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**Abstract**

Seemingly unrelated regression (SUR) models are used in studying the interactions among economic variables of interest. In a high dimensional setting and when applied to large panel of time series, these models have a large number of parameters to be estimated and suffer from inferential problems.

We propose a Bayesian nonparametric hierarchical model for multivariate time series in order to avoid overparametrization and overfitting issues and to allow for shrinkage toward multiple prior means with unknown location, scale and shape parameters. We propose a two-stage hierarchical prior distribution. 

The first stage of the hierarchy consists in a lasso conditionally independent prior distribution of the Normal-Gamma family for the SUR coefficients. The second stage is given by a random mixture distribution for the Normal-Gamma hyperparameters, which allows for parameter parsimony through two components. The first one is a random Dirac point-mass distribution, which induces sparsity in the SUR coefficients; the second is a Dirichlet process prior, which allows for clustering of the SUR coefficients.

We provide a Gibbs sampler for posterior approximations based on introduction of auxiliary variables. Some simulated examples show the efficiency of the proposed methods. We study the effectiveness of our model and inference approach with an application to macroeconomics.

**Keywords:** Bayesian nonparametrics; Large VAR; MCMC; Mixture model; Dirichlet Process; Graphical Representation

**JEL Codes:** C11, C13, C14, C32, C51, E17

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1 Introduction

In the last decade, high dimensional models and large datasets have increased their importance in different fields, especially in economics and finance. The use of large dataset has been proved to improve the forecasts in large macroeconomics and financial models (see, Banbura et al. (2010), Carriero et al. (2013), Koop (2013), Stock and Watson (2012)). For analyzing and better forecasting them, seemingly unrelated regression (the so called SUR) models have been introduced, but they required estimation of large number of parameters with few observations and the dimensionality of the parameters can be reduced through shrinkage methods. A SUR model is used in econometrics and financial modeling (Zellner, 1962, 1971) and analyzes the individual relationships linked by the fact that their disturbances are correlated. Moving in more complex econometric models, Markov Chain Monte Carlo methods are used for Bayesian analysis of the variations of the SUR models (Chib and Greenberg (1995)).

If we add lagged variables and deterministic terms as common regressor, we have a special case of SUR model, the Vector autoregressive (VAR) model. As in the case of SUR models, VAR models are useful for econometric modeling and for studying the interactions among the economic variables of interest. VAR models are a standard tool for structural analysis and forecasting in macroeconomics (Sims, 1980, 1992), on the other hand the use of a Bayesian approach to inference should be use to solve the problem of overfitting thanks to the introduction of prior on the VAR parameters. In fact, Litterman (1980) introduced a Bayesian approach to VAR estimation as a solution to the problem of overfitting. E.g. Litterman (1986), Doan et al. (1984) and Sims and Zha (1998) specified particular priors constraint on the VAR parameters for Bayesian VAR, while Canova and Ciccarelli (2004) discuss prior choice for panel VAR models.

Unfortunately the use of large SUR requires estimation of large number of parameters, hence we rely on two different streams of the literature: sparsity and nonparametrics. For solving this problem, in the literature, sparse SUR models have been introduced and, in particular, Wang (2010) develops a sparse SUR model with Gaussian errors, where the coefficients are near zero in both the regression coefficients matrix and the error precision matrix. On the other hand, Ahelgebey et al. (2014) propose an approach to handling multivariate
time series of high dimension by combining the notion of causality with the concept of sparsity in the graph structure. Moving in the graphical model representation of VAR, both the Stochastic Search Variable Selection (SSVS) of George et al. (2008) and the Bayesian graphical VAR (BGVAR) model of Ahelgebey et al. (2015) use two separate sets of restrictions for the contemporaneous and lagged interactions, where the SSVS used the reduced-form model, while in the BGVAR the restrictions are directly used in the structural model and help to solve the identification problem of the SVAR using the graph structures. Furthermore, the two models differ in the computational part, where George et al. (2008) use a single-move Gibbs sampler, while Ahelgebey et al. (2015) focus on a collapsed and multi-move Gibbs sampler.

Korobilis (2013) extended the use of SSVS to restricted VARs and particularly to select variable in linear and nonlinear VARs using Markov Chain Monte Carlo (MCMC) methods (see Koop and Korobilis (2010) for a textbook level treatment). Hence, Korobilis (2013) and Koop and Korobilis (2013) focused their analysis on the time-varying parameters VAR (TVP-VAR) for measuring monetary policy.

In this paper we propose a Bayesian nonparametric hierarchical model for multivariate time series to allow shrinkage of the SUR coefficients to multiple locations, which means with multiple means with the location, scale and shape parameters of these means unknown.

We introduce a class of sparse SUR (sSUR), where many of the SUR coefficients shrink to zero, due to the shrinking properties of our model. Hence, the sSUR models attempt to improve efficiency of parameters estimation, prediction accuracy and interpretation of the temporal dependence structure of the time series.

We use the Bayesian Lasso prior, which allows us to reformulate the SUR model as a penalized regression problem, in order to determine which SUR coefficients shrink to zero.

The use of shrinkage can help the estimation performance and reduce the mean square errors. Their estimators have an important role in the Bayesian framework thanks to the paper of Tibshirani (1996), where he introduced the Lasso procedure, which stays for 'least absolute shrinkage and selection operator' or penalized estimators methods. He proposed a new method for estimation in linear models, therefore the lasso procedure minimizes the residual sum of squares subject to the sum of the absolute value of the coefficient less than
a constant. In addition, Tibshirani (1996) used a double exponential prior concentrated around zero with heavier tails, which force sparsity in the model.


As far as we know, most of the papers in this fields work on the sVAR in a parametric view and/or in the Bayesian framework, our work develops a nonparametric prior for the analysis of the sSUR using a Bayesian approach. We build on Bassetti et al. (2014), which propose a vector of dependent Dirichlet process prior to capture similarities in clustering effects across time series and on MacLehose and Dunson (2010), which propose a Bayesian semiparametric approach that allows shrinkage to multiple locations using a mixture of double exponential priors with location and scale parameters assigned through a Dirichlet process hyperpriors to allow groups of coefficients to be shrunk toward the same mean.

Hence, after the seminal papers of Ferguson (1973), Lo (1984) and Sethuraman (1994), Dirichlet process priors and their multivariate extensions (e.g., see Müller et al. (2004), Griffin and Steel (2006), Hatjispyros et al. (2011), Hjort et al. (2010) for a review of Bayesian nonparametrics), are now widely used due to the availability of efficient algorithms for posterior computations (Escobar and West, 1995; MacEachern and Müller, 1998; Papaspiliopoulos and Roberts, 2008; Walker, 2007; Kalli et al., 2011), including but not limited to applications in time series settings (Hirano, 2002; Chib and Hamilton, 2002; Rodriguez and ter Horst, 2008; Jensen and Maheu, 2010; Griffin, 2011; Griffin and Steel, 2011; Bassetti et al., 2014; Jochmann, 2015).

