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Abstract
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Keywords
Performance Measure, Omega, Return Distribution, Risk, Stochastic Dominance.

JEL Codes
C10, C11, G12.

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"On the (Ab)Use of Omega?"☆

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1. Introduction

A relatively wide stream of the financial economics literature focuses on performance measurement with two main motivations: the introduction of measures capturing stylized facts of financial returns such as asymmetry or non-Gaussian densities, among many others; the use of performance measures in empirical studies dealing with managed portfolio evaluation or asset allocation. The interest in this research field started with the seminal contribution of Sharpe in 1966. An increasing number of studies appeared in the following decades (see the survey by Caporin et al., 2013). One of the most studied topics is the performance evaluation of active management, which probably represents the main element pushing a renewed interest in performance measurement. Among the most recent contributions, we can quote Cherny and Madan (2009), Capocci (2009), Darolles et al. (2009), Jha et al. (2009), Jiang and Zhu (2009), Stavetski (2009), Zakamouline and Koekebakker (2009), Darolles and Gouriéroux (2010), Glawischnig and Sommersguter-Reichmannn (2010), Billio et al. (2012a), Billio et al. (2012b), Cremers et al. (2013), Kapsos et al. (2014), Weng (2014), and Billio et al. (2014).

Performance evaluation has relevant implications from both a theoretical point of view, as it allows us to understand agent choices, and an empirical one, since practitioners are interested in ranking assets or managed portfolios according to a specific non-subjective criterion. As an example, financial advisors often rank mutual funds according to a specific performance measure. Moreover, when rankings produced by advisors are recognized as a reference by the investors, changes in the rankings might influence inflows and outflows (see Hendricks et al., 1993; Blake and Morey, 2000; Powell et al., 2002; Del Guercio and Tkac, 2008; Jagannathan et al., 2010).

A large number of performance measures have already been proposed and, as a consequence, related ranks can sensibly vary across different financial advisors, portfolio managers and investment institutions. The identification of the most appropriate performance measure depends on several elements. Among them, we cite the investors’ preferences and the properties or features of the analyzed asset/portfolio returns. Furthermore, the choice of the “optimal” performance measure, across a number of alternatives, also depends on
the purpose of the analysis which might be, for instance, one of the following: an investment decision, the evaluation of managers’ abilities, the identification of management strategies and of their impact, either in terms of deviations from the benchmark or in terms of risk/return.

Despite some limitations, the Sharpe (1966) ratio is still considered as the reference performance measure. If we derive this ratio within a Markowitz framework, it shares the same drawbacks as the Mean-Variance model, where the representative investor is characterized by a quadratic utility function and/or the portfolio returns are assumed to be Gaussian.\textsuperscript{1} However, it is well known that financial returns are not Gaussian, also due to investment strategies based on derivatives with time-varying exposures and leverage effects. An incorrect assumption of Gaussianity may lead to an underestimation of the portfolio total risk (see Fung and Hsieh, 2001; Lo, 2001; Mitchell and Pulvino, 2001; Brooks and Kat, 2002; Agarwal and Naik, 2004) and, thus, to biased investment rankings (a downward biased risk evaluation induces an upward biased Sharpe ratio). Thus, the Sharpe ratio does not completely reflect the attitude towards risk for all categories of investors. Furthermore, the standard deviation equally weights positive and negative excess returns. In addition, it has been shown that volatility can be subject to manipulations (see Ingersoll et al., 2007). Using this quantity to evaluate the risk of assets with low liquidity can also be another issue (see Getmansky et al., 2004). Ultimately, the pertinence of using this performance measure relies heavily on the accuracy and stability of the first and second moment estimations (Merton, 1981; Engle and Bollerslev, 1986). Moreover, the so-called “µ – σ Paradox” illustrates that the Sharpe ratio is not consistent, in the sense of the Second-order Stochastic Dominance criterion as introduced by Hadar and Russell (1969), when return distributions are not Gaussian (see Copeland and Weston, 1998; Hodges, 1998).

Based on these critics, academics and practitioners proposed alternative criteria that can be adapted to more complex settings than the Gaussian one. In recent years, different authors widely used the performance measure introduced by Keating and Shadwick

\textsuperscript{1}The Gaussianity assumption might be relaxed when considering, for instance, a Student-\(t\) density characterized by a specific degree of freedom.
(2002), called *Omega*, when ordering investment portfolios according to a rational criterion. The first ones consider *Omega* in the evaluation of active management strategies in contrast to the well-known Sharpe ratio, supporting their choice by the non-Gaussianity of returns and by the inappropriateness of volatility as a risk measure when strategies are non-linear (*e.g.* Eling and Schuhmacher, 2007; Annaert et al., 2009; Hamidi et al., 2009; Bertrand and Prigent, 2011; Ornelas et al., 2012; Zieling et al., 2014; Hamidi et al., 2014).

A second group of articles focuses on *Omega* as an objective function for portfolio optimization in order to introduce downside risk in the estimation of optimal portfolio weights (*e.g.* Mausser et al., 2006; Farinelli et al., 2008, 2009; Kane et al., 2009; Gilli and Schumann, 2010; Hentati et al., 2010; Gilli et al., 2011). The maximization program in an *Omega* paradigm is indeed, *per* nature, a non-convex non-linear one. As formulated in Mausser et al. (2006), the investor’s problem in finding the portfolio that optimizes *Omega* can be written, under some mild assumptions, as a linear one, after some variable changes, inspired by the works of Charnes and Cooper (1962) in the field of production optimization (see Appendix A). More generally, the investor problem of the maximization of *Omega* can be solved using specialized software for non-linear programming, which looks for a portfolio satisfying a system of inequalities (related to Kuhn-Tucker conditions) that are necessary (but not, in general, sufficient) for optimality. Such solvers can only find a “local” maximum, not necessarily the portfolio giving the true “global” solution (that is, in our case, the portfolio having the largest *Omega*). Some empirical studies report that this measure is more stable than other risk measures (*e.g.* Hentati et al., 2010), but has many local solutions because of the non-convexity of the *Omega* function. The resolution of the global optimum is proposed by Kane et al. (2009) using the Huyer and Neumaier MCS method. Based on another approach, Hentati and Prigent (2011) introduce Gaussian mixtures to model empirical distributions of financial assets and solve the portfolio optimization problem in a static way, taking account of discrete time portfolio rebalancing.

Similar to Copeland and Weston (1998) and Hodges (1998), who illustrate the in-

\(^2\)See also Bernardo and Ledoit (2000).
consistency of the Sharpe ratio, we show in the present article that the Omega measure has some severe drawbacks. In fact, it might conduct investors to misleading rankings of risky assets, even in an extremely simplified framework such as the Gaussian one.\(^3\) The main ideas that we present hereafter are: 1) the Omega measure is biased in some cases due to the importance of the mean return in its computation and 2) since gains can be compensated by losses, two densities with (almost) the same means can give identical Omega measures, even when the risk related to one asset is obviously higher than the other one. These problems mainly come from the fact that Omega does not comply, in general, with the Strict Inferior Second-order Stochastic Dominance criterion\(^4\) (SISSD, in short), which is entailed by traditional rational agent behavior. This might imply an indifference across assets with the same expected return but with different risk levels.

The paper proceeds as follows. Section 2 introduces the Omega function in the general context of performance measurement and presents some of its properties. Section 3 demonstrates that the Omega measure is inconsistent with the Strict Inferior Second-order Stochastic Dominance criterion. Then, we show that the trade-off of the Omega ratio – between return and risk – is mostly guided by the first moment of portfolio returns. In Section 4, we provide numerous illustrations, based on realistic simulations and several asset classes, showing incoherences in fund analyses when considering various (static and dynamic) optimization settings. Section 5 presents some robustness checks based on different long-only asset and hedge fund databases. The last section concludes. We also include complementary results in the appendices dedicated to the simulation schemes used in the corpus of the text, proofs and some extra robustness results.

2. Return Density-based Performance and the Omega Measures

The Omega measure belongs to a general class of performance measures based on features of the analyzed return density. Following the taxonomy proposed by Caporin

\(^3\)Note here that considering more general skewed and/or leptokurtic distributions will lead to the same qualitative conclusions.

\(^4\)Cf. footnotes 7 and 8, and Appendix B for the precise definitions of the Strict Inferior First-order and Second-order Stochastic Dominance criteria (denoted hereafter, respectively, SIFSD and SISSD).
et al. (2013), these measures can be represented with the following general notations:

\[ PM_p = P_p^+ (r) \times [ P_p^- (r) ]^{-1}, \]  

where \( P_p^+ (.) \) and \( P_p^- (.) \) are two functions associated with the right and left parts of the support of the density of returns.

