Dynamic Asymmetric GARCH∗

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Abstract

This paper develops the Dynamic Asymmetric GARCH (or DA-GARCH) model that generalises asymmetric GARCH models such as the GJR model, making the asymmetric effect time dependent. We provide the stationarity conditions and show how GJR can be obtained as a special case of DAGARCH. An application to daily stock market indices is also presented to demonstrate the usefulness of the new model.

Keywords: Asymmetric volatility, threshold GARCH, DAGARCH, stationarity conditions

1 Introduction

The idea that positive and negative shocks have different impacts on dynamic volatility was incorporated in the GARCH model (see Engle (1982))

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and Bollerslev (1986)) by Glosten, Jagannathan and Runkle (1993) (GJR), and generalised by Rabemananjara and Zakoian (1993) and Zakoian (1994) to the threshold (or TARCH) model. Several empirical studies have emphasised the relevance of the signs of the shocks on conditional volatility, but no particular attention has been given to the sizes of these shocks (for an exception, see the EGARCH model of Nelson (1991)).

In this paper we provide a different solution, including a size effect in a special type of threshold GARCH model which generalises GJR to allow for multiple thresholds. Furthermore, we provide a second innovation by adding dynamics in the threshold structure to enforce variance persistence that is directly related to both the size and sign of shocks.

This paper develops the Dynamic Asymmetric GARCH (DAGARCH) model. We provide the stationarity conditions of the model and an empirical example for stock market indices to demonstrate the usefulness of the new model.

2 Dynamic Asymmetric GARCH: DAGARCH

There is no theoretical or empirical reason to assume that the conditional distribution of asset returns is symmetric, with positive and negative shocks having the same impact on conditional variances. Furthermore, a similar assumption cannot be sustained for the size of shocks. In the class of GARCH models that deals explicitly with volatility asymmetry, we present the Dynamic Asymmetric GARCH (DAGARCH) model which can be used to anal-
yse the size and sign effects within a GARCH structure. Furthermore, the DAGARCH model enables an analysis of the persistence of the size and sign effects of shocks by adding dynamics to volatility asymmetry.

The conditional mean and conditional variance of DAGARCH(1,1) are given as follows:

\[ y_t - \mu \left( I_t^{t-1} \right) = \varepsilon_t = z_t \sigma_t \]  
\[ \sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \gamma_{t-1} \varepsilon_{t-1}^2 \]  
\[ \gamma_t = \begin{cases} 
\bar{\gamma}_1 + \phi_1 \gamma_{t-2} & \varepsilon_{t-1} > d_1 \\
\bar{\gamma}_2 + \phi_2 \gamma_{t-2} & d_2 < \varepsilon_{t-1} < d_1 \\
\vdots & \vdots \\
\bar{\gamma}_{m-1} + \phi_{m-1} \gamma_{t-2} & d_{m-1} < \varepsilon_{t-1} < d_{m-2} \\
\bar{\gamma}_m + \phi_m \gamma_{t-2} & \varepsilon_{t-1} < d_{m-1} 
\end{cases} \]  
\[ \omega > 0, \quad \beta > 0, \quad d_1 > d_2 > \ldots d_{m-2} > d_{m-1} \]  

We do not make any assumptions regarding the dynamic evolution of the mean. Equation (1) defines the relation between the observed series, the conditional variances \( \sigma_t^2 \), and the standardised residuals \( z_t \). Equation (2) refers to the dynamic evolution of the conditional variances, while equation (3) focuses on the dynamic asymmetry.
Assumption 1: The standardised residuals are independently and identically distributed, with probability density function \( f(z_t) \), cumulative distribution function \( F(z_t) \), satisfying \( E[z_t] = 0, E[z_t^2] = 1 \), and with a non-degenerate distribution.

