Dependency of insurance risks and the Burr distribution

Antonella Campana

Dipartimento di Scienze Economiche, Gestionali e Sociali
Università degli Studi del Molise
Via F. De Sanctis – 86100 Campobasso, Italy
campana@unimol.it

Abstract. Recently, several authors in the actuarial literature have derived approximations for sums of random variables when the distributions of terms are known, but the stochastic dependence structure between them is unknown or too cumbersome to work with. The approximations obtained are bounds in the sense of convex order. In this paper, we first give an overview of the recent actuarial literature on this topic. Secondly, we derive convex lower and upper bounds for sums of Burr distributed random variables.

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1 Introduction

In an insurance context, one is often interested in the distribution function of the aggregate claims of an insurance portfolio over a certain time period. The usual assumption of mutual independence between the payments related to a single policy is very convenient from a computational point of view, but sometimes it is not realistic. Recently, several authors in the actuarial literature have derived lower and upper bounds in the sense of convex order for sums of random variables when marginal distributions are given, but their joint distribution is either unspecified or too cumbersome to work with (see [1, 2]). The unknown or complex nature of the dependence between the random variables is the reason why it is often impossible to exactly derive the distributions of the sums exactly.

This paper is organized as follows. In Section 2, we introduce the notions (and their characterizations) of comonotonicity and mutual exclusivity of risks in a Fréchet space. In Section 3, we briefly recall the main results about the safest dependence structure and the most dangerous mutual dependence between the risks in terms of stop-loss and convex order; hence, we show how to improve, under some circumstances, convex upper bounds. In Section 4, we examine the Burr distribution as a mixture of models. In Section 5, we derive convex lower and upper bounds for sums of Burr distributed random variables. Finally, in Section 6 we give some final remarks.
2 Fréchet spaces and Fréchet bounds

Let $X_1, X_2, \ldots, X_m$ be non-negative random variables (rv’s, in short) with finite expectations, further called risks.

Let $F_{X_1}, F_{X_2}, \ldots, F_{X_m}$ be their cumulative distribution functions (cdf’s, in short). Consider the Fréchet space $R_m(F_{X_1}, F_{X_2}, \ldots, F_{X_m})$ consisting of all $m$-dimensional cdf’s $F_X$ (or equivalently of all the $m$-dimensional random vectors $X = (X_1, X_2, \ldots, X_m)$ possessing $F_{X_1}, F_{X_2}, \ldots, F_{X_m}$ as marginal cdf’s.

For all $X$ in $R_m(F_{X_1}, F_{X_2}, \ldots, F_{X_m})$, the following inequalities hold:

$$L_m(x) \leq F_X(x) \leq W_m(x) \quad \text{for all } x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m,$$

where $L_m$ (called the Fréchet lower bound of $R_m(F_{X_1}, F_{X_2}, \ldots, F_{X_m})$) is defined by

$$L_m(x) = \max\left\{\sum_{i=1}^m F_{X_i}(x_i) + 1 - m, 0\right\}, \quad x \in \mathbb{R}^m,$$

while $W_m$ (called the Fréchet upper bound of $R_m(F_{X_1}, F_{X_2}, \ldots, F_{X_m})$) is defined by

$$W_m(x) = \min\{F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_m}(x_m)\}, \quad x \in \mathbb{R}^m.$$

A multivariate risk $X$ in $R_m(F_{X_1}, F_{X_2}, \ldots, F_{X_m})$ possessing $W_m$ as cdf is said to be comonotonic.

Let $X^c$ be the comonotonic version of $X$. Let $S^c$ be the sum of its components. It is proved that the generalized inverse distribution of $S^c$, defined by

$$F_{S^c}^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_{S^c}(x) \geq p\}, \quad p \in (0, 1),$$

is given by the sum of the generalized inverse distributions of the marginal distributions (see [2]):

$$F_{S^c_m}^{-1}(p) = \sum_{i=1}^m F_{X_i}^{-1}(p), \quad p \in (0, 1). \quad (1)$$

Differently from $W_m$, the lower bound $L_m$ is not always reachable in $R_m(F_{X_1}, F_{X_2}, \ldots, F_{X_m})$: for $m \geq 3$, Tchen [8] showed, by a counterexample, that $L_m$ cannot be always a proper cdf.