In most cases, measures belonging to this class can be re-defined as ratios of two probability-weighted Power Expected Shortfalls (or Generalized Higher/Lower Partial Moments), which read:

\[
PM_p = \left[ E_p (|\tau_1 - r|^{o_1} | r > \tau_3) \right]^{(k_1)}^{-1} \times \left\{ \left[ E_p (|\tau_2 - r|^{o_2} | r \leq \tau_4) \right]^{(k_2)}^{-1} \right\}^{-1} \times \psi_p [f_p (r)],
\]

where \( \tau_1 \) is a threshold (a reserve return, a Minimum Acceptable Return – MAR, the null return, the risk-free rate \( r_f \)... for computing gains; \( \tau_2 \) is a similar threshold for calculating losses or risk; \( \tau_3 \) and \( \tau_4 \) are thresholds allowing the performance measure to focus more or less on the upper/lower part of the support density; constants \( o_1 \) and \( o_2 \) are intensification parameters monitoring the attitude of the investor toward gains and losses, while \( k_1 \) and \( k_2 \) are positive normalizing constants; finally \( \psi_p [f_p (r)] \) is a factor depending on the density of the portfolio returns, \( f_p (r) \), where \( r \) denotes the portfolio returns.

Note that some thresholds might be defined based on quantile regressions (i.e. Value-at-Risk), such as \( \tau_3 = VaR_{r_p, a_1} \) and \( \tau_4 = VaR_{r_p, a_2} \) with \( a_1 \) and \( a_2 \) being confidence levels.

Equation (2) thus highlights that measures belonging to this class capture features of the return density that are going beyond the first two moments.\(^6\) The Keating and Shadwick (2002) \textit{Omega} measure, following Bernardo and Ledoit (2000), corresponds to

\[^5\]The most general form of this factor is \( [1 - F_p (\tau)]^{k_3} / [F_p (\tau)]^{k_4} \) where \( k_3 \) and \( k_4 \) are normalizing constants.

\(^6\)See Caporin et al. (2013) for further discussions on this class of measures and for a list of derived measures.
the case where $\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau$ and $o_1 = o_2 = k_1 = k_2 = 1$. We have thus a unique threshold and all intensification and normalizing constants are fixed at 1. *Omega* indeed writes:\footnote{Several alternative expressions of the *Omega* measure exist in the literature (Cf. Appendix C).}

$$
\Omega_p(\tau) = (GHPM_{r_p,\tau,\tau,1}) \times (GLPM_{r_p,\tau,\tau,1})^{-1}
= E_p[r - \tau \mid r > \tau] \times \{E_p[\tau - r \mid r \leq \tau]\}^{-1} \times \left[ \frac{1 - F_p(\tau)}{F_p(\tau)} \right],
$$

where $GHPM_{r_p,\tau,\tau,1}$, $GLPM_{r_p,\tau,\tau,1}$ and $E[\cdot \cdot \cdot]$ are, respectively, the Higher/Lower Partial Moments and the conditional expectation operator, and $F_p(\cdot)$ is the Cumulative Density Function of the return $r$.

We see, here, that the *Omega* ratio separately considers, in a simple and intuitive way, favorable and unfavorable potential excess returns with respect to a threshold that has to be given (arbitrary). More precisely, as noted by Kazemi et al. (2004) and Bertrand and Prigent (2011), the *Omega* measure is equal to the ratio of the expectations (under the historical probability $P$) of a call option to a put option written on the risky reference asset with a strike price corresponding to the threshold. The main advantage of the *Omega* measure is that it incorporates, in some ways, some features of the return distribution, such as moments, including skewness and kurtosis. A ranking is theoretically always possible, whatever the threshold, in contrast to the Sharpe ratio where the threshold is fixed and equal to the riskless return. Furthermore, it displays some properties such as (see Kazemi et al., 2004; Bertrand and Prigent, 2011):

- for any portfolio $p$ (with a symmetric return distribution), $\Omega_p(\tau) = 1$ when $\tau = E_p[r]$;

- for any portfolio $p$ (and for all distributions), $\Omega_p(\cdot)$ is a monotonous decreasing function in $\tau \in \mathbb{R}$;

- for any couple of portfolios $p = \{A, B\}$, $\Omega_A(\cdot) = \Omega_B(\cdot)$ for all $\tau \in \mathbb{R}$, if and only if $F_A(\cdot) = F_B(\cdot)$, where functions $F_p(\cdot)$ is the Cumulative Density Function of the returns on a portfolio $p$;
• for any portfolio $p$ and if there exists one risk-free asset $p_0 = 0$ with return $r_{p_0}$, then

$$\Omega_p(\cdot) < \Omega_{p_0}(\cdot) \text{ for all } \tau \leq r_{p_0}, \text{ with } \Omega_{p_0}(\cdot) = +\infty.$$  

In this setting, the threshold $\tau$ must be exogenously specified as it may vary according to investment objectives and individual preferences. As mentioned by Unser (2000), we are often only interested in the evaluation of “risky” outcomes that reflect the attitude towards downside risk. Usually, their values are smaller than a given target, which is, for example, the riskless asset or the rate of a financial index (benchmark). Downside risk measures have been examined, for instance, in Ebert (2005), and are linked to the measures introduced by Fishburn (1977 and 1984). For the background literature on risk measures and their applications to finance and insurance, we refer to Kaas et al. (2004), Denuit et al. (2006) and Prigent (2007). Regarding performances measures, Omega and Kappa measures are among those that are very often used in financial applications and involve downside risk measures (Cf. Caporin et al., 2013). For Omega, the downside risk is clearly associated with the denominator of the general formulation given in Equation (3). Furthermore, the links between risk measures (and, thus, implicitly also performance measures coherent with Equation (2)) and the Cumulative Prospect Theory (CPT in short; Cf. Tversky and Kahneman, 1992) have been studied by several authors (see Jarrow and Zhao, 2006; De Giorgi et al., 2008). The former use the lower partial moments as a risk measure for downside loss-aversion, while the latter develop a behavioral risk-reward model based on the gain-loss trade-off in the CPT framework. As mentioned in Bernard and Ghossoub (2010), the Omega measure is, hence, a ratio of a probability-weighted sum of gains out of a probability-weighted sum of losses, relatively to a threshold. In particular, their studies report that in a one-period economy with one risky asset and one risk-free asset the optimal holding of a “CPT-investor” is a function of a generalized version of the Omega ratio.

The first purpose of this article is to highlight a severe drawback that characterizes the Omega measure. We refer, here, to the possibility that the use of Omega leads investors to create risky asset rankings which are misleading and not compatible with rational behavior. Such an event might be realized in simplified frameworks, such as the
Gaussian one, but also under less stringent hypotheses on the features of risky asset return densities. In order to introduce the Omega “curse”, we borrow hereafter the definition of consistency in the sense of the Stochastic Dominance\(^8\) (in a strict version hereafter), proposed by Danielsson et al. (2008), that says that a risk measure denoted \(\rho\):

- is consistent with the Strict Superior Stochastic Dominance criterion,\(^9\) if and only if \(A \succ_{SD} B\) implies \(A \preceq_{\rho} B\);
- is consistent with the Strict Inferior Stochastic Dominance criterion if and only if \(A \preceq_{\rho} B\) implies \(A \succ_{SD} B\);
- is consistent with the Strict (Complete) Stochastic Dominance criterion if and only if \(\rho\) is consistent with both Strict Inferior and Strict Superior Stochastic Dominance criteria.

Concerning these three notions, consistency with the Strict Inferior Stochastic Dominance criterion, applied here to a risk measure, is of great importance for risk managers. Let us suppose that a risk manager relies on a risk measure denoted \(\rho\), which is not consistent with the Strict Inferior Stochastic Dominance criterion. We assume that he has the choice between a fund \(A\) and a fund \(B\), which are characterized by identical mean returns such as \(E_A[r] = E_B[r]\). Even though he might be confident that fund \(A\) is less risky than fund \(B\), he would not be able to conclude that the investors would necessarily agree with his choice, i.e. that \(A\) is preferred to \(B\). For a measure of risk that is consistent with the Strict Inferior Stochastic Dominance criterion, he would have this certainty.

\(\text{Cf.} \) Quirk and Saposnik (1962), Hadar and Russell (1969), Hadar and Russell (1971), Hanoch and Levy (1969), Rothschild and Stiglitz (1970) for the traditional Stochastic Dominance criterion. See also De Giorgi (2005), Levy (1992) and Levy (1998) for surveys on Stochastic Dominance (Cf. Appendix B for definitions). See also Leshno and Levy (2002) and Tzeng et al. (2013) for other types of stochastic dominance such as the Almost Stochastic Dominance.