Let \( c(z_t) = \beta + \gamma_t z_{t-1}^2 \). Using the results of Ling and McAleer (2002a,b), which extend and correct the results in He and Terasvirta (1999a,b), and the results in McAleer, Chan and Marinova (2003), we can state the following:

**Theorem 1** Under Assumption 1, if \( E[c(z_t)]^\delta < 1 \) for some \( \delta \in (0,1] \), then there exists a unique, strictly stationary and ergodic solution of the DAGARCH model, with the following causal expansion:

\[
\sigma_t^2 = \omega \left[ 1 + \sum_{k=0}^{\infty} \prod_{j=0}^{k} c(z_t) \right],
\]

where the infinite sum converges almost surely.

**Proof.** Define the function \( \Psi(\delta) = E \left\{ [c(z_t)]^\delta \right\} \). Under Assumption 1, the standardised residuals have finite second moment, so that the function \( \Psi(\delta) \) is at least twice differentiable. Furthermore,

\[
\Psi'(\delta) = \frac{d}{d\delta} \int [c(z_t)]^\delta f(z_t) \, dz_t = \frac{d}{d\delta} [c(z_t)]^\delta f(z_t) \, dz_t = E \left\{ \ln [c(z_t)] [c(z_t)]^\delta \right\}
\]
\[ \Psi''(\delta) = E \left\{ (\ln[c(z_t)])^2 [c(z_t)]^{\delta} \right\}. \]

Noting that \( \Psi''(\delta) > 0 \), since its components take only positive values, and that \( E \left\{ \ln[c(z_t)] [c(z_t)]^{\delta} \right\} \) collapses to \( E \{ \ln[c(z_t)] \} \) when \( \delta = 0 \), we can verify the following:

i) by Jensen’s inequality, it follows that \( E \{ \ln[c(z_t)] \} < 0 \) when \( \delta = 0 \);

ii) by the fact that \( \Psi(0) = 1 \), for small \( \delta \), the function \( \Psi(\delta) \) must be less than 1 (given convexity and first derivative), so that there exists some \( \delta \in (0, 1] \) that satisfies \( E[c(z_t)]^\delta < 1 \);

iii) finally, by applying Theorem 2.1 in Ling and McAleer (2002a), we obtain the stated result.

Let us redefine the thresholds by using a compact formulation:

\[ S_j = I (d_j < \varepsilon_{t-1} < d_{j-1}), \] where \( I (\cdot) \) is the indicator function.

Remark 1 Using \( d_j^* \sigma_{t-1} = d_j \), where \( d_j^* \) is an unobservable ‘standardised’ threshold and \( r_j^* \) a set defined over the \( d_j^* \) values, the following equalities hold:

\[ S_j = I (d_j < \varepsilon_{t-1} < d_{j-1}) = I (d_j < \sigma_{t-1}z_{t-1} < d_{j-1}) = I (d_j^* < \sigma_{t-1} \sigma_{t-1} < d_{j-1}^*) = I (d_j^* < z_{t-1} < d_{j-1}^*) = S_j^* \]

and

\[ S_j^* = I (z_{t-1} \in r_j^*). \]
Remark 2 Whenever the model is re-cast in the standard GARCH representation, that makes explicit the ARCH parameter, a zero identification restriction must be imposed on one of the $\bar{\gamma}_j$ coefficients.

Remark 3 A set of sufficient conditions for the positivity of the conditional variances is the following:

$$\omega > 0, \quad \beta \geq 0, \quad \bar{\gamma}_j \geq 0 \quad \text{and} \quad \phi_j \geq 0 \quad \text{for} \quad j = 1, 2, ..., m.$$  

Assumption 2: The thresholds $d^*_j$ are known (or fixed) a priori or, alternatively, it is assumed that the standardised residuals follow a given distribution such as a standardised normal, Student-t with $k$ degrees of freedom, or GED.

The $d^*_j$ thresholds can be set in order to identify the tails of the standardised distribution, or to select positive and negative values. As an example, the lower threshold $d^*_{m-1}$ can be set to $F^{-1}(\alpha)$ to identify the lower $\alpha$-tail of the distribution.