In this work, we define a novel Bayesian nonparametric hierarchical model that allows shrinkage to multiple locations using a Normal-Gamma distribution with location, scale and shape parameters unknown. The second stage of the hierarchy is given by a mixture of hyperprior distributions for the Normal-Gamma hyperparameters, which allows for shrinkage of different locations. This mixture consists of two different components, where we assigned a Dirichlet process hyperpriors, which allows to achieve parameters parsimony due to clustering of the SUR coefficients. We rely on MCMC algorithm on slice sampling by Kalli et al. (2011), which is an improved version
of the algorithm of Walker (2007) and on the paper of Hatjispyros et al. (2011), where they present an approach to modeling dependent nonparametric random density functions through mixture of DP model.

In this paper we will contribute to the literature of financial and macroeconomic connectedness (Demirer et al., 2015; Diebold and Yilmaz, 2014). We are interested in the estimation of an empirical network through the definition of an adjacency matrix, which allows us to estimate the possible measure of contagion between different countries in the estimated network. We allow the measurement of the level of contagion through the use of a network representation. In particular, a network could be connected or less connected depending on the assumption and on the construction of the adjacency matrix. In our work, the sparsity of a graph allows us to construct a less connected network with respect to the non-sparse graph. With the notion of sparsity of a graph, we mean that the vertices of a graph and consequently the edges have less connections between them. The network connectedness has a central role in the financial, systemic and credit risk measurement and helps us to understand fundamental macroeconomic risks. In the last years the empirical and theoretical works have increased importance in the literature; see for example, Acharya et al. (2012), Billio et al. (2012), Diebold and Yilmaz (2015), Bianchi et al. (2015), Barigozzi and Brownlees (2016), Brownlees and Engle (2016), Diebold and Yilmaz (2016).

The paper is organized as follows. Section 2 introduces our sparse Bayesian SUR model and the prior assumptions on the hyperparameters. In Section 3 we explain the computational details of the model and the Gibbs sampling, while Section 4 illustrates the performance of the methodology through simulated results. Finally, Section 5 is devoted to the application of our methodology to the analysis of a macroeconomic dataset.

2 A sparse Bayesian SUR model

In this section, we review some preliminary notions about Seemingly unrelated regression models (SUR) and the generalized one, the Vector autoregressive model (VAR). Furthermore we focus on the prior specifications for our specific sparse SUR.
2.1 SUR and VAR models

Zellner (1962) paper introduced the seemingly unrelated regression (SUR) model and tried to analyze individual relationships that are linked by the fact that their disturbances are correlated. Hence, SUR models have many applications in different fields, for example demand functions can be estimated for different households for a given commodity or for different commodities.

In a SUR model with \( N \) units (or groups of cross-section observations) we consider a sequence of \( m_i \)-dimensional vectors of dependent variables, \( y_{i,t} \), that follow individual regressions:

\[
y_{i,t} = X_{i,t}\beta_i + \varepsilon_{i,t}, \quad t = 1, \ldots, T \quad i = 1, \ldots, N,
\]

where \( X_{i,t} \) is the \((m_i \times n_i)\)-matrix of observations on \( n_i \) explanatory variables with a possible constant term for individual \( i \) at time \( t \), \( \beta_i = (\beta_{i,1}, \ldots, \beta_{i,n_i}) \) is a \( n_i \)-vector of unknown coefficients, and \( \varepsilon_{i,t} \) is a random error. We write (1) in a stacked regression form:

\[
y_t = X_t\beta + \varepsilon_t \quad t = 1, \ldots, T,
\]

where \( y_t = (y_{1,t}', \ldots, y_{N,t}')' \) is the \( m \times 1 \) vector of observations, with \( m = \sum_{i=1}^{N} m_i \), \( X_t = \text{diag}(X_{1,t}, \ldots, X_{N,t}) \) is the \( m \times n \) matrix of observations on the explanatory variables at time \( t \) with \( n = \sum_{i=1}^{N} n_i \), \( \beta = (\beta_1', \ldots, \beta_N')' \), the \( n \)-vector of coefficients and \( \varepsilon_t = (\varepsilon_{1,t}', \ldots, \varepsilon_{m,t})' \) is the vector of errors distributed as \( \mathcal{N}_m(0, \Sigma) \), where \( \varepsilon_t \) and \( \varepsilon_s \) are independent for \( t \neq s \).

The use of SUR models is important to gain efficiency in estimation by combining different equations and to impose or test restrictions that involve parameters in different equations.

An important special case of the SUR model is the vector autoregressive (VAR) model. Due to the work of Sims (1980), VAR models have acquired a permanent place in the toolkit of applied macroeconomics to study the impact of a policy decision on the variables of interest. A VAR model of order \( p \) (VAR(p)) is defined as

\[
y_t = b + \sum_{i=1}^{p} B_i y_{t-i} + \varepsilon_t,
\]

for \( t = 1, \ldots, T \), where \( y_t = (y_{1,t}, \ldots, y_{m,t})' \), \( b = (b_1, \ldots, b_m)' \) and \( B_i \) is a \((m \times m)\) matrix of coefficients. We assume that \( \varepsilon_t = \)
$(\varepsilon_{1,t}, \ldots, \varepsilon_{m,t})'$ follows a Gaussian distribution $\mathcal{N}_m(0, \Sigma)$ with mean $0$ and covariance matrix $\Sigma$.

The VAR($p$) can be obtained as a special case of (2) by setting $N = 1$, $m = m_1$ and writing (3) in a stacked regression form:

$$y_t = (I_m \otimes x'_t)\beta + \varepsilon_t,$$

where $x_t = (1, y'_{t-1}, \ldots, y'_{t-p})'$ is the vector of predetermined variables, $\beta = \text{vec}(B)$, where $B = (b, B_1, \ldots, B_p)$, $\otimes$ is the Kronecker product and vec the column-wise vectorization operator that stacks the columns of a matrix in a column vector.

### 2.2 Prior assumption

The number of parameters to estimate in (2) is $q = r + (m + 1)m/2$, with $r = \sum_{i=1}^{N} r_i$, $r_i = n_i$. For large value of $m$, $q$ can be large and add some problems during the estimation, such as overfitting, or unstable predictions and difficult-to-interpret descriptions of the temporal dependence. In order to avoid overparameterization issues and the overfitting problem a hierarchical strategy in prior specification has been suggested in the Bayesian dynamic panel modelling literature (e.g., Canova and Ciccarelli (2004), Kaufmann (2010), and Bassetti et al. (2014)). The hierarchical prior can be used to incorporate cross-equation interdependences and various degrees of information pooling across units (e.g., see Chib and Greenberg (1995) and Min and Zellner (1993)), while a different stream of literature is using instead a prior model which induces sparsity (e.g., MacLehose and Dunson (2010), Wang (2010)).