\(\text{In a general context, we use } A \succ_X B \text{ (respectively, } \preceq_X, \prec_X, \ll_X\text{) when } A \text{ is strictly preferred to } B \text{ (respectively, } A \text{ is equivalent to } B, A \text{ is preferred or equivalent to } B, B \text{ is strictly preferred to } A, B \text{ is preferred or equivalent to } A \text{) according to a criterion } X.\) In a more specific framework, we will refer later to the Strict Inferior First-order Stochastic Dominance (SIFSD) and to the Strict Inferior Second-order Stochastic Dominance (SISSD) criteria, which imply a strict inequality instead of the non-strict inequality as presented in some of the definitions of the traditional First-order and Second-order Stochastic Dominance. Cf. Appendix B for more details.
But let us come back now to performance measurement (where the higher the measure, the better the investment). Considering *Omega* as an investment decision criterion, we will demonstrate below that this performance measure is inconsistent with the Strict Inferior Second-order Stochastic Dominance criterion, so that: on the one hand, we have \( A \preceq \Omega B \) (i.e. \( \Omega_A (\tau) \leq \Omega_B (\tau) \) for some thresholds \( \tau \)) and, on the other hand, we have \( A \succ_{SISSD} B \). More precisely, to illustrate this apparent opposition as a counter-example of the consistency of the *Omega* criterion, we will first show in the following section, posing \( E_A [r] = E_B [r] \) for the sake of simplicity, that we can have both \( \Omega_A (\tau) = \Omega_B (\tau) \) for some thresholds \( \tau \), and \( A \succ_{SISSD} B \), implying here as the SISSD requires that fund \( B \) be objectively riskier than fund \( A \).

3. Inconsistency of the *Omega* when Ranking Funds

Let us first grasp the intuition and consider the daily returns of fund \( A \) as illustrated in the following figures. In Figure 1 (Panel A), we represent the Probability Density Functions (PDF in short) of returns on fund \( A \) and fund \( B \) in the Gaussian paradigm (for the sake of simplicity in this example). Both funds have exactly the same average daily return, but the return distribution of fund \( B \) has twice the volatility of that of fund \( A \). Figure 1 (Panel B) also presents the Cumulative Distribution Function (CDF), the Cumulative Difference in CDF, as well as the explicit function of the *Omega* quantile function \(^{11}\) (Panel C) for both funds.

If we now study the Strict Inferior Stochastic Dominance criterion, Figure 1 (Panel B) shows that fund \( A \) does not dominate fund \( B \) according to the SIFSD criterion – since the two CDFs cross – but fund \( B \) is dominated by fund \( A \) according to the SISSD criterion, because the cumulative difference in CDF does not change sign (Cf. Appendix B).

\(^{10}\)See Footnote 9 for the notations.

\(^{11}\)The *Omega* Quantile Function highlights the relation existing between the *Omega* measure and the density assumed for the returns. Appendix C describes how the *Omega* measure can be expressed as a function of the return density, thus recovering the so-called *Omega* Quantile Function. The formula is provided for a generic density. Specific cases can be obtained by integrating the results reported in Appendix C with those of Andreev et al. (2005).
Figure 1: Gaussian Density Functions, Stochastic Dominance Criteria and Omega for Funds A and B (when $\tau = E_A(r) = E_B(r)$)

Panel A: Probability Density Functions

Panel B: Cumulative Density Functions

Panel C: Omega Quantile Functions

Source: Computation by the authors. The first Figure (Panel A) shows the Probability Density Functions of fund A (bold line) and fund B (thin line) returns where the x-axis represents the daily returns and the y-axis corresponds to the probabilities. Both fund returns follow Gaussian densities, such as the first two moments of the fund A distribution being calibrated on those of the daily Dow Jones Index over the period 1st January, 1900 to the 1st March, 2013 (data source: Datastream).

The second Figure (Panel B) pictures the Cumulative Density Functions of fund A (bold line) and fund B (thin line) returns on the (right) y-axis and the cumulative difference (line with markers) in CDF for the two funds on the (left) y-axis, where the x-axis displays the daily returns (quantile at the $\alpha$-confidence level). The third Figure (Panel C) represents the Omega quantile function corresponding to several thresholds (on the x-axis).
In this first simple illustration, we clearly see that fund $A$ should be preferred by any rational investor with a concave utility function, since the choice of fund $B$ goes with a higher risk for the same expected return. However, we note that the two assets have both an Omega equal to 1.00 when the threshold is .00% (Panel C). Furthermore, the Omega quantile functions reported in Panel C show that, depending on the threshold, fund $A$ is, or is not, preferred to fund $B$.

We have to admit that these facts are rather counter-intuitive and that Omega deserves some extra cautionary interest. Figure 1 illustrates that the Omega measure is inconsistent with the SISSD criterion in a simplified Gaussian framework. However, we can also provide the same qualitative conclusion within a more complex setting with asymmetric and leptokurtic (log-normal) densities. For instance, if we consider fund $A$ and fund $B$ with equal mean returns and a threshold below the means, the Omega measure will still be inconsistent with the SISSD criterion. Indeed, if we compute the Omega ratio for two series with some peculiar features, we can show that the asset ordering is inconsistent with the SISSD criterion as presented in the following proposition.

**Proposition 1. Inconsistency of the Omega Ranking**

Consider returns on two assets $A$ and $B$, with non-degenerated densities, with asset $A$ being preferred to asset $B$ according to the Strict Inferior Second-order Stochastic Dominance criterion (denoted $A \succ_{\text{SISSD}} B$). The ranking in terms of Omega of the two assets satisfies the following statements:

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12 Complementary illustrations are available upon request with different hypotheses on densities and related moments (Cf. Appendix G).

13 Several complementary tests using various densities lead to the same qualitative results. More interestingly, if we hypothesize that the mean return of fund $A$ is superior to that of fund $B$ by 1.00%, which is higher than the threshold, Omegas of the two funds will be equal, despite a trade-off of 1 against 10 in terms of expected return versus volatility. Complementary results are available upon request (Cf. Appendix G).

14 These statements are in coherence with Darinos and Satchell (2004) and Kazemi et al. (2004), since we can show that, under some crucial conditions on the threshold, the traditional (non-strict) second-order stochastic dominance, such that $A \succeq_{\text{SSD}} B$, implies a (non-strict) Omega (and Sharpe-Omega) dominance such that $A \succeq_{\Omega} B$ (and $A \succeq_{S-\Omega} B$), see appendix B, C and F for some comments.
1. If \( \int_{-\infty}^{+\infty} [F_B(r) - F_A(r)] \, dr = 0 \), a necessary and sufficient condition for having \( A \prec_{\Omega} B \)

is that \( \int_{-\infty}^{+\infty} [1 - F_A(r)] \, dr - \int_{-\infty}^{+\infty} F_A(r) \, dr < 0 \);

2. If \( \int_{-\infty}^{+\infty} [F_B(r) - F_A(r)] \, dr > 0 \), a sufficient condition for having \( A \prec_{\Omega} B \)

is that \( \int_{-\infty}^{+\infty} F_B(r) \, dr - \int_{-\infty}^{+\infty} [1 - F_B(r)] \, dr < 0 \);

3. If \( \int_{-\infty}^{+\infty} [F_B(r) - F_A(r)] \, dr > 0 \), a sufficient condition for having \( A \succ_{\Omega} B \)

is that \( \int_{-\infty}^{+\infty} [1 - F_A(r)] \, dr - \int_{-\infty}^{+\infty} F_A(r) \, dr > 0 \);

and thus Omega is inconsistent with the Strict Inferior Second-order Stochastic Dominance criterion (SISSD) when some conditions are met (in the first two cases).

Proof: see Appendix D.

Through Proposition 1, we notice that the Omega measure is inconsistent with the SISSD criterion in the first two cases since fund \( B \) is always chosen by Omega while fund \( A \) is Strictly Inferior Second-order Stochastically dominating fund \( B \) by assumption. In case 1, fund \( B \) is Omega-preferred because relative gains on the portfolio \( A \) are strictly lower than its relative losses in terms of cumulative densities. In case 2, fund \( B \) is Omega-preferred because relative gains on the portfolio \( B \) are strictly higher than its relative losses with regard to cumulative densities. In case 3, fund \( A \) is better ranked by Omega since relative gains on fund \( A \) are strictly higher than its losses in terms of cumulative densities.

Furthermore, we observe here that the choice of funds according to Omega is directly dependent on the threshold, which leads us to present the following corollary:

**Corollary 1. Inconsistency of the Omega Ranking under Symmetry of Densities and Equality of Means**

If the two assets of Proposition 1 have returns with a density of the same (elliptical) family, with identical means \( E_A(r) = E_B(r) = \mu \), but different volatilities such as \( \sigma_A(r) < \sigma_B(r) \), then, the Strict Second-order Stochastic Dominance criterion implies that
the ranking in terms of Omega of the two assets will be:

1. if \( \tau = \mu \), \( A \approx_\Omega B \);
2. if \( \tau > \mu \), \( A \prec_\Omega B \);
3. if \( \tau < \mu \), \( A \succ_\Omega B \);

and thus Omega is inconsistent with the Strict Inferior Second-order Stochastic Dominance criterion (SISSD in cases 1 and 2).\(^{15}\)

Proof: see Appendix D.

