\}\}}} d\sigma_i(s_i) = 1. \end{aligned}$$

Both sets of mixed strategies are written in terms of initial strategies due to a better visuability. A relation between mixed strategies on initial strategies

and their transformations in terms of partition-specific strategies is written down further.

Sets of mixed strategies in the nested games $\Gamma(K = 1)$, $\Gamma(K = 2)$, $\Gamma(K = 3)$ are different due to nested properties of strategy sets. We do not specify partition specific mixed strategies here as they have more complicated representation. Relation between two types of mixed strategies is discussed further.

3.5 An equilibrium

Definition 2. *K-Nash equilibrium for K simultaneously (or parallel) deviators is a mixed strategy profile $\sigma^*(K) = \left(\sigma_i^*(K)\right)_{i \in N}$ such that for every $i \in N$ and for every $\sigma_i(K) \neq \sigma_i^*(K)$ there is*

$$EU_i^{\mathcal{R}(K)}\left(\sigma_i^*(K), \sigma_{-i}^*(K)\right) \geq EU_i^{\mathcal{R}(K)}\left(\sigma_i(K), \sigma_{-i}^*(K)\right).$$

Theorem 1. *K-Nash equilibrium exists.*

For every i the space $\Delta\left(S_i(K)\right)$ is bounded, closed, complete and convex. It also contains concentrated distributions, and is uniformly convex. Bounded convex set of probability measures is also uniform bounded and uniform continuous, a necessary and sufficient condition for weak convergence of mixed strategies (probability measures).

Uniform continuity of the set $EU_i^{\mathcal{R}(K)}$ over mixed strategies is required to

ensure that i 's best response

$$\arg \max_{\sigma_i(K) \in \Delta(S_i(K))} EU_i^{\mathcal{R}(K)}(\sigma_i(K), \sigma_{-i}(K))$$

is a continuous, bounded and compact mapping.

Existence of an equilibrium follows immediately from the properties of compactness, completeness and uniform continuity of every mappings, $i = 1, N$:

$$\Psi_i = \arg \max_{\sigma_i(K) \in \Delta(S_i(K))} EU_i^{\mathcal{R}(K)}(\sigma_i(K), \sigma_{-i}(K)) \mapsto \Delta(S_i(K))$$

$$\text{for } \forall \sigma_{-i}(K) \in \left(\Delta(S_j(K)) \right)_{j \neq i}$$

with a final application of the Shauder fixed point theorem as a mapping of a compact and continuous operator $(\Psi_1, \dots, \Psi_N)$ to a continuous, convex set $(\Delta(S_1(K)), \dots, \Delta(S_N(K)))$:

$$(\Psi_1, \dots, \Psi_N) \left(\Delta(S_1(K)), \dots, \Delta(S_N(K)) \right) \mapsto \left(\Delta(S_1(K)), \dots, \Delta(S_N(K)) \right).$$

Introduced K -Nash equilibrium concepts differs from the strong Nash equilibrium (Aumann, 1950) in few respects - there is no restriction that all deviators move as one group even if $K = N$, the players have externalities between each other and with those, who do not deviate, there are explicit individual payoffs. There is no assumption about super-additivity of payoffs,

hence the game allows to study an equilibrium and efficiency separately.

The game includes inter and intra-group externalities between players, mentioned by Maskin (2011) as a requirement to study non-cooperative group formation. There are also differences with coalition-proof equilibrium (Bernheim, Peleg Whinston, 1987) and k -equilibrium: these concepts assume deviation of a group as a whole unit. The proposed concept assumes deviation of a set, which may have an internal coalition structure, which can not be described only by it's size.

Approach of Mertens and Zamir (1978) established the link between mixed strategies and types of players. The presented game can be reconstructed also as a game of private information, where a player's type is his choice of players to deviate with and a strategy to choose.

3.5.1 An example of equilibrium mixed strategies

It is clear that in the example above every player will choose a unit quantity to submit to a chosen partition. But for values $K = 2, 3$ a player needs to randomize between partition choices. Probabilities for different partition choices are presented in Table 4.

In a general case calculation of mixed strategies may become complicated.