In this paper we combine the two strategies and define a hierarchical prior distribution which induces sparsity on the vector of coefficients $\beta$. In order to regularize (2) we incorporate a penalty using a lasso prior $f(\beta) = \prod_{j=1}^{r} \mathcal{N}\mathcal{G}(\beta_j|0, \gamma, \tau)$, where $\mathcal{N}\mathcal{G}(\beta|\mu, \gamma, \tau)$ denotes the normal-gamma distribution with location parameter $\mu$, shape parameter $\gamma > 0$ and scale parameter $\tau > 0$. The normal-gamma distribution has density function

$$f(\beta|\mu, \gamma, \tau) = \frac{\tau^{\gamma+1}}{2^{\gamma+\tau}\Gamma(\gamma)}|\beta - \mu|^{\gamma+1/2} K_{\gamma+1/2}(\sqrt{\tau} |\beta - \mu|),$$

where $K_{\nu}(\cdot)$ represents the modified Bessel function of the second kind with the index $\gamma$ (see Abramowitz and Stegun (1972)). The
normal-gamma distribution has the double exponential distribution as a special case for $\gamma = 0$ and can be represented as a scale mixture of normals (see Andrews and Mallows (1974) and Griffin and Brown (2006)):

$$\mathcal{NG}(\beta|\mu, \gamma, \tau) = \int_{0}^{+\infty} \mathcal{N}(\beta|\mu, \lambda) \mathcal{G}(\lambda|\gamma/2) d\lambda,$$

(5)

where $\mathcal{G}(\cdot|a, b)$ denotes a gamma distribution.$^1$

The normal-gamma distribution in (5) induces shrinkage toward the prior mean of $\mu$, but we can extend the lasso model specification by introducing a mixture prior with separate location parameter $\mu_j^*$, separate shape parameter $\gamma_j^*$ and separate scale parameter $\tau_j^*$ such that: $f(\beta) = \prod_j \mathcal{NG}(\beta_j|\mu_j^*, \gamma_j^*, \tau_j^*)$. In our paper, we favor the sparsity of the parameters through the use of carefully tailored hyperprior and we use a nonparametric Dirichlet process prior (DPP), which reduces the overfitting problem and the curse of dimensionality by allowing for parameters clustering due to the concentration parameter and the base measure choice.

Also, following Bassetti et al. (2014), we assume that $N$ blocks of parameters can be exogenously defined. The blocks correspond to series from different countries which share a sparse component but have possibly different clustering features. Our framework can be extended to include dependence in the clustering features (Bassetti et al., 2014; Taddy, 2010; Griffin and Steel, 2011).

In our case we define $\theta^* = (\mu^*, \gamma^*, \tau^*)$ as the parameters of the Normal-Gamma distribution, and assume a prior $Q_t$ for $\theta_{ij}^*$, that is

$$\beta_j \overset{\text{i.i.d.}}{\sim} \mathcal{NG}(\beta_j|\mu_j^*, \gamma_j^*, \tau_j^*),$$

(6)

$$\theta_{ij}^* \overset{\text{i.i.d.}}{\sim} Q_t,$$

(7)

for $j = 1, \ldots, r_t$ and $t = 1, \ldots, N$.

Following a construction of the hierarchical prior similar to the one proposed in Hatjispyros et al. (2011) we define the vector of random

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$^1$The gamma distribution of $\tau$ ($\tau \sim \mathcal{G}(a, b)$) used in this paper is parametrized as:

$$f(\tau|a, b) = \frac{b^a}{\Gamma(a)} \tau^{a-1} \exp(-b\tau) I_{(0, +\infty)}(\tau)$$
measures
\[ Q_1(\theta_1) = \pi_1 P_0(\theta_1) + (1 - \pi_1) P_1(\theta_1), \]
\[ \vdots \]
\[ Q_N(\theta_N) = \pi_N P_0(\theta_N) + (1 - \pi_N) P_N(\theta_N), \]
with the same sparse component \( P_0 \) in each equation and with the following hierarchical construction as previously explained,
\[ P_0(\theta) \sim \delta(0, \tau_0) \det (\mu, \gamma, \tau), \]
\[ P_l(\theta) \sim_{i.i.d.} \text{DP}(\tilde{\alpha}, G_0), \quad l = 1, \ldots, N, \]
\[ \tau_l \sim_{i.i.d.} \text{Be}(\pi_l|1, \alpha_l), \quad l = 1, \ldots, N, \]
\[ (\tau_0, \tau_n) \sim g(\tau_0, \tau_n|\nu_0, p_0, s_0, n_0), \]
\[ G_0 \sim N(\mu|c, d) \times g(\gamma, \tau|\nu_1, p_1, s_1, n_1) \]

where \( \delta(\psi_0) (\psi) \) denotes the Dirac measure indicating that the random vector \( \psi \) has a degenerate distribution with mass at the location \( \psi_0 \), and \( g(\gamma_0, \tau_0) \) is the conjugate joint prior distribution (see Miller (1980)) with density
\[ g(\gamma, \tau|\nu_0, p_0, s_0, n_0) \propto \tau_0^{\nu_0 - 1} p_0^{\gamma_0 - 1} \exp\{-s_0 \tau_0\} \frac{1}{\Gamma(\gamma_0)^{n_0}}, \]
and hyperparameters fixed such that \( \nu_0 > 0, p_0 > 0, s_0 > 0 \) and \( n_0 > 0 \). From Miller (1980), we construct the gamma two-parameters \( g(\gamma, \tau) = g(\tau|\gamma) g(\gamma) \), where \( g(\tau|\gamma) \sim g(a, \nu_0, \gamma, s_0) \) and we marginalized out such that:
\[ g(\gamma) = \int_0^\infty g(\gamma, \tau) d\tau = C \frac{\Gamma(\nu_0 \gamma) p_0^{\gamma_0 - 1}}{\Gamma(\gamma_0)^{n_0} s_0^{\nu_0 \gamma}}, \]
\[ g(\tau|\gamma) = \frac{g(\gamma, \tau)}{g(\gamma)} = \frac{\tau_0^{\nu_0 \gamma - 1} e^{-s_0 \tau}}{\Gamma(\nu_0 \gamma)} s_0^{\nu_0 \gamma}, \]
with a normalizing constant \( C \) such that \( 1 = \int_0^\infty g(\gamma) d\gamma \). Based on MacLehose and Dunson (2010) and on our computational experiments, we assume the following parameter setting for the sparse and nonsparse component in the gamma two parameters distribution, \( g(\gamma, \tau) \),
\[ \nu_0 = 30 \quad s_0 = 1/30 \quad p_0 = 0.5 \quad n_0 = 18, \]
\[ \nu_1 = 3 \quad s_1 = 1/3 \quad p_1 = 0.5 \quad n_1 = 10. \]
As described in the hierarchical prior representations in (8) and in (9), with probability \( \pi \) (distributed as a beta\(^2\)) a coefficient, \( \beta_j \) is shrunk toward zero as in standard lasso, while with probability \( 1 - \pi \) the coefficient is distributed as a \( DP(\alpha, G_0) \). The amount of shrinkage is determined by the shape and scale parameter \((\gamma, \tau)\), which moves as a two-parameters gamma (Miller (1980)).