We indeed show that the Omega function is not consistent with the SISSD criterion in general. In fact, we can derive conditions (sufficient and, in some specific cases, also necessary) for having asset rankings inconsistent with the SISSD criterion. In the peculiar case of two funds with equal means and symmetric distributions, we can obtain any ranking in terms of Omega depending on the threshold (below, equal or higher than the mean return), as briefly mentioned by Bertrand and Prigent (2011).\(^ {16}\)

There is a very counter-intuitive result, since we notice here that the center of the distribution, where returns are close to 0 (i.e. when nothing happens in the market), has a significant impact on the Omega measure. If we compare two Omega ratios for the same portfolio, and move the threshold of the second Omega slightly in the right (respectively, in the left) of the mean return of the underlying distribution, this will transform relative gains (respectively, losses) into relative losses (respectively, gains), and thus inverse the portfolio ordering. This drawback may have severe consequences in terms of investment choices when comparing several portfolios.

As a result of Corollary 1, if one investor makes decisions based on Omega, he will prefer the safest asset for thresholds below the mean, and the riskiest asset for thresholds above the mean. This phenomenon is associated with choices that are driven by the

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\(^{15}\) However, having in mind that \( \tau \) is a “Minimum Acceptable Return” (MAR in short), cases 1 and 2 may appear unrealistic when comparing two portfolios. Probably only funds \( p \) with an \( \Omega_p(\tau) \) that is much greater than 1 (first part of a ranking) and a \( \mu_p \) that is much greater than \( \tau \) would be considered (focusing on large losses). Nevertheless, these cases are frequent when all assets are considered in an optimization program (see the next section).

\(^{16}\) Cf. Bertrand and Prigent (2011), Proposition 1 in section 2.3 on page 1.815.
probability of obtaining positive outcomes with high thresholds. In fact, an agent prefers
the asset that has the highest probability of being above the threshold, without giving
much importance to the risk associated with his choice. On the contrary, if the threshold
is below the mean, the choice made by the agent will focus more on risk.

We point out that our result on the inconsistency of Omega with the SISSD criterion refines the findings of Darsinos and Satchell (2004) and Kazemi et al. (2004). See Appendix F in which we prove that our results are valid within their framework.

We can thus derive here some preliminary conclusions. First, the Omega ratio is
inconsistent in the sense of the SISSD. Secondly, the threshold should be linked to the
agent’s preferences since it determines the ranking of the funds and reflects her investment
choice. In other words, the Omega is not a “universal” performance measure. More
specifically, when studying the case of two symmetric densities (simple illustration of
Gaussian laws) such as $E[r_A] = E[r_B] = \mu$, we may face an irrational ordering when
the chosen threshold is high. We can also show that even in a more complex setting
based on two asymmetric and leptokurtic (log-normal) densities (with some uncertainty),
the results remain similar.\(^{17}\) The Omega ratios of two (or several) funds, which are
characterized by similar mean returns but different volatilities, will be equal.

This latter fact leads us to a more general study on the trade-off between expected
return and risk when Omega is driving allocation and investment choices. Even if the
Omega ratio was truly compatible with the SISSD criterion in most cases (but not all
as shown earlier), what really matters is the price of exchange between performance and
risk that the Omega function values.

Based on the simulation scheme used in Ingersoll et al. (2007), Figure 2 below rep-
resents the Iso-Omega curves for various thresholds\(^{18}\) $\tau$, set to .00%, 5.00%, 7.00% and
10.00% (see Keating and Shadwick, 2002; Kane et al., 2009; Hentati et al., 2010; Bertrand
and Prigent, 2011; Gilli et al., 2011).

\(^{17}\) Another counter-intuitive example with a log-normal (asymmetric) density is available upon request (Cf. Appendix G).

\(^{18}\) For the sake of realism, we herein restrict our next analysis to the various values of thresholds we
found in the literature. However, results with other values (very low or very high) are in line with those
presented below.
These curves are realized by comparing 10,000 pairs of portfolios obtained from the following algorithm.\textsuperscript{19} We start by randomly choosing one time-series of returns among thousands simulated according to the scheme used in Ingersoll et al. (2007). This (initial) portfolio is composed by 250 returns distributed according to a Gaussian law characterized by an annual mean return equal to 17\% and an annual volatility set to 20\% as in Ingersoll et al. (2007). Next, we distort the return distribution of this initial portfolio in two different directions. First, we gradually increase its mean return, while keeping (by rescaling) its annual volatility equal to 20\% until we obtain 10,000 new portfolios. Secondly, we gradually decrease the volatility of the initial portfolio, without modifying (by rescaling) the mean return equal to 17\% until we obtain 10,000 other portfolios. Finally, for a given threshold $\tau$, we compute the Omega ratio for the two set of 10,000 simulated portfolios in order to estimate the amount of extra unit of performance, for a given amount of over-volatility, that is required to reverse the Omega ranking between two funds. Thus, for a given threshold, an Iso-Omega curve corresponds to identical Omega levels for various portfolios characterized by different mean returns and volatilities.\textsuperscript{20} For the sake of simplicity, let us compare a fund A with a fund B characterized by the same volatilities. We can then distinguish three main cases. In the first one, if the funds A and B are located on the same Iso-Omega curve, this will imply that they have identical Omega-rankings. In the second case, if fund A is above fund B (\textit{ceteris paribus}), this means that Omega will prefer fund A. Finally, in the third case, we assume that fund A is below fund B (\textit{ceteris paribus}), then the Omega ratio will prefer fund B.

In Figure 2, we note that when the threshold is equal to .00\%, the trade-off is very close to 1, as for the Sharpe ratio, since we require 100 basis points of extra over-performance for the same amount of over-volatility to reverse the fund rankings obtained with the Omega measure (slope of the iso-Omega curve being equal to 1 for the threshold on this sample). For a threshold equal to 10.00\%, we only require 100 basis points of extra

\textsuperscript{19}See Appendix E for a full sketch of the algorithm.
\textsuperscript{20}The trade-off between risk and return observed in Figure 2, when drawing the Iso-Omega curves, is based on the simulation scheme used in Ingersoll et al. (2007). Please see the source of Figure 2 and Appendix E for more details on the sketch of the algorithm. Other complementary tests on various databases show that these trade-offs may vary according to the statistical properties of the underlying return series.
Figure 2: *Iso-Omega* Curves displaying the Quantity of Over-volatility required for a given Over-performance for reversing the *Omega* Ranking

Source: Simulations by the authors. Illustration of *Iso-Omega* curves, i.e. over-performances (y-axis) versus over-volatilities (x-axis) yielding the same ranking according to the *Omega*; both are expressed in percentage terms. We represent by a blue star the amount of over-performance of fund $B$ compared to fund $A$, for a given level of over-volatility. The ranking curves (solid lines) are computed for the *Omega* measure (Cf. Keating and Shadwick, 2002) when the threshold is equal to .00%, 5.00%, 7.00% and 10.00%. Each *Iso*-curve represents the amount of over-performance of a portfolio compared to another one, for a given over-volatility, required to reverse the *Omega* ranking between these two funds. The dashed bold line is the *Iso*-Sharpe (1966) ratio curve assuming that a unit of extra over-volatility – for a unit of a given over-performance – is required to inverse the ranking between fund $A$ and fund $B$.

over-performance for 400 basis points of over-volatility. This last case can be associated with the behavior of a greedy agent. Finally, we see here that the lower the threshold, the closer the rankings between the Sharpe ratio and the *Omega* criterion.

However, we have seen previously that the threshold should not be too high; otherwise the conclusion drawn from the *Omega* measure ranking might be just... wrong.

Let us investigate more precisely the risk of being irrational when choosing to adopt the *Omega* measure in comparing two funds. In the following, the first one, named again fund $A$ (run by an informed portfolio manager), has a higher mean return than the second one (an uninformed manager), called fund $B$, but with a higher volatility (due to the noisy signal received by the informed manager). Table 1 displays the frequency at which the Sharpe ratio is higher for fund $A$ than for fund $B$ (group of S) and the frequency at which the *Omega* measure concludes the opposite (third to fifth groups of columns), when setting the thresholds equal to 10.00%, 5.00% and .00%.