Remark 4 It follows from Theorem 2.1 of Ling and McAleer (2002a) that the necessary and sufficient condition for the existence of the second moment solution is $E[c(z_t)] < 1$. 

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Theorem 2 The necessary and sufficient condition for the existence of the second moment of DAGARCH(1,1) is

\[
\beta + \sum_{j=1}^{m} \bar{\gamma}_j e_j + G \sum_{j=1}^{m} \phi_j e_j < 1
\]

where

\[
G = \frac{\sum_{j=1}^{m} \bar{\gamma}_j m_j}{1 - \sum_{j=1}^{m} \phi_j m_j}
\]

\[
m_j = E[S^*_j]
\]

\[
e_j = E[z_{t}^{2} S^*_j].
\]

Note that the last two equations define scalars that depend on the assumptions for the standardised residual distribution and on the thresholds.

Proof. Recall that \(c(z_t) = \beta + \gamma_t z_t^2\). Taking expectations of both sides gives

\[
E[c(z_t)] = \beta + E[\gamma_t z_t^2].
\]
Focusing on the expectation of the dynamic asymmetry term, we have

\[
E[\gamma_t z_t^2] = E \left\{ \sum_{j=1}^{m} [\bar{\gamma}_j + \phi_j \gamma_{t-1}] z_t^2 I(z_t \in r^*_j) \right\}
\]

\[
= E \left\{ \sum_{j=1}^{m} [\bar{\gamma}_j + \phi_j \gamma_{t-1}] z_t^2 I_j(z_t) \right\}
\]

\[
= \sum_{j=1}^{m} \bar{\gamma}_j E[z_t^2 I_j(z_t)] + \sum_{j=1}^{m} \phi_j E[\gamma_{t-1} z_t^2 I_j(z_t)].
\]

As \(\gamma_{t-1}\) is a function of \(z_{t-1}\), given the assumption of temporal independence of the \(z_t\), it follows that

\[
E[\gamma_t z_t^2] = \sum_{j=1}^{m} \bar{\gamma}_j E[z_t^2 I_j(z_t)] + \sum_{j=1}^{m} \phi_j E[\gamma_{t-1} z_t^2 I_j(z_t)].
\]

Defining \(E[z_t^2 I_j(z_t)] = e_j\) (which is obtained by direct computation, given the assumption on the distribution of the standardised residuals and knowledge of the \(d_j^*\) thresholds), upon substitution yields:

\[
E[\gamma_t z_t^2] = \sum_{j=1}^{m} \bar{\gamma}_j e_j + \sum_{j=1}^{m} \phi_j e_j E[\gamma_{t-1}].
\]

We then compute the expectation of the Dynamic Asymmetry effect, which
is given by

\[
E[\gamma_t] = E\left\{ \sum_{j=1}^{m} [\bar{\gamma}_j + \phi_j \gamma_{t-1}] I_j(z_t) \right\}
\]

\[
= \sum_{j=1}^{m} E[\bar{\gamma}_j I_j(z_t)] + \sum_{j=1}^{m} E[\phi_j I_j(z_t) \gamma_{t-1}]
\]

\[
= \sum_{j=1}^{m} \bar{\gamma}_j E[I_j(z_t)] + \sum_{j=1}^{m} \phi_j E[I_j(z_t)] E[\gamma_{t-1}].
\]

Then, unconditionally \( E[\gamma_t] = E[\gamma_{t-j}] \) and, using \( E[I_j(z_t)] = m_j \) (that is, the probability of occurrence of each interval), gives

\[
E[\gamma_t] = \sum_{j=1}^{m} \bar{\gamma}_j m_j + E[\gamma_t] \sum_{j=1}^{m} \phi_j m_j,
\]

which can be solved as

\[
E[\gamma_t] = \frac{\sum_{j=1}^{m} \bar{\gamma}_j m_j}{1 - \sum_{j=1}^{m} \phi_j m_j} = G.
\]