A sufficient condition for $L_m$ to be a cdf in $R_m(F_{X_1}, F_{X_2}, \ldots, F_{X_m})$ is (see [4]):

$$\sum_{i=1}^m F_{X_i}(0) \geq m - 1. \quad (2)$$

The risks $X_1, X_2, \ldots, X_m$ are said to be mutually exclusive when at most one of them can be different from zero (see [1]):
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\[ P[X_i > 0, X_j > 0] = 0 \quad \text{for all } i \neq j . \]

It is shown in [1] that the elements of a Fréchet space \( R_m(F_{X_1}, F_{X_2}, \ldots, F_{X_m}) \) satisfying (2) are mutually exclusive if and only if they have a multivariate cdf given by \( L_m \).

Let \( S^e \) be the sum of the components of a mutually exclusive risk \( X^e \) in a Fréchet space \( R_m(F_{X_1}, F_{X_2}, \ldots, F_{X_m}) \) satisfying condition (2). The cdf of \( S^e \) is given by (see [1]):

\[ F_{S^e}(x) = \sum_{i=1}^{m} F_{X_i}(x) + 1 - m . \] (3)

There are several examples of comonotonic risks and mutually exclusive risks in actuarial sciences (see [1, 3]).

3 Extremal dependence structure

Let \( S \) be the sum of the components of a multivariate risk \( X \) in the Fréchet space \( R_m(F_{X_1}, F_{X_2}, \ldots, F_{X_m}) \).

In an insurance context the rv \( S \) is often used to represent the aggregate claims of an insurance portfolio over a certain time period. The calculation of the stop-loss premiums related to such a portfolio is one of the main topics of risk theory. Therefore not only the marginal distributions of the \( X_i \)'s have to be known, but also the knowledge of the dependence structure among the \( X_i \)'s is required. In practice, the problem is almost simplified by assuming that the \( X_i \)'s are mutually independent so that knowledge of the marginal distributions suffices to compute stop-loss premiums.

If we examine the dependence in a Fréchet space by looking for the element with the most dangerous mutual dependence between the risks, i.e. the one leading to the highest stop-loss premiums, we find that the most dangerous mutual structure is described by the Fréchet upper bound (see [5]).

By symmetry, when (2) is fulfilled, the Fréchet lower bound provides the least dangerous mutual dependence between risks, i.e. the one giving rise to the smallest stop-loss premiums (see [1]).

By using the standard notation for stop-loss order and convex order, we can write

\[ S^e \leq_{sl} S \leq_{sl} S^e \] (4)

or equivalently, as the rv’s have the same expectation,
If the only information available on the multivariate risk $X$ consists of the marginal cdf’s of the $X_i$’s, then the cdf of $S^c$ is a prudent choice for approximating the unknown cdf of $S$.

It is possible to improve upper bounds, in terms of convex order, for the aggregate claims $S$ if we assume to have some additional information concerning the stochastic nature of $X$.

More precisely, if we assume that there exists some rv $Λ$ with a given distribution function, such that we know the conditional cdf’s of the rv’s $X_i$, given $Λ = \lambda$, for all possible values of $\lambda$, then we can derive upper and lower bounds for $S$ in terms of convex order.

Let $F^{-1}_{X_i|Λ}(U)$ be the rv $f_i(U, Λ)$ where the function $f_i$ is defined by $f_i(u, Λ = \lambda) = F^{-1}_{X_i|Λ=\lambda}(u)$ and $U$ is the uniform rv in $(0, 1)$ independent of the rv $Λ$.

Let $S^*$ be defined by

$$S^* = F^{-1}_{X_1|Λ}(U) + F^{-1}_{X_2|Λ}(U) + \cdots + F^{-1}_{X_m|Λ}(U).$$

As shown in [5], we have

$$S \leq_{cx} S^*.$$  

The random vector $(F^{-1}_{X_1|Λ}(U), F^{-1}_{X_2|Λ}(U), \ldots, F^{-1}_{X_m|Λ}(U))$ has marginal distributions $F_{X_1}, F_{X_2}, \ldots, F_{X_m}$ and so it is a multivariate risk in $R_m(F_{X_1}, F_{X_2}, \ldots, F_{X_m})$. This implies that the upper bound derived $S^*$ is smaller in convex order than the comonotonic upper bound $S^c$:

$$S \leq_{cx} S^* \leq_{cx} S^c.$$  

In order to obtain a lower bound for $S$, in terms of convex order, we observe that the expectation of a rv is always smaller or equal in convex order (by Jensen’s inequality) than the rv itself and also that convex order is maintained under mixing.