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Table 1: Contradiction in Choices of Investments using various Criteria

<table>
<thead>
<tr>
<th>Dist. Coef.</th>
<th>Risk-free Rate</th>
<th>$S_B \leq S_A$</th>
<th>Cases when $A \succ E B$, $A \succ_{\text{Sharpe}} B$ and $A \prec_{\Omega} B$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10.00%</td>
<td>5.00%</td>
<td>.00%</td>
</tr>
<tr>
<td>$m = 1.00$</td>
<td>48.27%</td>
<td>47.47%</td>
<td>46.58%</td>
</tr>
<tr>
<td>$m = 1.20$</td>
<td>44.99%</td>
<td>42.29%</td>
<td>39.54%</td>
</tr>
<tr>
<td>$m = 1.40$</td>
<td>42.14%</td>
<td>37.92%</td>
<td>33.79%</td>
</tr>
<tr>
<td>$m = 1.60$</td>
<td>39.43%</td>
<td>33.88%</td>
<td>28.54%</td>
</tr>
<tr>
<td>$m = 1.80$</td>
<td>37.11%</td>
<td>30.62%</td>
<td>24.50%</td>
</tr>
<tr>
<td>$m = 2.00$</td>
<td>35.21%</td>
<td>27.97%</td>
<td>21.35%</td>
</tr>
</tbody>
</table>

Source: Simulations by the authors. This table displays the frequency to which the Sharpe ratio is higher for fund $A$ than for fund $B$ (group of S) and the frequency to which the Omega measure concludes the opposite (third to fifth groups of columns) according to several thresholds: 10.00%, 5.00% and .00%. Following the simulation scheme developed in Ingersoll et al. (2007), portfolio returns are defined such as: $\tilde{r}_p = \exp \left\{ \left[ \mu_m + \alpha_p - 5 \left( \sigma_m^2 + \upsilon_p^2 \right) \right] \Delta t + \left( \sigma_m \tilde{\varepsilon} + \upsilon_p \tilde{\eta} \right) \sqrt{\Delta t} \right\} - 1$, where $\mu_m$ is the market portfolio return, $\alpha_p$ is the extra-performance generated by the manager, $\sigma_m$ is the market portfolio total risk, $\upsilon_p$ corresponds to the residual portfolio specific risk, $\Delta t$ is the data frequency, $\tilde{\varepsilon}$ and $\tilde{\eta}$ are Gaussian random variables. We use here four different profiles of investors. We characterize the first one by setting $\alpha_p = .00%$ and $\upsilon_p = .20%$. The three other managers are defined such as $\alpha_p = 1.00%$ and the residual portfolio specific risks are respectively equal to $\upsilon_p = .20%$, $\upsilon_p = 2.00%$ and $\upsilon_p = 20.00%$. Then, we randomly choose two portfolios among all these profiles and order them according to their mean returns. Finally, we compute the associated Sharpe ratios and Omega measures for each threshold and determine how often these measures conclude as the ordering given by their mean returns. A supplementary check confirms that the risk premia for all portfolios are positive, which proves consistency between rankings with the Sharpe ratios. Our simulations are based on the comparison of 10,000 pairs of portfolios, with a 5-year return history, for each distortion coefficient $m$ varying from 1 to 2 with a step equal to .20. The market hypotheses are: risk free rate 5.00% per year, market premium 12.00%, market standard deviation 20.00%.
Lines in the table distinguish several cases corresponding to an increase in the volatility of portfolio returns, obtained by multiplying the probability density functions by a distortion coefficient $m$ varying from 1 to 2 with successive steps equal to .20 (third to eighth rows), without altering the associated mean returns.\textsuperscript{21}

In Table 1, we observe in the second column that the Sharpe ratio naturally decreases when the studied portfolio total risk increases, for a same amount of mean return. If we now focus on the results given by the Omega measure, we can see that the value of the threshold - equal to 10.00\%, 5.00\% and .00\% (third to fifth groups of columns), has a real impact on the final ranking. Obviously, the intuition tells us that the Sharpe and the Omega ratios are equal most of the time. One of the main reasons that can explain the rate of false answers is the value of the threshold. But we also note some significant differences between the Omega and the Sharpe ratios (even when the threshold is low), mainly because Omega is focusing too much on the difference of expected returns, disregarding in some sense the implied risks. Another potential explanation is in the relative value of the mean returns of the compared funds (above and/or below the threshold); in some limited cases, the Omega ratio will be biased, just as the Sharpe ratio, when considering portfolios with negative mean performances (see Israëlsen, 2005).

In a simple setting (a Gaussian case for a non-multiple comparison of two funds), we can thus say that the choice of the level of the threshold is of interest: if too “high”, some aberrant results may appear (see proposition 1) and if too “low” (compared to mean returns of investments considered), the risk of error is accentuated (see results in Table 1). A second rule of thumb is thus to choose a not too high or too low threshold.

We have seen previously that we may face, under some circumstances, misleading financial advice when using Omega to rank investments when the value of the threshold is too high or too low. But what is the typical behavior of portfolios optimized according to this criterion when the threshold is in a reasonable band? The answer to this question is the objective of the next section.

\textsuperscript{21}Still following the simulation scheme proposed in Ingersoll et al. (2007), our results are based on a random selection of 10,000 pairs of simulated portfolios (one uninformed portfolio return series against one informed manager series for all comparisons).
4. **Omega as an Optimization Criterion?**

In the previous section, we studied the inconsistency of the *Omega* ratio (*Cf.* proposition 1) through simple illustrations based on realistic simulations comparing two investment portfolios. The main and basic ideas were, first, to show the importance of the choice of the threshold (*Cf.* Figure 2 and Table 1), and secondly, to prove, in a simplified (Gaussian) framework, that when comparing funds with identical mean returns but different risk levels, the *Omega* ratio for these two portfolios is equal since losses are merely compensated by gains (*Cf.* Figure 1). We have thus shown that using *Omega* as a ranking criterion may be misleading in some cases since it is inconsistent with the SISSD criterion. We would like now to study the behavior of *Omega* in a static and dynamic optimization framework.

Next, we compare the performance of portfolios optimized according to the *Omega criterion* and other classical paradigms, in both static and dynamic ways. For the sake of robustness, we will present some empirical illustrations of properties based on, first, realistic simulations and, secondly, on three different market databases used in the literature on portfolio optimization (namely Hentati et al., 2010; Darolles et al., 2009; DeMiguel et al., 2009).

Since the *Omega* ratio can be inconsistent with the SISSD criterion in some cases, we show as a natural consequence, through Figure 3, that using this measure as an optimization objective function can also lead to some mismatches, even in the simple Gaussian framework. More precisely, we illustrate a typical case that can happen in a very common situation when we optimize a portfolio according to *Omega*. First, we simulate two Net Asset Value series (once again using the traditional Gaussian hypothesis on returns), corresponding to distinct risky assets, respectively characterized by annual mean returns (volatilities) equal to 9.96% (6.44%) for fund *A* and 10.44% (12.29%) for fund *B*. We consider the return of the riskless asset at 2.00% per year without loss of

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22 Figure 3 is just one mere illustration among thousands of tests, that exhibits the phenomenon. Our results are based on the simulation of 10,000 pairs of Gaussian distributions. We also obtain the same conclusion when using a log-normal law for simulating the NAV of fund *A* and fund *B* (results are available upon request).
generality. The threshold used for computing the Omega portfolio is set to 10.00%. In a static way, we then optimize portfolios according to the maximum Sharpe ratio, the maximum Omega and the minimum volatility criteria, and illustrate the results in Figure 3 (below).

Figure 3: An Illustration of a Misleading Choice of Omega (Static Analysis)

Source: Simulations by the authors. This figure illustrates over a 5-year period three static portfolios optimized according to the following paradigms: maximum Sharpe ratio (light line), maximum Omega ratio (bold line) and minimum volatility (thin line). The x-axis represents the number of simulated observations and the y-axis is the cumulative performance of optimal portfolios. Fund A and Fund B return densities are simulated according to Gaussian laws and are characterized, respectively, by annual mean returns (volatilities) equal to 9.96% (6.44%) and 10.44% (12.29%). We consider the return of the risk-free asset at 2.00% per year (fixed), without loss of generality (complementary results are available upon request - Cf. Appendix G).

We clearly see that the Omega-based optimal portfolio is misleading compared to the other ones (namely, maximum Sharpe and minimum volatility). Indeed, it is completely invested in Fund B that over-performs Fund A by only .48% but with a much higher volatility (12.29% versus 6.44%).\textsuperscript{23} Then, we can say that the Omega measure may not, always, allow us to rationally choose the best investment.

Nevertheless, real financial series are known to be non-Gaussian. We thus want to highlight some peculiarities of Omega-based strategies on a real dataset. For the sake of comparison, we refer in the following illustrations to the dataset used in Hentati et al. (2010) and based on the same methodology. This database is composed of five indexes,

\textsuperscript{23}Complementary results are available upon request (Cf. Appendix G).
expressed in USD with a monthly frequency, representing the most common asset classes often used in asset allocation optimization problems, namely: the HFRX Global Index for hedge funds, the UBS GC Index for convertibles, the JPM GBI Index for bonds, the MSCI World Free Equity Index for equities and the S&P GSCI Index for commodities.

Figure 4: Performance of the Five Indexes representing the Studied Asset Classes

Source: Bloomberg and Datastream (data sources); monthly data in USD from 31st August, 1997 to 31st August, 2007. The x-axis represents dates and the y-axis corresponds to cumulative monthly returns in base 100. This figure shows the performance of the five studied indexes, namely, the HFRX Global Hedge Fund Index (thin line with circular markers), the UBS Global Convertible Index (thin light line), the JPM Global Bond Index (bold light line), the MSCI World Equity Index (thin line with cross markers) and the S&P GS Commodity Index (bold dark line). Computations by the authors.