Collecting all terms gives

\[
E[\gamma_t^2] = \sum_{j=1}^{m} \bar{\gamma}_j e_j + G \sum_{j=1}^{m} \phi_j e_j
\]
and

\[ E[c(z_t)] = \beta + \sum_{j=1}^{m} \bar{\gamma}_j e_j + G \sum_{j=1}^{m} \phi_j e_j, \]

which completes the proof. ■

Remark 5 Note that when the dynamic asymmetric autoregressive terms are zero (that is, \( \phi_j = 0 \) for all \( j \)), this model collapses to a multiple threshold GJR-type representation. Furthermore, when the threshold is unique (that is, \( m = 1 \)), we can derive the GJR model (\( d_0 = 0 \)) as a special case by imposing symmetry on the standardised residual.

Claim 1 The ARMA representation of DAGARCH has a SETARMA structure.

Proof. Setting \( v_t = \varepsilon_t^2 - \sigma_t^2 \) and substituting gives

\[ \varepsilon_t^2 = \omega + \beta \varepsilon_{t-1}^2 + \sum_{j=1}^{m} \left[ \bar{\gamma}_j + \phi_j \gamma_{t-2} \right] \varepsilon_{t-1}^2 S_j + (1 - \beta L) v_t, \]

where the thresholds are self-exciting as \( d_j = d_j^* \sigma_{t-1}^2 = d_j^* (\varepsilon_{t-1}^2 - v_t) \). ■
Claim 2  The unconditional variance implied by DAGARCH is given by

\[ \hat{\sigma}^2 = \frac{\omega}{1 - \beta - \sum_{j=1}^{m} \bar{\gamma}_j e_j - G \sum_{j=1}^{m} \phi_j e_j}. \]

Proof. This follows from the stationarity condition and the SETARMA representation.

In the following, for practical purposes, we will label the model as DAGARCH(q,m,d) where \( q \) represents the GARCH order (while the ARCH order is constrained to be 1), \( m \) is the number of asymmetric constant terms, and \( d \) is the number of asymmetric dynamic coefficients (which can be assumed constant over the thresholds, leading to \( d \leq m \)). It is noted that DAGARCH(1,2,0) with the threshold set to zero is simply GJR.

3 Feasible estimation of DAGARCH

In the GJR model, the thresholds are stable over time as they are derived under the assumption of a symmetric distribution. In this case, we have

\[
E[\gamma_1 z_t^2] = E[\bar{\gamma}_1 z_t^2 I(z_t < 0) + \bar{\gamma}_2 z_t^2 I(z_t > 0)]
\]

\[
= \bar{\gamma}_1 E[z_t^2 I(z_t < 0)] + \bar{\gamma}_2 E[z_t^2 I(z_t > 0)],
\]
and given a symmetric $z_t$ density function, we can write

$$E \left[ z_t^2 \right] = E \left[ z_t^2 I (z_t < 0) \right] + E \left[ z_t^2 I (z_t > 0) \right]$$

$$= 2 \cdot E \left[ z_t^2 I (z_t < 0) \right],$$

$$E \left[ z_t^2 I (z_t < 0) \right] = E \left[ z_t^2 I (z_t > 0) \right] = \frac{1}{2} E \left[ z_t^2 \right],$$

which is then used to derive the standard GJR stationarity restriction. Furthermore, the same derivation can be recast using $\varepsilon_t^2$ instead of $z_t^2$ since the zero threshold is not influenced by the value of the conditional variance, that is:

$$E \left[ z_t^2 I (z_t < 0) \right] = E \left[ \varepsilon_t^2 I (\varepsilon_t < 0) \right].$$

In the DAGARCH model, the last inequality does not generally hold as the conditional variances are relevant in the definition of the thresholds. This can be observed directly in the $d_i$ thresholds, or indirectly in the $d_i^*$ thresholds through the derivation of the standardised residuals, which are not observable.