3.6 Measure transformation mapping $\mathcal{R}(K)$

Measure transportation property establishes consistency between initial individual mixed strategies from S_i (possibly defined in real terms) and partition-

Table 4: Equilibrium mixed strategies for players in the unanimous group formation game

partition	$i = 1$	$i = 2$	$i = 3$
$\{\{12\}, \{3\}\}$	$\frac{7}{5+6+7}$	$\frac{5}{5+6+7}$	$\frac{6}{5+6+7}$
$\{\{13\}, \{2\}\}$	$\frac{6}{5+6+7}$	$\frac{7}{5+6+7}$	$\frac{5}{5+6+7}$
$\{\{1\}, \{23\}\}$	$\frac{5}{5+6+7}$	$\frac{6}{5+6+7}$	$\frac{7}{5+6+7}$

specific mixed strategies, (see Billingsley (1965, 1999), Dudley (2005), Schilling (2002) on measure transportation).

In the example above a set of all strategies of a game $\Gamma(K)$ has two representations: one in terms of initial strategies of all players $S(K = 2) = \times_{i \in N} S_i(K = 2)$, another in terms of partition specific strategies $\cup_{\beta} S_{\beta} = S$, S_{β} , where $S_{\beta} = \{(s_{\beta_1}, \dots, s_{\beta_N})\}$, $S_{\beta} \subset S$, $\forall \tilde{\beta} \neq \tilde{\beta}, S_{\tilde{\beta}} \cap S_{\tilde{\beta}} = \emptyset$. A general property of the transformation $\mathcal{R}(K)$ is based on the property of strategies:

$$\times_{i \in N} S_i = S = \cup_{\beta} S_{\beta}.$$

In the example above the group formation rule $\mathcal{R}(K = 1)$ for $K = 1$ is trivial, and measure invertible, as it induces only a partition of singletons, $P^{\beta} = \{\{1\}, \{2\}, \{3\}\}$.

But $K = 2$ and $K = 3$ induce more than one final partition. In every case a final partition is *backward* invertible, as for every S_{β} from $\cup_{\beta} S_{\beta}$ there is

$$\mathcal{R}(K)\{\mathcal{R}^{-1}(K)(S_{\beta})\} = S_{\beta}.$$

However a reverse is not true, as $\mathcal{R}^{-1}(K)(S_\beta)$ may not be a unique for $K = 2, 3$, that is clear from the example above. For example, In Table 1 the final partition $\{\{1\}, \{2\}, \{3\}\}$ is induced by five different initial strategy sets.

Consider the problem of a mixed strategies over partition-specific strategy sets in more details. We have seen that for $K = 2, 3$ a partition $\{\{1\}, \{2\}, \{3\}\}$ can be appear from three different sets of strategies. Let $A_{\beta=\{\{1\}, \{2\}, \{3\}\}}$ be a set of strategies of all players in a partition $\{\{1\}, \{2\}, \{3\}\}$, such that

$$A_{\beta=\{\{1\}, \{2\}, \{3\}\}} = [0, a_1] \times [0, a_2] \times [0, a_3] \subset [0, 1]_{\{\{1\}, \{2\}, \{3\}\}}^3,$$

for $0 \leq a_i \leq 1, i = 1, 2, 3$. The partition $\{\{1\}, \{2\}, \{3\}\}$ can be formed either from a choice of all players or from a mismatch of choices of the players. We assume that players act independently. Probability of a mixed strategy profile of all three players in the partition $\{\{1\}, \{2\}, \{3\}\}$ can be reconstructed from mixed strategies profile from initial strategies of all three players as