In Figure 4, we represent the realized performance of the five studied indexes from 31st August, 1997 to 31st August, 2007. A brief analysis shows that only the commodity market suffered strongly from the Asian and the Russian crises. Indeed, the S&P GS Commodity Index fell by 40% between October 1997 and December 1998. In contrast, despite heterogeneous correlation levels between the studied markets, the successive financial turbulences that occurred in 2000 (Internet Bubble), 2001 (terrorist attack and Junk Bonds) and 2002 (Brazilian crisis) had a more global impact since the convertible, commodity and equity indexes, dropped between 20.00% and 40.00% within this time period.

In Figures 4 and 5, we present and compare our results obtained when using the Omega ratio and other common paradigms as optimization criteria.
Figure 5: Performance of Five Portfolios optimized according to the Omega Criterion and other Classical Paradigms (Static Analysis)

Source: Bloomberg and Datastream (data sources); monthly data in USD from 31st August, 1997 to 31st August, 2007. The x-axis corresponds to dates and the y-axis represents cumulative monthly returns in base 100. This figure presents a static analysis of the performance of portfolios optimized according to the following criteria: maximum mean return (line with square markers), maximum mean return under a volatility constraint (thin light line), maximum Sharpe ratio (bold light line), maximum Omega measure (bold dark line) and minimum volatility (thin line with cross markers). These five optimal portfolios are composed by the HFRX Global Hedge Fund Index, the UBS Global Convertible Index, the JPM Global Bond Index, the MSCI World Equity Index and the S&P GS Commodity Index (Cf. Figure 4). Computations by the authors.

We start our study within a static framework using a classical optimization algorithm over the five assets used by Hentati et al. (2010). In Figure 5, we represent five (long-only) optimal portfolios, each corresponding to a specific direction of an in-sample optimization. The first two maximize (only) mean return and mean return under a volatility constraint. The next two paradigms consist of maximizing the Sharpe (1966) and Omega (2002) ratios. Finally, the last considered portfolio minimizes volatility. The composition of these static portfolios is represented in Table 2.

We observe that the first one-period optimized portfolio, corresponding to the maximum mean return criterion, displays the highest performance, but also the strongest volatility since it totally disregards risk (or losses) by definition. In contrast, the other four static optimal portfolios have very similar profiles and smoothed performances in the sample. Then, the use of the Omega (2002) ratio as an optimization criterion does not present any major improvement compared to other classical directions (using this
Table 2: Composition of the Five Optimal Portfolios according to the Omega Criterion and other Classical Paradigms (Static Analysis)

<table>
<thead>
<tr>
<th>Optimization Paradigms</th>
<th>HFRX Hedge Fund Global</th>
<th>JPM Global Bond Index</th>
<th>MSCI World Equity Index</th>
<th>S&amp;P GS Com. Index</th>
<th>UBS Global Conv. Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max. Mean Return</td>
<td>.00%</td>
<td>.00%</td>
<td>.00%</td>
<td>.00%</td>
<td>100.00%</td>
</tr>
<tr>
<td>Max. Mean s.t. Vol. (5.00%)</td>
<td>54.20%</td>
<td>43.90%</td>
<td>.00%</td>
<td>1.90%</td>
<td>.00%</td>
</tr>
<tr>
<td>Max. Sharpe</td>
<td>28.37%</td>
<td>66.45%</td>
<td>.00%</td>
<td>5.18%</td>
<td>.00%</td>
</tr>
<tr>
<td>Max. Omega ($\tau = .00%$)</td>
<td>32.40%</td>
<td>60.15%</td>
<td>1.99%</td>
<td>.68%</td>
<td>4.77%</td>
</tr>
<tr>
<td>Min. Volatility</td>
<td>27.20%</td>
<td>67.47%</td>
<td>.00%</td>
<td>5.33%</td>
<td>.00%</td>
</tr>
</tbody>
</table>

Source: Bloomberg and Datastream (data sources); monthly data in USD from 31st August, 1997 to 31st August, 2007. This table shows the one-period optimal weights of portfolios corresponding to the following directions: maximum mean return, maximum mean return under a volatility constraint, maximum Sharpe, maximum Omega and minimum volatility; and for each studied index, namely, the HFRX Global Hedge Fund Index, the JPM Global Bond Index, the MSCI World Equity Index, the S&P GS Commodity Index and the UBS Global Convertible Index. Computations by the authors.

In order to complement the previous static analysis presented in Figure 5, we show in Table 2 the one-period optimal allocation for each portfolio (reported in the first column) over the five studied indexes (second to sixth columns).

Here, we read results that are perfectly in line with the given optimization paradigms and our previous illustration. The first optimal portfolio, corresponding to the maximum mean return criterion, naturally overweights in a static framework the indexes displaying the highest final performance (see Figure 4 and 5). The one based on the Omega ratio is very slightly more diversified but it is fundamentally almost identical to the three other remaining optimal portfolios that disregard the equity and convertible bond indexes.

In Figure 6, we now propose an illustration of a dynamic out-of-sample analysis of the performance of the five portfolios previously studied. The rolling window for estimations is based on two years and we assume a monthly rebalancing, without specific constraints, during the period of time that ran from 31st August, 1999 to 31st August, 2007.

This figure shows that the dynamic portfolio maximizing (only) the mean return displays (almost) the highest terminal performance, but also the largest volatility compared to the other strategies. In contrast, the less risky optimal portfolio is the one minimizing volatility that presents the lowest but smoothed performance. If we now focus on the
Figure 6: Performance of Five Portfolios optimized according to the Omega Criterion and other Classical Paradigms (Dynamic Analysis)

Source: Bloomberg and Datastream (data sources); monthly data in USD from 31st August, 1997 to 31st August, 2007. The x-axis represents dates and the y-axis corresponds to cumulative monthly returns in base 100. This figure presents a dynamic analysis of the performance of portfolios optimized according to the following criteria: maximum mean return (line with square markers), maximum mean return under a volatility constraint (thin light line), maximum Sharpe ratio (bold light line), maximum Omega measure (bold dark line) and minimum volatility (thin line with cross markers). We consider a two-year in-sample window and a monthly rebalancing for the projection period. These five optimal portfolios are initially composed of the HFRX Global Hedge Fund Index, the UBS Global Convertible Index, the JPM Global Bond Index, the MSCI World Equity Index and the S&P GS Commodity Index (Cf. Figure 4). Computations by the authors.

portfolio maximizing the Omega ratio, we observe that it underperforms almost all the other ones, and more precisely, those based on the maximum mean return, the maximum Sharpe ratio and the maximum mean return under a volatility constraint.

We now focus our study on the use of the Omega ratio as an optimization criterion. Similar to our previous analysis, we thus present, first, a static framework, in which we examine the evolution of the performance of optimal portfolios using the Omega optimization criterion according to several thresholds, respectively, equal to .00%, 5.00%, 7.00% and 10.00%. Figure 7 illustrates our results.

As shown in Figure 7, the higher the value of the threshold, the riskier the profile of the optimal portfolio. For instance, the one considering a null threshold presents the lowest but the most stable performance with a very low risk, whereas the optimal portfolio based on a threshold set to 10.00% displays a slightly better performance, but for a much
Figure 7: Performance of Four Portfolios optimized according to the Omega Ratio when varying the Threshold (Static Analysis)

Source: Bloomberg and Datastream (data sources); monthly data in USD from 31st August, 1997 to 31st August, 2007. The x-axis represents dates and the y-axis corresponds to cumulative monthly returns in base 100. This figure presents a static analysis of the performance of portfolios optimized according to the Omega criteria when varying the thresholds, respectively, equal to .00% (bold dark line), 5.00% (bold light line), 7.00% (thin light line) and 10.00% (thin light line with cross markers). These four optimal portfolios are initially composed by the HFRX Global Hedge Fund Index, the UBS Global Convertible Index, the JPM Global Bond Index, the MSCI World Equity Index and the S&P GS Commodity Index (Cf. Figure 4). Computations by the authors.

more important risk level. However, we can wonder about the (rational) maximum value of the threshold since performances of static optimal portfolios based on rates equal to 5.00%, 7.00% and 10.00% are almost identical. The following illustration allows us to answer this question.

In Figure 8, we now compare the performance of four dynamic portfolios optimized according to the same setting of thresholds used in the previous analysis.

As expected, this figure shows that the dynamic portfolio based on a threshold equal to 10.00% clearly overperforms the optimal portfolio for which the threshold is set to .00%. This last observation raises an important question of the usefulness and justification in optimization problems of the Omega, and confirms our intuition about its sensitivity to the value of the threshold. The next section is dedicated to some robustness checks.
5. Robustness Checks

We present, in this last section, some robustness checks in a static and a dynamic framework, when optimizing portfolios according to the Omega ratio. First, we use a sub-sample of hedge funds as in Darolles et al. (2009). Secondly, we calculate the Sharpe ratios of several strategies based on the methodology and the datasets used in DeMiguel et al. (2009).