Therefore, the thresholds for the observable $\varepsilon_t$ are time varying, creating a relevant computational problem in that the standard recursive estimation
approaches of GARCH-type models cannot be used. Furthermore, the $d_i$ thresholds depend both on the conditional variances and on the $d_i^*$ thresholds. In fact, as shown in Remark 1, $d_i = d_i^*\sigma_{t-1}$, and the thresholds on the observable are time dependent. These $d_i^*$ values can be determined only on the basis of a distributional assumption on the $z_t$ and should be fixed, considering the research purposes. Assuming that the interest is in comparing the effects of the tails and/or the positive and negative observable innovations, the various $d_i^*$ should be able to select innovations in the upper (or lower) $\alpha\%$ tail. Consequently, we have a computational problem that is caused by two factors, namely an assumption on the standardised residuals which are not directly observed, and a set of time-varying thresholds which depends on the model parameter and specification.

In order to solve both of these problems, we propose two solutions in the form of estimation algorithms.

**Algorithm A.** Simplifying the DAGARCH thresholds:

1) make the $d_i$ thresholds dependent on the unconditional observable variance, that is, $d_i = d_i^*\left(\frac{1}{N}\sum_{j=1}^{N} z_j^2\right)$ and assume a priori a Student-t or a GED distribution for the standardised residuals;

2) compute the $d_i^*$ thresholds under the assumed distribution;

3) estimate DAGARCH;

4) test the assumed distribution for the obtained standardised residual and, if necessary, repeat the procedure from step 2).

**Algorithm B.** Using a procedure that computes thresholds on the basis
of a simpler previously estimated model:

1) assume a Student-t or a GED distribution for the standardised residuals and compute the $d^*_i$ thresholds;

2a) estimate a standard GARCH model under the distributional assumption of step 1) and save the estimated conditional variances as $GARCH^2_i$;

2b) test the distributional assumption of the standardised residuals and, if necessary, update the $d^*_i$ thresholds;

2c) compute the $d_i$ thresholds using $GARCH^2_i$ and $d^*_i$;

3a) estimate DAGARCH with a zero restriction on the dynamic asymmetric autoregressive terms and save the conditional variances as $DAGARCH,1^2_i$;

3b) test the distributional assumption of standardised residuals and, if necessary, update the $d^*_i$ thresholds;

3c) compute the $d_i$ thresholds using $DAGARCH,1^2_i$ and $d^*_i$;

4a) estimate DAGARCH with an equality restriction on the dynamic asymmetric autoregressive terms and save the conditional variances as $DAGARCH,2^2_i$;

4b) test the distributional assumption of standardised residuals and, if necessary, update the $d^*_i$ thresholds;

4c) compute the $d_i$ thresholds using $DAGARCH,2^2_i$ and $d^*_i$;

5a) estimate a full DAGARCH and save the conditional variances as $DAGARCH,3^2_i$;

5b) test the distributional assumption of standardised residuals and, if necessary, update the $d^*_i$ thresholds;

5c) compute the $d_i$ thresholds using $DAGARCH,3^2_i$ and $d^*_i$;

6) if necessary, repeat step 5) until convergence is achieved.
Clearly, the full estimation of the model is possible, even though computationally demanding. The second solution is computationally more intensive but it allows a comparison of the likelihoods obtained by the different approaches. Furthermore, the error included in the time-varying threshold will be lower than in the first case which uses static thresholds. Finally, the result of step 6) can be compared with the estimation output of an unrestricted DAGARCH model with endogenous determination of the thresholds (which can be viewed as a possible step 7) with parameter starting values given by the step 6) results).

4 Dynamic asymmetry in daily stock indices

This section provides an empirical analysis of the variance asymmetry of stock market indices. We consider the following daily indices: Dow Jones industrials (DJ), Nikkei 225 (NK), Eurostoxx 50 (EX) and FTSE 100. Table 1 reports the sample period, the number of observations (obs.) and the sample moments of the logarithmic returns. The means and standard deviations (SD) of all four series are similar; the skewness of DJ is similar to that of FTSE, and that of EX is similar to NK; and the kurtosis of FTSE is similar to that of NK.