Let $S^l$ be defined by

$$S^l = E[X_1 | Λ] + E[X_2 | Λ] + \cdots + E[X_m | Λ].$$

Even if condition (2) is not fulfilled, we have (see [5]):

$$S^l \leq_{cx} S.$$  

In the next section, we will describe how to construct convex lower and upper bounds for $S$ when the risks $X_i$, given $Λ = \lambda$, have a Weibull distribution and the rv $Λ$ has a gamma distribution. In this case, the unconditional distribution of $X_i$ is a Burr distribution which represents an interesting model for losses in insurance.
4 The Weibull and the Burr distribution

Let us now assume that the random variable \( \Lambda \) has a gamma distribution with positive parameters \( \alpha \) and \( \delta \) (and we write \( \Lambda \sim G(\alpha, \delta) \)). Thus the probability density function (pdf, in short) of \( \Lambda \) is

\[
f_{\Lambda}(\lambda) = \frac{\delta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} \exp(-\delta \lambda), \quad \lambda > 0 .
\] (10)

Further, we assume that, for any possible outcome \( \lambda \) of \( \Lambda \), conditionally given \( \Lambda = \lambda \), the risks \( X_i \) are identically distributed and their conditional distribution is a Weibull distribution with positive parameters \( \tau \) and \( \lambda \) (and we write \( X_i \mid \Lambda = \lambda \sim W(\tau, \lambda) \)):

\[
f_{X_i \mid \Lambda = \lambda}(x \mid \lambda) = \lambda \tau x^{\tau-1} \exp(-\lambda x^{\tau}), \quad x > 0 .
\] (11)

The Weibull distribution provides a good model for size of claims in casualty insurance (malpractice, windstorms, etc.) and this is particularly significative when \( \tau < 1 \) (see [7]).

We observe that there are two cases. If \( \tau < 1 \), we obtain a distribution with an upper tail which is intermediate in weight between the exponential and the Pareto distribution; conversely, if \( \tau > 1 \) the upper tail is lighter than the exponential (\( \tau = 1 \) is, of course, the exponential distribution).

The moments about zero of the Weibull distribution are given by

\[
E[X_i^n \mid \Lambda = \lambda] = \lambda^{-\frac{n}{\tau}} \Gamma \left(1 + \frac{n}{\tau}\right), \quad n = 1, 2, \ldots .
\] (12)

Thus, the mean and the variance are respectively obtained by the following expressions:

\[
E[X_i \mid \Lambda = \lambda] = \lambda^{-\frac{1}{\tau}} \Gamma \left(1 + \frac{1}{\tau}\right),
\] (13)

\[
Var[X_i \mid \Lambda = \lambda] = \lambda^{-\frac{2}{\tau}} \left[ \Gamma \left(1 + \frac{2}{\tau}\right) - \left(\Gamma \left(1 + \frac{1}{\tau}\right)\right)^2 \right]
\] (14)

for any \( i \) (\( i = 1, 2, \ldots , m \)).

By compounding, we now develop the unconditional pdf of \( X_i \):

\[
f_{X_i}(x) = \int_0^\infty f_{X_i \mid \Lambda = \lambda}(x \mid \lambda) f_{\Lambda}(\lambda) \, d\lambda
= \frac{\tau \delta^\alpha x^{\tau-1}}{\Gamma(\alpha)} \int_0^\infty \lambda^{\alpha-1} \exp(-\lambda (\delta + x^{\tau})) \, d\lambda .
\]
By comparing this integrand with the gamma pdf with parameters $\alpha + 1$ and $\delta + x^\tau$, we get

$$f_{X_i}(x) = \frac{\alpha \tau \delta^\alpha x^{\tau-1}}{(\delta + x^\tau)^{\alpha+1}}, \quad x > 0. \quad (15)$$

This is the pdf of the Burr distribution with parameters $\alpha$, $\delta$ and $\tau$ (and we write $X_i \sim B(\alpha, \delta, \tau)$). The corresponding cdf is

$$F_{X_i}(x) = 1 - \left(\frac{\delta}{\delta + x^\tau}\right)^\alpha, \quad x > 0. \quad (16)$$

Clearly, when $\tau = 1$, we have the Pareto distribution with parameters $\alpha$ and $\delta$ and we write $X_i \sim P(\alpha, \delta)$.