The Dirichlet Process, \( DP(\alpha, G_0) \), can be defined by using the stick-breaking representation (Sethuraman (1994)) given by:

\[
P_l(\cdot) = \sum_{j=1}^{\infty} w_{lj} \delta_{\{\theta_{lj}\}}(\cdot) \quad l = 1, \ldots, N. \tag{13}
\]

Following the definition of the dependent stick-breaking processes, proposed by MacEachern (1999) and MacEachern (2001) the atoms \( \theta_{lj} \) and the weights \( w_{lj} \) (for \( l = 1, \ldots, N \)) are stochastically independent and satisfy the following hypothesis:

- \( \theta_{lj} \) is an independent and identically distributed (i.i.d.) sequence

\(^2\)The beta distribution for \( x \ (x \sim \text{Be}(\alpha, \beta)) \) used in this paper is parametrized as follows:

\[
f(x|\alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}I_{[0,1]}(x)
\]

where \( B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta) \) and \( \alpha, \beta > 0 \)
of random elements with common probability distribution $G_0$ ($\theta_{ij} \sim G_0$);

- the weights $(w_{lj})$ are determined through the stick-breaking construction for $j > 1$, while for $j = 1$ $w_{l1} = v_{l1}$:

$$w_{lj} = v_{lj} \prod_{k=1}^{j-1} (1 - v_{lk}) \quad l = 1, \ldots, N$$

with $v_j = (v_{lj}, \ldots, v_{Nj})$ independent random variables taking values in $[0,1]^N$ distributed as a $\text{Be}(1,\tilde{\alpha})$ such that $\sum_{j \geq 1} w_{lj} = 1$ almost surely for every $l = 1, \ldots, N$.

After this definition, we are able to construct a random density function $f(\beta|\mathcal{P})$ based on an infinite mixture representation similar to the well known Dirichlet process mixture model (Lo (1984)):

$$f_l(\beta|\tilde{\mathcal{P}}_l) = \int K(\beta|\theta)\tilde{\mathcal{P}}_l(d\theta), \quad (14)$$

where $K(\beta|\theta)$ is a density for each $\theta \in \Theta$, the so called density kernel and $\tilde{\mathcal{P}}_l$ is a random measure. In our paper, the density kernel is defined as $K(\beta|\theta) = N\mathcal{G}(\beta|\mu,\gamma,\tau)$. Following the definition of the density kernel and using the representation as infinite mixture, we have that, for each $l = 1, \ldots, N$, the equation (14) has the following representation

$$f_l(\beta|Q_l) = \pi_l f(\beta|\mathcal{P}_0) + (1 - \pi_l) f(\beta|\mathcal{P}_l) = \pi_l \int N\mathcal{G}(\beta|\mu,\gamma,\tau)\mathcal{P}_0(d(\mu,\gamma,\tau))$$

$$+ (1 - \pi_l) \int N\mathcal{G}(\beta|\mu,\gamma,\tau)\mathcal{P}_l(d(\mu,\gamma,\tau))$$

$$= \pi_l N\mathcal{G}(\beta|0,\gamma_0,\tau_0) + (1 - \pi_l) \sum_{k=1}^{\infty} w_{lk} N\mathcal{G}(\beta|\mu_{lk},\gamma_{lk},\tau_{lk})$$

$$= \sum_{k=0}^{\infty} \tilde{w}_{lk} N\mathcal{G}(\beta|\tilde{\theta}_{lk}),$$

where

$$\tilde{w}_{lk} = \begin{cases} 
\pi_l, & k = 0 \\
(1 - \pi_l) w_{lk}, & k > 0 
\end{cases} \quad \tilde{\theta}_{lk} = \begin{cases} 
(0, \gamma_0, \tau_0), & k = 0 \\
(\mu_{lk}, \gamma_{lk}, \tau_{lk}), & k > 0. 
\end{cases}$$

As regards to the choice of the prior for $\Sigma$, we model it by considering its restrictions induced by a graphical model structuring.
A graph $G$ is defined by the pair $(L, E)$, where $L$ is the vertex set and $E$ is the edge-set, or the set of linkages. In our case the prior over the graph structure is defined as a Bernoulli distribution with parameter $\psi$, which is the probability of having an edge. That is, a $m$ node graph $G = (L, E)$, with $|L|$ the cardinality of the set of nodes and with $|E|$ edges has a prior probability:

$$p(G) \propto \prod_{i,j} \psi^{e_{ij}} (1 - \psi)^{1 - e_{ij}} = \psi^{|E|} (1 - \psi)^{|T| - |E|},$$  \hspace{1cm} (15)

with $e_{ij} = 1$ if $(i, j) \in E$ and $|T| = |L|/2$ is the maximum number of edges, while to induce sparsity we choose $\psi = 2/(p - 1)$ which would provide a prior mode at $p$ edges. Conditional on a specified graph $G$ we assume a Hyper Inverse Wishart prior distribution for $\Sigma$ that is:

$$\Sigma \sim \mathcal{HIW}(b, L),$$  \hspace{1cm} (16)

where $b$ means the degrees of freedom and $L$ is the scale hyperparameters. The density function of the $\mathcal{HIW}$ is represented in the Appendix A.

### 3 Computational details

In this section we will develop a Gibbs sampler algorithm in order to approximate the posterior distribution. For simplicity of notations we will focus on the bivariate case, $N = 2$ and consequently $l = 1, 2$, and, without loss of generality, we can extend the following representation to the multivariate case.