$$\begin{aligned}
\sigma(A_{\beta=\{\{\{1\},\{2\},\{3\}\}}}) = & \\
& \sigma_1(a_1)_{\beta_1=\{\{\{1\},\{2\},\{3\}\}}}\sigma_2(a_2)_{\beta_2=\{\{\{1\},\{2\},\{3\}\}}}\sigma_3(a_3)_{\beta_3=any} \\
& + \sigma_1(a_1)_{\beta_1=\{\{\{1\},\{2\},\{3\}\}}}\sigma_2(a_2)_{\beta_2=any}\sigma_3(a_3)_{\beta_3=\{\{\{1\},\{2\},\{3\}\}}} \\
& + \sigma_1(a_1)_{\beta_1=any}\sigma_2(a_2)_{\beta_2=\{\{\{1\},\{2\},\{3\}\}}}\sigma_3(a_3)_{\beta_3=\{\{\{1\},\{2\},\{3\}\}}} \\
& + \sigma_1(a_1)_{\beta_1=\{\{1,2\},\{3\}\}}\sigma_2(a_2)_{\beta_2=\{\{1\},\{2,3\}\}}\sigma_3(a_3)_{\beta_3=\{\{1,3\},\{2\}\}} \\
& + \sigma_1(a_1)_{\beta_1=\{\{1,3\},\{2\}\}}\sigma_2(a_2)_{\beta_2=\{\{1,2\},\{3\}\}}\sigma_3(a_3)_{\beta_3=\{\{1,2\},\{3\}\}}.
\end{aligned}$$

Probability measure for other partitions are constructed simpler, for example:

$$\sigma(A_{\beta=\{\{\{12\},\{3\}\}}}) = \sigma_1(a_1)_{\beta_1=\{\{\{12\},\{3\}\}}}\sigma_2(a_2)_{\beta_2=\{\{\{12\},\{3\}\}}}\sigma_3(a_3)_{\beta_3=any}.$$

Everywhere above used the notation $\beta = any \equiv \beta \in \{1, 2, 3, 4\}$. We leave for future detailed investigation of measure transportation issues, as it requires additional topics from measure and ergodic theories.

4 Equilibrium *vs.* stability

The game $\Gamma(K)$ has an additional degree of flexibility in comparison to original non-cooperative game of Nash: a number of deviators. Every equilibrium in $\Gamma(K)$ is parametrized by a number of deviators K . Within the suggested

game we can ask a question: what a player knows about a number of deviators in a game $\Gamma(K)$ after the equilibrium is formed, can a number of deviators increase? ⁸ A rational player should anticipate this before accepting an equilibrium with K deviators. This is what can be called as an opportunistic behavior.

We can ask the question differently: an equilibrium strategy profile $\sigma^*(K)$ is robust to a change in K or not?

Let there is a list of games, $\{\Gamma(K), K = 1, \dots, N\}$. The games are related to each other with the properties: $\mathcal{P}(K = 1) \subset \dots \subset \mathcal{P}(K = N)$ and $S(K = 1) \subset \dots \subset S(K = N)$. An above example with a unanimous group formation rule demonstrated them.

Let $\sigma^*(K)$ be an equilibrium strategy profile for a game $\Gamma(K)$. A Nash equilibrium is

$$EU_i^{\mathcal{R}(K)}(\sigma_i^*(K), \sigma_{-i}^*(K)) \geq EU_i^{\mathcal{R}(K)}(\sigma_i(K), \sigma_{-i}^*(K)).$$

Now let us reconsider it as an incentive compatibility inequality of Hurwicz (1972). The left side is an expected payoff if i follows an equilibrium strategy σ_i^* within a group size no greater than K . Let $K = K_{IN}$ for construction of an equilibrium, and $\sigma^*(K_{IN}) = \left(\sigma_i^*(K_{IN})\right)_{i \in N}$ be an induced equilibrium strategy profile with expected payoff $EU_i^{\mathcal{R}(K_{IN})}(\sigma_i^*(K_{IN}), \sigma_{-i}^*(K_{IN}))$.

The suggestion is to consider the right-side, $EU_i^{\mathcal{R}(K)}(\sigma_i(K), \sigma_{-i}^*(K))$, as

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⁸We can address this property as an opportunistic behavior.

an expected payoff when i deviates in groups size different from K . Let $K = K_{OUT}$ for a deviation from the equilibrium, and $\sigma^*(K_{OUT}) = \left(\sigma_i^*(K_{OUT})\right)_{i \in N}$ be an a corresponding equilibrium strategy profile in a game $\Gamma(K_{OUT}) \neq \Gamma(K_{IN})$. An equilibrium expected payoff in this case will be

$$EU_i^{\mathcal{R}(K_{OUT})}\left(\sigma_i^*(K_{OUT}), \sigma_{-i}^*(K_{OUT})\right)$$

.