The Burr distribution is an useful long-tailed distribution. It is also called transformed Pareto distribution. Indeed, it can be easily founded by letting $X_i = Y^{1/\tau}$ with $Y \sim P(\alpha, \delta)$.

By using the formula $E[X_i^n] = E[E[X_i^n | A]]$ or directly, we find

$$E[X_i^n] = \frac{\delta^{\frac{n}{\tau}}}{\Gamma(\alpha)} \Gamma\left(\alpha - \frac{n}{\tau}\right) \Gamma\left(1 + \frac{n}{\tau}\right), \quad (17)$$

provided $\alpha \tau > n$.

Hence, the variance of the individual risks $X_i$ is given by

$$Var[X_i] = \frac{\delta^{\frac{2}{\tau}}}{\Gamma(\alpha)} \left[\Gamma\left(\alpha - \frac{2}{\tau}\right) \Gamma\left(1 + \frac{2}{\tau}\right) - \left(\frac{\Gamma(\alpha - \frac{1}{\tau}) \Gamma(1 + \frac{1}{\tau})}{\Gamma(\alpha)}\right)^2\right]. \quad (18)$$

5 Convex bounds

In order to derive the cdf of the comonotonic upper bound $S^c$, we see that from (1), by hypotheses on $X_i$ of the previous paragraph, it follows:

$$F_{S^c}^{-1}(p) = m F_{X_1}^{-1}(p), \quad p \in (0, 1).$$

Furthermore, by the theorem concerning quantiles of transformed rv’s proved in [2] we have $m F_{X_1}^{-1}(p) = F_{mX_1}^{-1}(p)$ and so it follows:

$$S^c \overset{d}{=} m X_1, \quad (19)$$

where the symbol $\overset{d}{=} \text{ is used to indicate equality in distribution.}$
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By using (19) we can easily prove that the rv $mX_1$ has a Burr distribution with parameters $\alpha$, $m^\tau \delta$ and $\tau$, i.e. $S^c \sim B(\alpha, m^\tau \delta, \tau)$.

For the improved upper bound $S^*$, we have $S^* \overset{d}{=} m F_{X_1|A}^{-1}(U)$. In [5] it is shown that $F_{X_1|A}^{-1}(U) \overset{d}{=} X_1$ holds, then we find

$$S^* \overset{d}{=} mX_1.$$  

It follows: $S^* \overset{d}{=} S^c$.

We observe that condition (2) is not satisfied. We consider the lower bound $S^I$. From (8) and (13) we derive

$$S^I \overset{d}{=} m E[X_1 | A] \overset{d}{=} m A^{-\frac{1}{\tau}} \Gamma\left(1 + \frac{1}{\tau}\right).$$  \hspace{1cm} (20)

Let us now consider the variance of the stochastic upper and lower bounds $S^c$ and $S^I$.

From (18), provided $\alpha \tau > 2$, by replacing $\delta$ by $m^\tau \delta$ we have

$$\text{Var}[S^c] = \frac{m^2 \delta^\frac{2}{\tau}}{\Gamma(\alpha)} \left[ \Gamma\left(\alpha - \frac{2}{\tau}\right) - \frac{\left(\Gamma\left(\alpha - \frac{1}{\tau}\right) \Gamma\left(1 + \frac{1}{\tau}\right)\right)^2}{\Gamma(\alpha)} \right].$$  \hspace{1cm} (21)

In order to derive the variance of lower bound $S^I$, from (20) we note that first $\text{Var}[A^{-\frac{1}{\tau}}]$ is to be determined.

By hypothesis $A \sim G(\alpha, \delta)$, so we find

$$\text{Var}[A^{-\frac{1}{\tau}}] = \frac{\delta^\frac{2}{\tau}}{\Gamma(\alpha)} \left[ \Gamma\left(\alpha - \frac{2}{\tau}\right) - \frac{(\Gamma\left(\alpha - \frac{1}{\tau}\right))^2}{\Gamma(\alpha)} \right]$$  \hspace{1cm} (22)

and then we have

$$\text{Var}[S^I] = \frac{m^2 \delta^\frac{2}{\tau} \left((\Gamma(1 + \frac{1}{\tau}))^2\right)}{\Gamma(\alpha)} \left[ \Gamma\left(\alpha - \frac{2}{\tau}\right) - \frac{\left(\Gamma(\alpha - \frac{1}{\tau})\right)^2}{\Gamma(\alpha)} \right].$$  \hspace{1cm} (23)