First of all, we focus on the slice latent variables for $l = 1, 2$ through the introduction of the latent variables, $u_{ij}, j = 1, \ldots, r_1$, for $f_l$, which allows us to write the infinite mixture model in an easy way. Hence we represent the full conditional of $\beta_{1j}$ as follows,

$$f_1(\beta_{1j}, u_{1j}|(\mu_1, \gamma_1, \tau_1), w_1) = \pi_1 \sum_{k=0}^{\infty} \mathbb{I}(u_{1j} < \bar{w}_{1k}) \mathcal{N}(\beta_{1j}|0, \gamma_{1k}, \tau_{1k}) + \pi_1 \sum_{k=1}^{\infty} \mathbb{I}(u_{1j} < w_{1k}) \mathcal{N}(\beta_{1j}|\mu_{1k}, \gamma_{1k}, \tau_{1k}) + (1 - \pi_1) \sum_{k=1}^{\infty} \mathbb{I}(u_{1j} < \bar{w}_0) \mathcal{N}(\beta_{1j}|0, \gamma_0, \tau_0) + (1 - \pi_1) \sum_{k=1}^{\infty} \mathbb{I}(u_{1j} < w_{1k}) \mathcal{N}(\beta_{1j}|\mu_{1k}, \gamma_{1k}, \tau_{1k}).$$
where we introduce a variable $\tilde{w}_{1k}$ such that we can apply the slice sampler and then we assume $\tilde{w}_{1k} = \tilde{w}_0 = 1$ if $k = 0$ and $\tilde{w}_{1k} = 0$ for $k > 0$ and, for simplicity of notations, we denote $(0, \gamma_1, 0, \tau_1, 0) = (0, \gamma_0, \tau_0)$.

Moving to the density function $f_2$, we introduce the latent variables $u_{2j}$, $j = 1, \ldots, r_2$, which allows us to write the following density:

$$f_2(\beta_{2j}, u_{2j}|(\mu_2, \gamma_2, \tau_2), w_2) = \pi_2 \mathbb{I}(u_{2j} < \tilde{w}_0) \mathcal{NG}(\beta_{2j}|(0, \gamma_0, \tau_0)) +$$

$$+ (1 - \pi_2) \sum_{k=1}^{\infty} \mathbb{I}(u_{2j} < w_{2k}) \mathcal{NG}(\beta_{2j} | \mu_{2k}, \gamma_{2k}, \tau_{2k}).$$

The introduction of the slice variables $(u_{1j}, u_{2j})$ allows us to reduce the dimensionality of the problem from a mixture with an infinite number of components to a similar finite mixture model. In particular, letting

$$A_{w_1}(u_{1j}) = \{k : u_{1j} < w_{1k}\}, \quad j = 1, \ldots, r_1,$$

$$A_{w_2}(u_{2j}) = \{k : u_{2j} < w_{2k}\}, \quad j = 1, \ldots, r_2,$$

then it can be proved that the cardinality of the sets $(A_{w_1}, A_{w_2})$ is almost surely finite.

Therefore, we express $f_1$ and $f_2$ as an augmented random joint probability density function for $\beta_{1j}, \beta_{2j}$ and $u_{1j}, u_{2j}$

$$f_1(\beta_{1j}, u_{1j}|(\mu_1, \gamma_1, \tau_1), w_1) = \pi_l \mathbb{I}(u_{1j} < \tilde{w}_0) \mathcal{NG}(\beta_{1j}|(0, \gamma_0, \tau_0))$$

$$+ (1 - \pi_l) \sum_{k \in A_{w_1}(u_{1j})} \mathcal{NG}(\beta_{1j} | \mu_{lk}, \gamma_{lk}, \tau_{lk}).$$

We iterate the data augmentation principle for each $f_l$ (with $l = 1, 2$) through the introduction of two auxiliary variables, the latent variables $\delta_{lj}$ ($j = 1, \ldots, r_l$) and the allocation variables $d_{lj}$ ($j = 1, \ldots, r_l$). The first variable described above selects one of the two random measures $\mathbb{P}_0$ and $\mathbb{P}_l$, hence, when $\delta_{lj}$ is equal to one, we choose the sparse component $\mathbb{P}_0$, while if it is zero, we choose the nonsparse component $\mathbb{P}_l$ and we need to introduce the allocation variables. The second variable of interest, $d_{lj}$, selects the components of the Dirichlet mixture $\mathbb{P}_l$ to which each single coefficient $\beta_{lj}$ is allocated to. Then the density function can be expressed as

$$f_l(\beta_{lj}, u_{lj}, d_{lj}, \delta_{lj}) = \left(\mathbb{I}(u_{lj} < \tilde{w}_{dlj}) \mathcal{NG}(\beta_{lj}|(0, \gamma_0, \tau_0))\right)^{1-\delta_{lj}} \times$$

$$\left(\mathbb{I}(u_{lj} < w_{ldlj}) \mathcal{NG}(\beta_{lj}|(\mu_{ldlj}, \gamma_{ldlj}, \tau_{ldlj}))\right)^{\delta_{lj}} \frac{1}{\pi_l^{1-\delta_{lj}}(1 - \pi_l)^{\delta_{lj}}}.$$
From (5), we demarginalize the Normal-Gamma distribution by introducing a latent variable $\lambda_{ij}$ for each $\beta_{ij}$ such that the joint distribution has the following representation:

$$f_l(\beta_{ij}, \lambda_{ij}, u_{ij}, d_{ij}, \delta_{ij}) =$$

$$= \left( \mathbb{I}(u_{ij} < \bar{w}_{d_{ij}}) \mathcal{N}(\beta_{ij}|0, \lambda_{ij}) \mathcal{G}(\lambda_{ij} | \gamma_0, \tau_0 / 2) \right)^{1-\delta_{ij}} \times$$

$$\left( \mathbb{I}(u_{ij} < \bar{w}_{l_{d_{ij}}}) \mathcal{N}(\beta_{ij}| \mu_{l_{d_{ij}}}, \lambda_{ij}) \mathcal{G}(\lambda_{ij} | \gamma_{l_{d_{ij}}}, \tau_{l_{d_{ij}}} / 2) \right)^{\delta_{ij}} \frac{1-\delta_{ij}}{\pi_l} (1 - \pi_l)^{\delta_{ij}}.$$

Hence, we describe the joint posterior distribution based on the distribution previously defined as follows

$$f(\Theta, \Sigma, \Lambda, U, D, V, \Delta | Y) \propto$$

$$\prod_{l=1}^{T} (2\pi |\Sigma|)^{-1/2} \exp \left( -\frac{1}{2} (y_{t} - X'_l \beta)^\prime \Sigma^{-1} (y_{t} - X'_l \beta) \right) \times$$

$$\prod_{j=1}^{r_l} f_1(\beta_{ij}, \lambda_{ij}, u_{ij}, d_{ij}, \delta_{ij}) \prod_{j=1}^{r_j} f_2(\beta_{ij}, \lambda_{ij}, u_{ij}, d_{ij}, \delta_{ij}) \times$$