[Insert Table 1 here]

Using these logarithmic returns series, we estimated several specifications of the DAGARCH model and compared them with the standard GARCH(1,1)
and GJR(1,1) models. In particular, we considered GARCH, GJR (labelled DAGARCH(1,2,0)), GJR with common or specific dynamic asymmetry (DAGARCH(1,2,1) and DAGARCH(1,2,2), respectively), and two DAGARCH models with three thresholds (DAGARCH(1,4,0) and DAGARCH(1,4,1)). The DAGARCH(1,4,4) model was also considered but is not reported here as the dynamics appeared to be stable over the different thresholds.

The following tables report the various estimated models for the indices given in Table 1. Not all the estimated models are reported. In some cases, DAGARCH with three thresholds is preferred to the GJR representation, while in other cases the generalisation of the GJR with asymmetric dynamic is the preferred model. Figure 1 reports a comparison of the asymmetric impacts with and without dynamics. The models are preferred in terms of likelihoods using the standard LR test, as all models are nested. An interesting result concerns the asymmetric dynamics which are present for the negative returns values.

[Insert Figure 1 here]

[Insert Tables 2-5 here]

5 Concluding remarks

This paper developed the Dynamic Asymmetric GARCH (or DAGARCH) model which generalises the GJR model to allow for multiple thresholds and dynamics in the asymmetric conditional variance. The DAGARCH model
accommodates both the size and sign effects of shocks and induces some
dynamics in the variance asymmetry. The stationarity conditions are derived,
as well as two algorithms for model estimation. An empirical application is
also provided, with several DAGARCH specifications being estimated using
the daily returns of the Dow Jones, Nikkei225, FTSE100 and EuroStoxx50
indices to show the usefulness of the new model.

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Figure 1: FTSE Index - Dynamic Evolution of $\gamma_t$ for the period November 2002 - March 2004
Table 1: Descriptive Statistics of Logarithmic Returns

<table>
<thead>
<tr>
<th>Index</th>
<th>Sample</th>
<th>Obs.</th>
<th>Mean</th>
<th>SD</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
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<tbody>
<tr>
<td>DJ</td>
<td>02/01-30/04</td>
<td>18664</td>
<td>0.00020</td>
<td>0.01122</td>
<td>-0.54372</td>
<td>29.122</td>
</tr>
<tr>
<td>FTSE</td>
<td>02/04-08/04</td>
<td>5059</td>
<td>0.00028</td>
<td>0.01057</td>
<td>-0.53799</td>
<td>10.781</td>
</tr>
<tr>
<td>EX</td>
<td>01/01-08/04</td>
<td>3144</td>
<td>0.00033</td>
<td>0.01346</td>
<td>-0.11034</td>
<td>6.548</td>
</tr>
<tr>
<td>NK</td>
<td>04/01-11/05</td>
<td>5008</td>
<td>0.00001</td>
<td>0.01402</td>
<td>-0.10905</td>
<td>10.642</td>
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</tbody>
</table>
Table 2: Nikkei estimates (insignificant estimates are in bold)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>GARCH(1,1)</th>
<th>DAGARCH(1,2,0)</th>
<th>DAGARCH(1,2,1)</th>
<th>DAGARCH(1,2,2)</th>
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<td>ω</td>
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<td>0.01629</td>
<td>0.01669</td>
<td>0.01646</td>
</tr>
<tr>
<td></td>
<td>0.00618</td>
<td>0.00543</td>
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<tr>
<td>β</td>
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<td>0.86261</td>
<td>0.86135</td>
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<tr>
<td></td>
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<td>0.03269</td>
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<td>γ₁ (α)</td>
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<td>0.22243</td>
<td>0.20472</td>
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</tr>
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<td></td>
<td>0.04436</td>
<td>0.06840</td>
<td>0.05761</td>
<td>0.04027</td>
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<tr>
<td>γ₂</td>
<td></td>
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<td></td>
</tr>
<tr>
<td></td>
<td>0.04890</td>
<td>0.03774</td>
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<td></td>
<td>0.01478</td>
<td>0.01220</td>
<td>0.03104</td>
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<td>φ₁</td>
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<td>-6329.016</td>
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</table>

γ₁ (α) reports the ARCH(1) parameter in GARCH(1,1), the constant asymmetry parameter for negative returns values in DAGARCH(1,2,j), and the constant asymmetry parameter for large negative returns in DAGARCH(1,4,j).