Let us start by the study of a sample of hedge funds (Cf. Darolles et al., 2009). Quotations are expressed in US dollars on a monthly basis. At inception, hedge funds are self-declared in one or several categories, called styles. We have then chosen 18 hedge funds\textsuperscript{24} that represent various styles and report the information on the fund name, the self-declared strategy and the management company in Table 3.

\textsuperscript{24}We have selected here (from the HFR database) the hedge fund quotes for funds reported in the database used in Darolles et al. (2009) (in the same period: from the the 31st May, 2004 to the 31st July, 2007), that were still alive in 2012 (and added the Fairfield fund NAV series).
Table 3: Hedge Funds Classified by Style

<table>
<thead>
<tr>
<th>No</th>
<th>Fund Name</th>
<th>Style</th>
<th>Company Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Exane Investors Gulliver Fund</td>
<td>Equity Hedge</td>
<td>Exane Structured Asset Mgt.</td>
</tr>
<tr>
<td>2</td>
<td>Ibis Capital, LP</td>
<td>Equity Hedge</td>
<td>Ibis Management, LLC</td>
</tr>
<tr>
<td>3</td>
<td>Odey European Inc.</td>
<td>Equity Hedge</td>
<td>Odey Asset Mgt. Limited</td>
</tr>
<tr>
<td>4</td>
<td>Platinum Fund Ltd.</td>
<td>Equity Hedge</td>
<td>Optima Fund Mgt.</td>
</tr>
<tr>
<td>5</td>
<td>RAB Europe Fund</td>
<td>Equity Hedge</td>
<td>RAB Capital PLC</td>
</tr>
<tr>
<td>6</td>
<td>Robbins Capital Partners, L.P.</td>
<td>Equity Hedge</td>
<td>T. Robbins Capital Mgt., LLC</td>
</tr>
<tr>
<td>7</td>
<td>Invesco QLS Equity</td>
<td>Equity Mkt. Neutral</td>
<td>Invesco Structured Pdts Grp.</td>
</tr>
<tr>
<td>8</td>
<td>Fairfield Sentry Ltd</td>
<td>Equity Mkt. Neutral</td>
<td>Fairfield Greenwich Group</td>
</tr>
<tr>
<td>9</td>
<td>Thames River European Fund</td>
<td>Equity non-hedge</td>
<td>Thames River Capital LLP</td>
</tr>
<tr>
<td>10</td>
<td>Permal Global Opportunities Ltd.</td>
<td>Macro</td>
<td>Permal Asset Mgt., Inc</td>
</tr>
<tr>
<td>11</td>
<td>Aspect Diversified Fund Limited</td>
<td>Macro</td>
<td>Aspect Capital Limited</td>
</tr>
<tr>
<td>13</td>
<td>Haidar Jupiter International, Ltd.</td>
<td>Macro</td>
<td>Haidar Capital Mgt., LLC</td>
</tr>
<tr>
<td>14</td>
<td>Winton Futures Fund</td>
<td>Macro</td>
<td>Winton Capital Mgt.</td>
</tr>
<tr>
<td>15</td>
<td>Aristeia International, Ltd.</td>
<td>Convertible Arbitrage</td>
<td>Aristeia Capital LLC</td>
</tr>
<tr>
<td>16</td>
<td>Advent Conv. Arbitrage Fund, L.P.</td>
<td>Convertible Arbitrage</td>
<td>Advent Capital Mgt., LLC</td>
</tr>
<tr>
<td>17</td>
<td>Paulson International Ltd.</td>
<td>Merger Arbitrage</td>
<td>Paulson &amp; Co., Inc.</td>
</tr>
<tr>
<td>18</td>
<td>Courage Special Offshore Fund, Ltd.</td>
<td>Event-Driven</td>
<td>Courage Capital Mgt., LLC</td>
</tr>
</tbody>
</table>

Source: HFR (data source); this table presents the names, the styles and the companies of 18 hedge funds used in Darolles et al. (2009), including the Fairfield Sentry Ltd (Madoff’s fund).

Based on the hedge fund database presented in Table 3, we illustrate in the following figure a static analysis by comparing one portfolio optimized according to the \( \Omega \) ratio, with a threshold equal to 5.00%, and four others based on the same paradigms as described in the previous analyses. The static optimization is realized from 31st May, 2004 to 31st July, 2007.

As expected, we observe in Figure 9 that the portfolio maximizing the mean return displays the highest performance and that the \( \Omega \)-based optimal portfolio has the lowest one.\(^{25}\)

These results obtained with a hedge fund database confirm our previous finding: using the \( \Omega \) measure as an optimization criterion may yield to non optimal investment choices.

In Figure 10, we present a dynamic analysis of five portfolios optimized according to the same paradigms as those used in the previous figure, namely: the maximum mean return, the maximum mean return under a volatility constraint, the maximum Sharpe

\(^{25}\) Complementary analyses of the different optimal portfolios are available upon request (Cf. Appendix G).
Figure 9: Performance of Five Portfolios optimized according to the Omega Ratio and other Classical Paradigms on Hedge Funds (Static Analysis)

[Graph showing performance metrics over time]

Source: HFR (data source); monthly data in USD from 31st May, 2004 to 31st July, 2007. The x-axis represents dates and the y-axis corresponds to cumulative monthly returns in base 100. This figure presents a static analysis of the performance of portfolios optimized according to the following criteria: maximum mean return (line with square markers), maximum mean return under a volatility constraint (thin light line), maximum Sharpe ratio (bold light line), maximum Omega measure (bold dark line) and minimum volatility (thin line with cross markers). These five optimal portfolios are initially composed by the 18 hedge funds (Cf. Table 3). Computations by the authors.

ratio, the maximum Omega ratio with a threshold equal to 5.00% and the minimum volatility. In this setting, we consider a one-year in-sample period and we assume a monthly rebalancing, without specific constraints, during the projection period that lies from 31st, July 2005 to 31st, July 2007.

This illustration of dynamic optimizations leads us to identical conclusions as those stated in the previous analyses (Cf. Figure 6 and 8). The performance of portfolios optimized according to the Omega ratio corresponds to that of portfolios maximizing the mean return for high thresholds (as for instance, 10.00%), and minimizing the volatility for low thresholds (close to .00%).

We can thus conclude here that using the Omega ratio as an optimization function does not display any real improvements.

For the sake of comparison, once again, we use in Table 4 the same datasets\textsuperscript{26} as in DeMiguel et al. (2009) and based on an identical methodology.\textsuperscript{27} We consider six

\textsuperscript{26}See also DeMiguel et al. (2014).
\textsuperscript{27}Cf. Table 3 in DeMiguel et al. (2009) on page 1,931.
Figure 10: Performance of Five Portfolios optimized according to the Omega Ratio and other Classical Paradigms on Hedge Funds (Dynamic Analysis)

Source: HFR (data source); monthly data in USD from 31st May, 2004 to 31st July, 2007. The x-axis represents dates and the y-axis corresponds to cumulative monthly returns in base 100. This figure presents a dynamic analysis of the performance of portfolios optimized according to the following criteria: maximum mean return (line with square markers), maximum mean return under a volatility constraint (thin light line), maximum Sharpe ratio (bold light line), maximum Omega measure (bold dark line) and minimum volatility (thin line with cross markers). We consider a one-year in-sample window and a monthly rebalancing for the projection period. These five optimal portfolios are initially composed by the 18 hedge funds (Cf. Table 3). Computations by the authors.

different databases, corresponding to “Industry Portfolios”, “International Portfolios”, “MKT/SMB/ HML”, “FF 1-factor”, “FF 3-factor” and “FF 4-factor”. Each dataset, that we briefly describe hereafter, consists of excess monthly returns over the 90-day T-bill (obtained from the Kenneth R. French Website).

The first database is “Industry Portfolios” that contains ten US industry portfolios, which are “Consumer-Discretionary”, “Consumer-Staples”, “Manufacturing”, “Energy”, “High-Tech”, “Telecommunication”, “Wholesale and Retail”, “Health”, “Utilities”, and “Others”. The period of interest is from July, 1963 to November, 2004. As in DeMiguel et al. (2009), we added into this dataset an extra factor that is the excess return on the US equity market portfolio. The second set of data is named “International Portfolios” that includes eight international equity indices (Canada, France, Germany, Italy, Japan, Switzerland, the UK and the US) and the World Equity Index. Excess-returns are computed for the period from January, 1970 to July, 2001. The third database is titled “MKT/SMB/HML” (see also Pástor, 2000) and represented by three broad portfolios

The last three datasets (see Wang, 2005) contain 20 portfolios sorted by size and book-to-market from July, 1963 to December, 2004 to which we add the market portfolio for “FF-1-factor”, the zero-cost portfolios HML and SMB for “FF-3-factor” and the momentum portfolio for “FF-4-factor”.

Table 4 reports the monthly Sharpe ratios of nine portfolios, respectively, statically (Panel A) and dynamically (Panel B) optimized on the six datasets previously described (in columns) according to the following strategies (in rows): 1/N, maximum mean return, maximum Sharpe ratio, maximum Omega ratio for various thresholds (-2.00%, .00%, 5.00%, 7.00%, 10.00%) and minimum volatility. Furthermore, we also report the p-value (in italic) of the difference between the Sharpe ratios of each strategy and the one of the 1/N strategy, which is computed using the Jobson and Korkie (1981) methodology.