(17)

$$\prod_{k=1}^{k>1} \mathcal{B}(\nu_{1k}, 1, \alpha) \mathcal{B}(\nu_{2k}, 1, \alpha) \mathcal{H}(\nu_{1k}, \nu_{2k}) \mathcal{D}(b, L) \times g(\gamma_0, \tau_0 | \nu_0, p_0, s_0, n_0) \times$$

$$\prod_{k=1}^{k>1} \mathcal{N}(\mu_{1k}, c, d) g(\gamma_{1k}, \tau_{1k} | \nu_1, p_1, s_1, n_1) \mathcal{N}(\mu_{2k}, c, d) g(\gamma_{2k}, \tau_{2k} | \nu_1, p_1, s_1, n_1).$$

The distribution defined in (17) is not tractable thus we apply Gibbs sampling to draw random numbers from it. We use the notation $U = \{u_{ij} : j = 1, 2, \ldots, r_l \text{ and } l = 1, 2, \ldots, N\}$, $V = \{v_{ij} : j = 1, 2, \ldots \text{ and } l = 1, 2, \ldots, N\}$ to describe the latent variables and the stick-breaking components; $D = \{d_{ij} : j = 1, 2, \ldots, r_l \text{ and } l = 1, 2, \ldots, N\}$ and $\Delta = \{\delta_{ij} : j = 1, 2, \ldots, r_l \text{ and } l = 1, 2, \ldots, N\}$ to describe the new variables that we have introduced in this section. The Gibbs sampler iterates over the following steps using the conditional independence between the different variables as seen in the appendix:

1. The stick-breaking and the latent variables $U, V$ are updated given $[\Theta, \beta, \Sigma, G, \Lambda, D, \Delta, \pi, Y]$;

2. The latent variable $\Lambda$ is updated given $[\Theta, \beta, \Sigma, G, U, V, D, \Delta, \pi, Y]$;

3. The parameters of the Normal-Gamma distribution $\Theta$ are updated given $[\beta, \Sigma, G, \Lambda, U, V, D, \Delta, \pi, Y]$;

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4. The coefficients $\beta$ of the SUR model are updated given $[\Theta, \Sigma, G, \Lambda, U, V, D, \Delta, \pi, Y]$;
5. The matrix of variance-covariance $\Sigma$ is updated given $[\Theta, \beta, G, \Lambda, U, V, D, \Delta, \pi, Y]$;
6. The Graph $G$ is updated given $[\Theta, \beta, \Sigma, \Lambda, U, V, D, \Delta, \pi, Y]$;
7. The allocation and the latent variables $D, \Delta$ are updated given $[\Theta, \beta, \Sigma, G, \Lambda, U, V, \pi, Y]$;
8. The probability of being sparse $\pi$ is updated given $[\Theta, \beta, \Sigma, G, \Lambda, U, V, D, \Delta, Y]$.

The full conditional distributions of the Gibbs sampler and the sampling methods are discussed in Appendix A.

4 Simulation Experiments

This section illustrates the performance of our Bayesian nonparametric sparse model with simulated data. We generate different datasets sample size $T = 100$ from a VAR model with lag $p = 1$:

$$y_t = B y_{t-1} + \varepsilon_t \quad \text{for } t = 1, \ldots, 100,$$

where the dimension of $y_t$ and of the square matrix of coefficients $B$ takes different values, $m = 20$ (small dimension), $m = 40$ (medium dimension), $m = 80$ (big dimension). Furthermore, the matrix of coefficients has different construction, from a block-diagonal to a random form, as follows:

- if $m = 20$, the matrix of coefficients $B = \text{diag}\{B_1, \ldots, B_5\} \in \mathcal{M}_{(20,20)}$ is a block-diagonal matrix with blocks $B_j$ ($j = 1, \ldots, 5$) of $(4 \times 4)$ matrices on the main diagonal:

$$B_j = \begin{pmatrix} b_{11,j} & \cdots & b_{14,j} \\ \vdots & \ddots & \vdots \\ b_{41,j} & \cdots & b_{44,j} \end{pmatrix},$$

where the elements are randomly taken from an uniform distribution $U(-1.4, 1.4)$ and then checked for the stationarity conditions;
Table 1: Summary statistics of the number of clusters with different dimensions $m$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>mean</th>
<th>mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>9.48</td>
<td>9</td>
</tr>
<tr>
<td>40</td>
<td>12.32</td>
<td>12</td>
</tr>
<tr>
<td>80 (random)</td>
<td>11.49</td>
<td>11</td>
</tr>
<tr>
<td>80 (blocks)</td>
<td>11.29</td>
<td>12</td>
</tr>
</tbody>
</table>

- if $m = 40$, the matrix of coefficients $B = \text{diag}(B_1, \ldots, B_{10})$ is a block-diagonal matrix with blocks $B_j$ of $(4 \times 4)$ matrices on the main diagonal:

$$
B_j = \begin{pmatrix}
    b_{11,j} & \cdots & b_{14,j} \\
    \vdots & \ddots & \vdots \\
    b_{41,j} & \cdots & b_{44,j}
\end{pmatrix},
$$

where the elements are randomly taken from an uniform distribution $U(-1.4, 1.4)$ and then checked for the stationarity conditions;

- if $m = 80$, we analyse two different situations, when
  - the matrix of coefficients $B = \text{diag}(B_1, \ldots, B_{20})$ is a block-diagonal matrix with blocks $B_j$ of $(4 \times 4)$ matrices on the main diagonal:

$$
B_j = \begin{pmatrix}
    b_{11,j} & \cdots & b_{14,j} \\
    \vdots & \ddots & \vdots \\
    b_{41,j} & \cdots & b_{44,j}
\end{pmatrix},
$$

where the elements are randomly taken from an uniform distribution $U(-1.4, 1.4)$ and then checked for the stationarity conditions;

  - the $(80 \times 80)$ matrix of coefficients has 150 elements randomly chosen from an uniform distribution $U(-1.4, 1.4)$ and then checked for the stationarity conditions.

For all the cases, we run the Gibbs sampler algorithm described in Section 3 and sample from the posterior distribution via Monte Carlo methods with 5,000 iterations and a burn-in period of 500 iterations. Furthermore, we have chosen the hyperparameters for the sparse and
non-sparse components as in Section 2.2 and the hyperparameters of the Hyper-inverse Wishart as in Section 2.2, where the degree of freedom is $b_0 = 3$ and the scale matrix $L = I_n$. Figure B.1 shows the histograms for the posterior distribution of the number of clusters for each sample sizes, the comparison between the construction of our simulated outputs and the posterior of the number of clusters highlights the good fit of our Bayesian nonparametric hierarchical model, which is also confirmed by the mean and the mode of the number of cluster for every sample sizes (see Table 1).