Thus with the K -Nash equilibrium we can ask a question about a robustness of an equilibrium strategy profile $\sigma^*(K_{IN})$ to a variation in K , i.e. for cases when $K_{IN} \neq K_{OUT}$.

For any $K_1, K_2, K_1 < K_2$, the domains of pure strategies are nested as $S_i(K_1) \subset S_i(K_2), i \in N$. This is clear from the example above. Hence a decrease in a number of potentially deviating players can not decrease expected utility. Hence if $K_{IN} \geq K_{OUT}$, then there is always

$$EU_i^{\mathcal{R}(K_{IN})}\left(\sigma_i^*(K_{IN}), \sigma_{-i}^*(K_{IN})\right) \geq EU_i^{\mathcal{R}(K_{OUT})}\left(\sigma_i^*(K_{OUT}), \sigma_{-i}^*(K_{OUT})\right),$$

meaning that the equilibrium strategy profile $\sigma^*(K_{IN})$ is also an equilibrium for any $K_{OUT}, K_{OUT} < K_{IN}$. However if $K_{IN} < K_{OUT}$ this statement may be no longer true.

We can consider a maximum number of deviators, when an equilibrium

still holds true, i.e.

$$K^* = \max_{\substack{K_{IN} < K \\ i \in N}} \left\{ EU_i^{\mathcal{R}(K_{IN})} \left(\sigma_i^*(K_{IN}), \sigma_{-i}^*(K_{IN}) \right) \geq EU_i^{\mathcal{R}(K)} \left(\sigma_i^*(K), \sigma_{-i}^*(K) \right) \right\}$$

as **a confidence** of all players to an induced equilibrium $\left(\sigma_i^*(K_{IN}) \right)_{i \in N}$ (other suggested terms: s trust to an equilibrium, a measure of commitment or a self-enforcement equilibrium property). K^* is a maximum number of deviators, which still does not break the equilibrium $\sigma^*(K_{IN})$.

Now we can reconsider the example with three players above. A change in K changes a structure of a game. A change in the structure of the game means a change in how players will play against each other. If an increase in K adds new information in terms of payoffs, then players change their incentives. For $K = 1$ every player is guaranteed that any deviating group can not exceed size one. For $K = 2$ there is an additional information and payoffs increase. For $K = 3$ there is an increase in strategy options, but neither player can be better off, thus $K^* = 3$ even an equilibrium strategy profile was constructed for $K = 2$, as $\sigma^*(K = 2) = \sigma^*(K = 3)$.

We can summarize by constructing a network of equilibria for the game in the example. Equilibrium strategy profiles are ordered as $\sigma^*(K = 1) \prec \sigma^*(K = 2) = \sigma^*(K = 3)$, meaning an order of expected payoffs in different games $EU^{\mathcal{R}(K=1)} \prec EU^{\mathcal{R}(K=2)} = EU^{\mathcal{R}(K=3)}$.

4.0.1 An application to self-enforcing property of an equilibrium

The case when a change in a number of deviators changes a number of equilibria was already discussed in literature in terms of a “stag and hare” game. Take an example from Aumann (1991), a response to D.Kreps.

Table 5: A variant of “a stag and a hare” game

	”c” together	separately
”c” together	(100;100)	(0;8)
separately	(0;8)	(7;7)

If every player is allowed to deviate unilaterally, than there are two equilibria in pure strategies and one in mixed. Two equilibria in pure strategies are ranked by gain sizes. But it is impossible to move from one to another pure strategies equilibrium by a one-step unilateral deviation of either player. However for $K = 2$ there is only one equilibrium in pure strategies (100; 100). An increase in a number of deviators supports only one equilibrium in pure strategies. Thus a game with two deviators allows immediately to reach (100; 100).

An increase from $K = 1$ to $K = 2$ eliminates the necessity to control a number of deviators and justifies the conclusion of Aumann (1991) that the strategy profile (c, c) : ”almost surely will be played without agreement”. We may assume the Schelling’s concept of ”focal point” to compare equilibria in terms of payoffs for every player is implicitly based on letting more than one player to deviate. The suggested approach offers a way for selection of

multiple equilibria ordered by payoffs.

The uncertainty comes from the side - how can players be sure that each of them plays a game with two deviators, but not with one? In other words, every player may have an uncertainty, which game is played, $\Gamma(K = 1)$ or $\Gamma(K = 2)$. This ruins an equilibrium in pure strategies again.