In view of (7) and (9), the exact (and unknown) variance of $S$ satisfies the inequalities

$$\text{Var}[S^I] \leq \text{Var}[S] \leq \text{Var}[S^c].$$  \hspace{1cm} (24)

Let us now further assume that, conditionally given $A = \lambda$, the risks $X_i$ are not only identically distributed, but also independent.
Theorem 1 Let $\Lambda$ have a gamma distribution with positive parameters $\alpha, \beta$. Let $X_i$ be i.i.d random variables conditionally given $\Lambda = \lambda$ and let the conditional distribution be a Weibull distribution with parameters $\tau$ and $\lambda$ (i.e. $X_i \mid \Lambda = \lambda \sim W(\tau, \lambda)$). Then

$$\text{Var}[S^I] = \rho \text{Var}[S]$$

and

$$\text{Var}[S^c] = [\rho + (1 - \rho)m]\text{Var}[S]$$

where $\rho$ is defined as follows

$$\rho = \left(1 + \frac{\Gamma(\alpha - \frac{2}{\tau}) \left[\Gamma(1 + \frac{2}{\tau}) - (\Gamma(1 + \frac{1}{\tau}))^2\right]}{m \left(\Gamma(1 + \frac{1}{\tau})\right)^2 \left[\Gamma(\alpha - \frac{2}{\tau}) - \left(\frac{\Gamma(\alpha - \frac{1}{\tau})}{\Gamma(\alpha)}\right)^2\right]}\right)^{-1}.$$

Proof

The variance of the rv $S$ can be decomposed in two parts (called the between-variance and the within-variance respectively) as follows

$$\text{Var}[S] = \text{Var}[E(S \mid \Lambda)] + E[\text{Var}(S \mid \Lambda)].$$

By hypothesis on $X_i$, we have that

$$\text{Var}[E(S \mid \Lambda)] = \text{Var}[S^I],$$

$$E[\text{Var}(S \mid \Lambda)] = m E[\text{Var}(X_1 \mid \Lambda)].$$

Analogously to (27), the variance of the upper bound $S^c$ can be decomposed as follows

$$\text{Var}[S^c] = \text{Var}[E(S^c \mid \Lambda)] + E[\text{Var}(S^c \mid \Lambda)]$$

and, by using (19), we can write

$$\text{Var}[S^c] = \text{Var}[S^I] + m^2 E[\text{Var}(X_1 \mid \Lambda)].$$

The variance of the lower bound $S^I$ is given by (23) while the expected value $E[\text{Var}(X_1 \mid \Lambda)]$ can be calculated from (14), i.e.

$$E[\text{Var}(X_1 \mid \Lambda)] = E[\Lambda^{-\frac{2}{\tau}}] \left[\Gamma\left(1 + \frac{2}{\tau}\right) - \left(\Gamma\left(1 + \frac{1}{\tau}\right)\right)^2\right].$$
By hypothesis $\Lambda \sim G(\alpha, \delta)$, so we obtain

$$E[\Lambda^{-\frac{2}{\tau}}] = \delta^2 \frac{\Gamma(\alpha - \frac{2}{\tau})}{\Gamma(\alpha)}.$$  \hspace{1cm} (33)

Then it follows

$$E[\text{Var}[X_i | \Lambda]] = \delta^2 \frac{\Gamma\left(\alpha - \frac{2}{\tau}\right)}{\Gamma\left(\alpha\right)} \left[ \frac{\Gamma\left(1 + \frac{2}{\tau}\right)}{\Gamma\left(1\right)} - \left(\frac{\Gamma\left(\alpha - \frac{2}{\tau}\right)}{\Gamma\left(\alpha\right)}\right)^2 \right].$$  \hspace{1cm} (34)

By setting

$$\rho = \frac{\text{Var}[S^I]}{\text{Var}[S] + m E[\text{Var}(X_1 | A)]},$$

the results follow.

As explained in [3], the factor $\rho \in [0, 1]$ can be interpreted as a measure for the goodness-of-fit when $S$ is replaced by $S^I$: the larger is $\rho$, the better the lower bound performs. Maximum performance (i.e. $\rho = 1$) is achieved if $m \to \infty$ or when $E[\text{Var}(X_1 | A)] = 0$ (i.e. $\text{Var}(X_1) = \text{Var}[E(X_1 | A)]$). Therefore, for a sufficiently large portfolio the lower bound $S^I$ will perform very well.