The summary statistics of the posteriors of some elements of the matrix of coefficients $B$ for different sample sizes are reported in Table 2, where we have the mean, the standard deviation and the 95% credible intervals for $B$.

<table>
<thead>
<tr>
<th></th>
<th>real $B$</th>
<th>Mean</th>
<th>Std. Dev</th>
<th>95% C. I.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 20$</td>
<td>1.3993</td>
<td>1.3894</td>
<td>0.0124</td>
<td>(1.3652, 1.4130)</td>
</tr>
<tr>
<td>$m = 20$</td>
<td>-1.1148</td>
<td>-1.1105</td>
<td>0.0672</td>
<td>(-1.2422, -0.9781)</td>
</tr>
<tr>
<td>$m = 40$</td>
<td>1.066</td>
<td>1.051</td>
<td>0.0874</td>
<td>(0.894, 1.2033)</td>
</tr>
<tr>
<td>$m = 40$</td>
<td>-1.0737</td>
<td>-1.0768</td>
<td>0.0532</td>
<td>(-1.1839, -0.9731)</td>
</tr>
<tr>
<td>$m = 80$ (random)</td>
<td>1.3786</td>
<td>1.3870</td>
<td>0.0458</td>
<td>(1.2852, 1.4668)</td>
</tr>
<tr>
<td>$m = 80$ (random)</td>
<td>-1.3377</td>
<td>-1.3338</td>
<td>0.0327</td>
<td>(-1.3982, -1.2696)</td>
</tr>
<tr>
<td>$m = 80$ (blocks)</td>
<td>1.3604</td>
<td>1.3559</td>
<td>0.0426</td>
<td>(1.2716, 1.4408)</td>
</tr>
<tr>
<td>$m = 80$ (blocks)</td>
<td>-1.1776</td>
<td>-1.1974</td>
<td>0.0652</td>
<td>(-1.3284, -1.0735)</td>
</tr>
</tbody>
</table>

Table 2: Summary statistics of the posterior distributions for different values of $B$ and different dimensions $m$.

We evaluate the accuracy of our estimates by using the Hamming distance for the matrix of coefficients, which is the difference between the real values of the matrix of coefficients $B$ and the posterior values of it. In definition, the Hamming distance is $|a - b|_H = |\{i | a_i \neq b_i\}|$, where the difference $a - b$ contains negative values corresponding to points where $b_i > a_i$. Figure B.2 shows this difference for different sample sizes and it converges to zero, which means that our posteriors for the matrix of coefficients are exactly what we were expecting. In conclusion, Figure B.3 explains the posterior mean of the matrix of $\delta$, which shows us the choice of the components between the two random measures $P_0$ and $P_1$. In particular, we have that the white color explains if the coefficient $\delta$ is equal to zero, while the black one if the $\delta$ is equal to one. The representation in Figure B.3 correctly
explains the sparsity in the matrix of coefficients through the definition of the matrix of the latent variable $\delta$. Furthermore, Figure B.4 shows the graphical representation of the adjacency matrix of the estimated $\delta$ for all the four simulated examples and allows us to explain the different cliques composition. As known, the representation with block matrices confirms the presence of different cliques (e.g. for $n = 20$, we have exactly 5 cliques, while increasing the dimensionality, increase the number of cliques due to the construction of the coefficient matrix).

The model performance appears to be consistent between all the three different sizes and the different construction of the matrix $B$, demonstrating that the approach is suitable to model the sparsity in a model.

5 An Empirical Application

To illustrate the proposed Bayesian nonparametric sparse model, we analyse a macroeconomic dataset. In this section, we focus on a vector autoregressive (VAR) process $\{y_t\}$ with $p$ lags to investigate the possible relationships between the GDP of different countries, with a particular focus on the concept of sparsity.

Following the literature on international business cycles in large models (Kose et al., 2003, 2010; Del Negro and Otrok, 2008) we use a multi-country macroeconomic dataset as in Francis et al. (2012) and Kaufmann and Schumacher (2012), in which papers they investigate the role of global business cycles for many different countries in large factor models.

In our application we used a VAR($p$), with quarterly lags of interest, which means $p = 4$. We investigate the role of the global business cycles for every countries. For our analysis, we need the GDP growth rate, which is computed by taking the first differences of the logarithm of each GDP series. We apply our methodology to a dataset of the most important OECD countries, which will be described below, from the first quarter of 1961 to the second quarter of 2015 for a total of $T = 215$ observations. We represent the VAR($p$) in the following form:

$$Y_t = \sum_{i=1}^{p} B_i Y_{t-i} + \varepsilon_t.$$

Due to missing values in the GDP time series of some countries, we choose a subset of all the OECD countries, which is formed by
the most industrialised countries, and in particular we focus on two big macroareas, the European one and the rest of the world, where the latter is formed by the countries from Asia, Oceania, North and Central America and Africa. Hereafter, we describe more in details the two macroareas:

- Rest of the World - Australia, Canada, Japan, Mexico, South Africa, Turkey, United States;
- Europe - Austria, Belgium, Denmark, Finland, France, Germany, Greece, Ireland, Iceland, Italy, Luxembourg, Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, United Kingdom;

Based on empirical and computational experiments, we run the Gibbs sampling algorithm described in Section 3 for 4,000 iterations with a burn-in period of 500 iterations adopting the same priors of the simulation studies. The location of the posterior mode (value equals to 2) of the histograms in Figure 2 allows us to conclude that following our approach there is evidence in favour of two clusters. The results from our Bayesian nonparametric approach is interesting because it suggests a substantial evidence in favour of two mixtures components, which can be seen from the mean of the posterior number of clusters with value 2.6346. Figure 2 shows the posterior distribution of the number of clusters and identifies in particular the two different clusters.

![Figure 2: Posterior distribution of the number of clusters for the macroeconomic application.](image)
Figure B.6 and Figure B.7 show the predictive distributions (solid lines) generated by the nonparametric approach conditioning on all values of $Y_{it}$, where $t = 1, \ldots, T$ and $i = 1, \ldots, 25$ (the number of the states) and the best normal fits (dashed lines) for the empirical distributions of all the series. From a comparison with the empirical distribution, we note that the nonparametric approach is able to capture skewness and excess of kurtosis in the data. Furthermore, we observe that for the majority of the countries of interest, the predictive densities (solid lines) generated with our nonparametric sparse approach have fatter tails than the tails of the best normal (dashed lines) and they have long left tails. Our Bayesian nonparametric sparse model is suitable for describing and predicting these data thanks to these features.