γ₂ reports the constant asymmetry parameter for small negative returns values in DAGARCH(1,4,j).

γ₃ reports the constant asymmetry parameter for positive returns values in DAGARCH(1,2,j), and the constant asymmetry parameter for small positive returns values in DAGARCH(1,4,j) estimates.

γ₄ reports the constant asymmetry parameter for large positive returns values in DAGARCH(1,4,j).

φ₁ reports the dynamic asymmetry parameter in DAGARCH(1,j,1), and the dynamic asymmetry parameter for negative returns values in DAGARCH(1,2,2).

φ₂ reports the dynamic asymmetry parameter for positive returns values in DAGARCH(1,2,2).
Table 3: Dow Jones estimates (insignificant estimates are in bold)

<table>
<thead>
<tr>
<th>Parameters</th>
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<th>DAGARCH(1,2,1)</th>
<th>DAGARCH(1,2,2)</th>
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<td>0.01181</td>
<td>0.01169</td>
<td>0.01153</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.90409</td>
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<td>0.90897</td>
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<tr>
<td>$\gamma_1 (\alpha)$</td>
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<td>0.01644</td>
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<tr>
<td>LogLik.</td>
<td>61857.507</td>
<td>62002.810</td>
<td>62005.037</td>
<td>62006.75</td>
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See the footnotes for Table 2.
Table 4: FTSE estimates (insignificant estimates are in bold)

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<tr>
<th>Parameters</th>
<th>GARCH(1,1)</th>
<th>DAGARCH(1,2,0)</th>
<th>DAGARCH(1,2,1)</th>
<th>DAGARCH(1,2,2)</th>
<th>DAGARCH(1,4,0)</th>
<th>DAGARCH(1,4,1)</th>
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</thead>
<tbody>
<tr>
<td>( \omega )</td>
<td>0.03440</td>
<td>0.03296</td>
<td>0.03363</td>
<td>0.03293</td>
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<td>( \beta )</td>
<td>0.01070</td>
<td>0.01000</td>
<td>0.00995</td>
<td>0.00972</td>
<td>0.01022</td>
<td>0.01004</td>
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<tr>
<td>( \gamma_1 (\alpha) )</td>
<td>0.85477</td>
<td>0.87046</td>
<td>0.86989</td>
<td>0.87325</td>
<td>0.86768</td>
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<td>0.02513</td>
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<td>0.02280</td>
<td>0.02279</td>
<td>0.02433</td>
<td>0.02368</td>
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<tr>
<td>( \gamma_2 )</td>
<td>0.01070</td>
<td>0.01000</td>
<td>0.00995</td>
<td>0.00972</td>
<td>0.01022</td>
<td>0.01004</td>
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<tr>
<td>( \gamma_3 )</td>
<td>0.01597</td>
<td>0.01997</td>
<td>0.01977</td>
<td>0.02738</td>
<td>0.02898</td>
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<td>( \gamma_4 )</td>
<td>0.10699</td>
<td>0.13197</td>
<td>0.09167</td>
<td>0.4840</td>
<td>0.12507</td>
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<td>( \phi_1 )</td>
<td>0.02172</td>
<td>0.02278</td>
<td>0.04990</td>
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<td>0.02255</td>
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<td>( \phi_2 )</td>
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</table>

See the footnotes for Table 2.
Table 5: Eurostoxx estimates (insignificant estimates are in bold)

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See the footnotes for Table 2.