We note that the ratio between $\text{Var}[S^c]$ and $\text{Var}[S]$ is an increasing function of the portfolio’s volume. This means that the larger the portfolio, the worse the relative performance of the comonotonic (and the improved) upper bound. For a portfolio of a given size $m$, we observe that the larger is $\rho$, the better the comonotonic upper bound will perform (see [3]).

By Theorem 1 we also obtain the following decomposition for $\text{Var}[S]$:

$$\text{Var}[S] = \frac{m^2 \delta^2}{\Gamma(\alpha)} \left[ \frac{\Gamma\left(\alpha - \frac{2}{\tau}\right)}{\Gamma\left(\alpha\right)} \right]^2 \left[ \frac{\Gamma\left(1 + \frac{2}{\tau}\right)}{\Gamma\left(1\right)} - \left(\frac{\Gamma\left(\alpha - \frac{2}{\tau}\right)}{\Gamma\left(\alpha\right)}\right)^2 \right] +$$

$$+ \frac{m \delta^2}{\Gamma(\alpha)} \left[ \frac{\Gamma\left(\alpha - \frac{2}{\tau}\right)}{\Gamma\left(\alpha\right)} \right]^2 \left[ \frac{\Gamma\left(1 + \frac{2}{\tau}\right)}{\Gamma\left(1\right)} - \left(\frac{\Gamma\left(\alpha - \frac{2}{\tau}\right)}{\Gamma\left(\alpha\right)}\right)^2 \right].$$  \hspace{1cm} (35)

It is proved in [6] that for two stop-loss ordered risks $U$ and $W$ with the same mean $\mu$, the stop-loss premiums for large retentions satisfy:

$$\frac{E[(U - d)_{+}]}{E[(W - d)_{+}]} \approx \frac{\text{Var}[U]}{\text{Var}[W]}, \text{ for all } d \geq \mu.$$  \hspace{1cm} (36)

Then, we can approximate stop-loss premium ratios for retentions $d \geq E[S]$, where $E[S]$ is given by

$$E[S] = \frac{m \delta^2}{\Gamma(\alpha)} \frac{\Gamma\left(\alpha - \frac{1}{\tau}\right)}{\Gamma\left(1 + \frac{1}{\tau}\right)}.$$
provided \( \alpha \tau > 1 \).

We consider, for example, the case Pareto (i.e. \( \tau = 1 \)). One can easily verify that the stop-loss premium with retention \( d \geq 0 \) is given by

\[
E[(S^c - d)_+] = \frac{m^\alpha \delta^\alpha}{(\alpha - 1)(m \delta + d)^{\alpha-1}}.
\]

From (35) it follows

\[
Var[S] = \frac{m^2 \delta^2 (m + \alpha - 1)}{(\alpha - 1)^2 (\alpha - 2)},
\]

while \( Var[S^c] \) can be obtained by (21):

\[
Var[S^c] = \frac{m^2 \delta^2 \alpha}{(\alpha - 1)^2 (\alpha - 2)}.
\]

Then, from (36) we find that the following approximation

\[
E[(S - d)_+] \approx \frac{m^\alpha \delta^\alpha}{(\alpha - 1)(m \delta + d)^{\alpha-1}} \cdot \frac{m + \alpha - 1}{m \alpha},
\]

holds for retentions \( d \geq \frac{m \delta}{\alpha - 1} \), if the assumptions made in Theorem 1 are satisfied.

6 Concluding remarks

In this paper, we considered approximations for sums of dependent random variables whose marginal distributions are known Burr distributions but with an unknown joint distribution. We used some simple yet powerful techniques described in \[2\]. Their central idea consists in replacing the original sum by another ones (with a simpler dependence structure) which are considered less or more favorable by all risk-averse decision makers.

The less favorable sum is an upper bound in terms of convex order and involves the components of the comonotonic version of the original random vector. We proved that, under our assumptions, also this extremal sum has a Burr distribution.

On the other hand, it is shown in \[5\] how a lower bound, in terms of convex order, can be obtained by conditioning the marginal distributions on some random variable.

In our case we used an additional information which is available. Indeed, the Burr distribution can be developed as a mixture of Weibull distributions with a gamma mixing distribution. Then, we derived a lower bound by conditioning on a random variable with a given gamma distribution.

Finally, we obtained approximations for the variance of the random sums considered.
References