The results given by Table 4 (Panel A and Panel B) do not show a very obvious intrinsic structure when, as done by DeMiguel et al. (2009), we only use the Sharpe ratio as the criterion: the various paradigms produce very similar financial outputs (except in some cases and depending on the database – see for instance the results of Omega optimization with a high threshold). However, the performance of each strategy on the six datasets confirms our previous results. Indeed, the Omega-based optimal portfolios, respectively, for a low and a high threshold, are very close to those obtained in minimum volatility and maximum mean return optimization paradigms. In other words, the use of the Omega ratio does not present here a true added value.
Table 4: Sharpe Ratios of Optimal Portfolios

### Panel A: Static Analysis

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Industry Portfolios</th>
<th>Inter'l Portfolios</th>
<th>Mkt/ SMB/HML Mkt/ SMB/HML</th>
<th>FF 1-factor 3-factor 4-factor</th>
<th>FF 1-factor 3-factor 4-factor</th>
<th>FF 1-factor 3-factor 4-factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/N</td>
<td>.1162 .2357 .1961</td>
<td>.2234 .2267 .2344</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
<tr>
<td>Max. mean return</td>
<td>.1186 .1684 .1053</td>
<td>.2749 .2749 .2749</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
<tr>
<td>Max. Sharpe</td>
<td>.1364 .2281 .2323</td>
<td>.2780 .3192 .3828</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
<tr>
<td>Max. Omega (τ = -2.00%)</td>
<td>.1458 .2583 .2332</td>
<td>.2915 .3436 .4234</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
<tr>
<td>Max. Omega (τ = 0.00%)</td>
<td>.1166 .2146 .1019</td>
<td>.2901 .3439 .4223</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
<tr>
<td>Max. Omega (τ = 5.00%)</td>
<td>.1455 .2581 .2335</td>
<td>.2901 .3439 .4223</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
<tr>
<td>Max. Omega (τ = 7.00%)</td>
<td>.1371 .2558 .1053</td>
<td>.2857 .2858 .3586</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
<tr>
<td>Min. volatility</td>
<td>.1157 .1935 .2335</td>
<td>.1931 .2651 .3386</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
</tbody>
</table>

### Panel B: Dynamic Analysis

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Industry Portfolios</th>
<th>Inter'l Portfolios</th>
<th>Mkt/ SMB/HML Mkt/ SMB/HML</th>
<th>FF 1-factor 3-factor 4-factor</th>
<th>FF 1-factor 3-factor 4-factor</th>
<th>FF 1-factor 3-factor 4-factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/N</td>
<td>.1356 .2577 .2389</td>
<td>.2567 .2610 .2682</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
<tr>
<td>Max. mean return</td>
<td>.0635 .1668 .0903</td>
<td>.2813 .2813 .2704</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
<tr>
<td>Max. Sharpe</td>
<td>.1031 .1945 .2223</td>
<td>.3162 .3103 .3335</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
<tr>
<td>Max. Omega (τ = -2.00%)</td>
<td>.1760 .3203 .2670</td>
<td>.3452 .4215 .4889</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
<tr>
<td>Max. Omega (τ = 0.00%)</td>
<td>.2378 .0626 .3877</td>
<td>.3480 .4177 .4992</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
<tr>
<td>Max. Omega (τ = 5.00%)</td>
<td>.2941 .0487 .3746</td>
<td>.3488 .3865 .4570</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
<tr>
<td>Max. Omega (τ = 7.00%)</td>
<td>.4469 .1181 .3795</td>
<td>.3460 .3492 .4216</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
<tr>
<td>Max. Omega (τ = 10.00%)</td>
<td>.5234 .1185 .0421</td>
<td>.3455 .3484 .3893</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
<tr>
<td>Min. volatility</td>
<td>.1688 .1842 .2207</td>
<td>.1377 .2301 .3073</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
<td>.0000 .0000 .0000</td>
</tr>
</tbody>
</table>

Source: Kenneth R. French Website (data source); monthly excess return in USD over the 90-day T-bill. These tables are an abstract of the Table 3 in DeMiguel et al. (2009). Panel A (static analysis) and Panel B (dynamic analysis) report the monthly Sharpe ratios for the various strategies (from 1/N to minimum volatility objective) and, in parentheses, the $p$-value of the difference between the Sharpe ratios of each strategy and those of the 1/N benchmark, computed using the Jobson and Korkie (1981) methodology. See also DeMiguel et al. (2009). Computations by the authors.
6. Conclusion

We recently observed that an increasing number of significant finance articles refer to the Omega measure for evaluating the performance of funds or of active strategies (e.g. Eling and Schuhmacher, 2007; Farinelli and Tibiletti, 2008; Annaert et al., 2009; Bertrand and Prigent, 2011; Zieling et al., 2014; Kapsos et al., 2014; Hamidi et al., 2014), since both the return distributions may not be Gaussian and the volatility may not be the relevant risk measure in such a case. The Omega measure is also used in some non-linear portfolio optimization problems when returns are characterized by severe downside risks (e.g. Mausser et al., 2006; Kane et al., 2009; Gilli et al., 2011).

Our article started with a first simple intuition, which was that the Omega measure might excessively privilege the performance regardless of the implied risk and thus exhibits inconsistency with the Stochastic Dominance criteria. The second intuition was on the role of the threshold that we should define for computing Omega, which might be very crucial, as already pointed out by Bertrand and Prigent (2011). Based on these two assumptions, we have studied more precisely the Omega ratio under two main aspects.

The first one is the relevance of using Omega as a portfolio ranking criterion. First, we show through a simple realistic illustration that the Omega measure can be equal for two portfolios, even when the total risk of the first fund is twice that of the second one. Secondly, we observe that for positive thresholds close to .00%, the trade-offs return/risk of the Omega and the Sharpe ratios are both almost equal to 1.00. This trade-off is inferior to 1.00 (for instance, equal to .25) when using high thresholds in the computation of the Omega measure (for example, 10.00%). The lower the threshold, the closer the trade-offs using the Sharpe ratio and the Omega criterion. We have thus demonstrated that portfolio rankings obtained according to this performance measure are seriously influenced by the chosen threshold, and more precisely by the difference between this threshold and the mean return of studied underlying return distributions. Choices of investments guided by the Omega measure may thus yield different investment decisions, depending on the threshold, that should not be too high or too low.

The second interest is the use of the Omega ratio within an optimization frame-
work. In a simple static Gaussian optimization setting, we first show that the \textit{Omega} ratio can lead to clear under-optimal solutions in some realistic cases (mainly when the given threshold is higher than the mean return of return distributions of some considered risky assets). Our next results are based on three different market databases used in publications, respectively, by Hentati et al. (2010), Darolles et al. (2009) and DeMiguel et al. (2009). The first dataset represents five main asset classes, the second database regroups 18 hedge funds and the third one corresponds to six sub-markets as classified by Kenneth French. Using these three datasets, we provide illustrations in which \textit{Omega}-based optimal portfolios are sensibly very similar to those obtained according to more classical criteria (namely the \textit{maximum} Sharpe ratio and the \textit{minimum} volatility). Furthermore, we show in our study that an \textit{Omega}-based optimal portfolio is similar to a \textit{maximum} mean return-based optimal portfolio for a high threshold, and to a \textit{minimum} volatility-based optimal portfolio for a low threshold. This confirms the intuition that a low threshold corresponds to risk-averse agents (when the risk-free rate is the preferred asset in the limit case) and a high threshold represents greedy investors (with a strong preference for performance).

We conclude that \textit{Omega} ratio can entail misleading financial decisions and lead us to question the use of this measure in any of the traditional finance applications it was designed for; we finally have some doubts about its accuracy, stability and overall usefulness.\footnote{See Hübner (2007) for a general discussion on the precision and stability of performance measures.}

Finally, the \textit{Omega} ratio belongs to a general class of performance measurement based on features of return densities. \textit{Omega} is, indeed, a simplified version of some other measures when varying the parameters in the general function proposed by Caporin et al. (2013). For instance, the Sharpe-\textit{Omega} ratio (Kazemi et al., 2004) is reported in some studies\footnote{See also Appendix C for a comparison between the \textit{Omega} and the Sharpe-\textit{Omega} ratios.} to be equal to $\Omega_p (\tau) - 1$; thus, it would also be interesting to challenge such measures. In the same vein, we should question other related measures such as, for instance, the Sortino-Meer-Plantinga (1999) Upside-Potential ratio, the Gemmill-Hwang-Salmon (2006) Loss-Averse Performance measures and the Farinelli-Tibiletti (2008) measure (\textit{Cf.})
Caporin et al., 2013), in order to see if they share, or not, the same drawbacks.

7. References


Appendices are available on demand to the authors.