The same example can be reconsidered in more details. It explicitly assumes that two-person hunting for every animal may occur alone or together.

Table 6: A variant of “a stag or a hare” hunting game. A strategy - a target animal and a group size of hunters

	hare, alone	stag, alone	hare, together	stag, together
hare, alone	(8;8)	(8;0)	(8;8)	(8;0)
stag, alone	(0;8)	(0;0)	(0;8)	(0;0)
hare, together	(8;8)	(8;0)	(4;4)	(8;0)
stag, together	(0;8)	(0;0)	(0;8)	(100;100)

We can see that the game $\Gamma(K = 1)$ is located at four left-top cells and has one equilibrium in weakly-dominating strategies. The game $\Gamma(K = 2)$ is presented by the whole Table 6 and it has one equilibria in strictly dominating strategies.

5 Conclusion

Myerson (p.370, 1991) noted that “we need some model of cooperative behavior that does not abandon the individual decision-theoretic foundations of game theory”. One of the key problems here is an individual uncertainty

about how many others can be cooperative or how many others may have an intention to deviate from a non-cooperative equilibrium. This paper suggests this number to be fixed, but can be different from one, what initially was assumed by John Nash. Thus the paper targets possibly cooperative behavior within a non-cooperative framework with a predefined mechanism design, where the individuals “rationally further their individual interests”(Olson, 1971). In other words the paper targets a non-cooperative equilibrium for a number of deviators ranging from one till everybody, or between Nash and strong Nash equilibria.

The justification of a chosen tool, a non-cooperative game, comes from Maskin (2011) and a remark of Serrano (2014), that for studying coalition formation “it may be worth to use strategic-form games, as proposed in the Nash program”.

The paper suggests a solution for the problem: how to modify a non-cooperative game and a corresponding equilibrium concept in such a way that we can study deviations of more than one player. A constructed non-cooperative simultaneous game with a modified strategy set (in comparison to Nash non-cooperative game) has a number of deviators as a parameter of the game as well as an embedded mechanism design feature.

Existence of a mixed strategy equilibrium follows from properties of bounded, continuous, compact and convex mappings in infinite-dimensional spaces of probability transferable Borel measures. The equilibrium concept is constructed as close as possible to the case, when “each players strategy is opti-

mal against those of the others” (Nash, 1950) in terms of mixed strategies.

The introduced family of equilibria concepts refines strong Nash, coalition-proof and k equilibrium concepts. The differences are: an explicit allocation of payoffs and a combined presence of intra- and inter- group externalities (the list is not complete). Differences from core-approach of Aumann (1960) are obvious - a presence of externalities, no restrictions that deviates only one group, no restrictions on a direction of a deviation (inside or outside), and a construction of individual payoffs from a strategy profile of all players.

The suggested approach is different in a role of a central planner offered by Nash, who “argued that cooperative actions are the result of some process of bargaining” Myerson (p.370, 1991). Central planner offers a predefined mechanism design framework, that includes a number of deviators, family of eligible partitions and a family of rules to construct these partitions from individual strategies.

Pre-announced family of group formation rules $\mathcal{R}$ can be considered as special case of networks (Myerson, (1977), Aumann and Myerson, (1978)). In the forthcoming paper the author addresses the problem. Alternative to fix a family of group formation rules $\mathcal{R}$ are more complicated, for example they are an uncertainty about group formation rules or an evolutionary approach for group formation rules.

The paper suggests a way to measure a trust to an equilibrium as a maximum number of deviators, which still does not destroy an equilibrium

strategy profile for an initially chosen number of deviators.⁹ Construction of the criterium does not depend on a family of group formation rules. This measure can also be considered as measure for self-enforcing quality of an equilibrium. It captures the idea of an opportunistic behavior in groups when a rational player before playing a game should anticipate number of potential deviators *ex post* groups are formed.

Direct next research includes network formation, general equilibrium and repeated games applications of the suggested model. Other important directions are a non-cooperative welfare theory with group formation and a non-cooperative mechanism studies.

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⁹The difference from traditional "blocking approach " is that here it comes on a non-cooperative base with details mentioned above.

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