Moving to the posterior predictive densities, Figure B.8 and Figure B.9 show the one-step-ahead posterior predictive densities for $Y_{it}$, where $t = 50, \ldots, T$ and $i = 1, \ldots, 25$, evaluated at the current values of the explanatory variables $Y_{it-1}, \ldots, Y_{it-p}$. In the same plot, the grey area represents the heatmap sequence of the 95% high probability density region of the predictive densities (darker colors represent higher density values). These densities have been used to predict the peaks and the troughs of the cycles in the OECD countries. In particular we can see troughs near the 1980s and 2009s near the crisis in the majority of the European countries.

Figure B.5 draws the network of the GDP connectivity between different countries with respect to different time lags (a) $t - 1$, (b) $t - 2$, (c) $t - 3$ and (d) $t - 4$. Table 3 shows the network statistics extracted from the four different graphs. Here, the average path length represents the average graph-distance between all pair of nodes, where connected nodes have graph distance 1. The first lag graph has the highest density (0.258), the highest number of links (155), followed by the fourth, the second and the third lag graph. The average path length is lowest for the first lag graph showing more connected graphs.

6 Conclusions

In this paper we have proposed a novel Bayesian nonparametric sparse model through the introduction of multiple shrinkage priors. In order to capture the sparsity structure in the model, we introduce two stages of the hierarchy for the prior choice, where the first one consists in a
Table 3: The network statistics for the 4 different lags. The average path length represents the average graph-distance between all pairs of nodes. Connected nodes have graph distance 1.

Bayesian lasso conditionally independent Normal-Gamma prior and the second one is given by a random mixture distribution for the hyperparameters of the Normal-Gamma distribution with a particular base measure, based on the two-parameters gamma developed by Miller (1980). The proposed hierarchical prior is used to proposed a Bayesian nonparametric model for VAR models. We provide an efficient Monte Carlo Markov Chain algorithm for the posterior computations and the effectiveness of this algorithm is assessed in simulation and real data exercises. These simulation studies illustrate the good performance of our model with different sample sizes and different constructions of the matrix of coefficients.

Besides through simulation studies, the application to the GDP growth rates in different OECD countries reveals the relations between two different clusters, the European and the rest of the world one. Furthermore we found evidence of good predictive abilities of our Bayesian nonparametric model.

We conclude the paper with the indication of some future research lines. Our hierarchical prior and our nonparametric approach can be extended to the graphical models for the study of the financial contagion with the introduction of different link functions (such as the probit or the logit function) or to the Factor autoregressive models (see Kaufmann and Schumacher (2012)) for the analysis of the stochastic volatility processes.

Acknowledgements

We would like to thank all the conference participants for helpful discussion at: “9th Annual RCEA Bayesian Econometric Workshop”
at Rimini Center for Economic Analysis; “Joint PhD Workshop Economics and Management” and “Internal Seminar” at Ca’ Foscari University of Venice; “Statistics Seminar” at University of Kent; “9th Computational and Financial Econometrics Conference (CFE 2015)” in London; “ISBA 2016 World Meeting” in Sardinia; “3rd Bayesian Young Statistician Meeting” at University of Florence. We would like to thank also Stefano Tonellato for his valuable comments to the manuscript and his constructive suggestions.

References


Griffin, J. E. (2011). Inference in infinite superpositions of non-Gaussian Ornstein-Uhlenbeck processes using Bayesian


A Gibbs sampling details

We introduce the following notations, for $k \geq 1$, and $l = 1, 2$,

$$
\mathcal{D}_{lk} = \{ j \in 1, \ldots, r_l : d_{lj} = k, \delta_{lj} = 1 \},
$$

$$
\mathcal{D}^* = \{ k | \mathcal{D}_{1k} \cup \mathcal{D}_{2k} \neq 0 \}, \quad D^* = \max_{l=1,2} \max_{j=1,\ldots,r_l} d_{lj},
$$

where $\mathcal{D}_k$ denotes the set of indexes of the coefficients allocated to the $k$-th component of the mixture and $\mathcal{D}^*$ the set of indexes of the non-empty mixture components, while $D^*$ is the number of stick-breaking components used in the mixture. As noted by Kalli et al. (2011), the sampling of infinitely many elements of $\Theta$ and $V$ is not necessarily, since only the elements in the full conditional probability density functions of $D, \Delta$ are needed.

The maximum number of atoms and stick-breaking components to sample is $N^* = \max\{ N_1^*, N_2^* \}$, where $N_l^*$ is the smallest integer such that $\sum_{k=1}^{N_l^*} u_{lk} > 1 - u_l^*$, where $u_l^* = \min_{1 \leq j \leq n_l} \{ u_{lj} \}$. In the following sections we explain in details all the steps of the Gibbs sampler, which is built on the slice sampler algorithm of Walker (2007) and of Kalli et al. (2011).

A.1 Update $V, U$

We treat $V$ as three blocks of random length: $V = (V^*, V^{**}, V^{***})$, where

$$
V^* = \{ V_k : k \in \mathcal{D}^* \} = (v_{k1}, \ldots, v_{kD^*}),
$$

$$
V^{**} = (v_{kD^*+1}, \ldots, v_{kN^*}), \quad V^{***} = \{ V_k : k > N^* \}.
$$

In order to sample from the conditional distribution of $(U, V)$ a further blocking is used:

i) Sampling from the full conditional posterior distribution of $V^*$, is obtained by drawing $v_{1k}, v_{2k}$, with $k \leq D^*$ from the full conditionals

$$
f(v_{1j} | \ldots) \propto \text{Be} \left( 1 + \sum_{j=1}^{r_1} \mathbb{I}(d_{1j} = d, \delta_{1j} = 1), \alpha + \sum_{j=1}^{r_1} \mathbb{I}(d_{1j} > d, \delta_{1j} = 1) \right),
$$

$$
f(v_{2j} | \ldots) \propto \text{Be} \left( 1 + \sum_{j=1}^{r_2} \mathbb{I}(d_{2j} = d, \delta_{2j} = 1), \alpha + \sum_{j=1}^{r_2} \mathbb{I}(d_{2j} > d, \delta_{2j} = 1) \right).
$$
ii) Sampling form the full conditional posterior distribution of $U$ is obtained by simulating from, for $1 \leq j \leq r_1$, 

$$f(u_{1j} | \ldots) \propto \begin{cases} 
I(u_{1j} < w_{1d_{1j}})^{\delta_{1j}} & \text{if } \delta_{1j} = 1, \\
I(u_{1j} < 1)^{1-\delta_{1j}} & \text{if } \delta_{1j} = 0,
\end{cases}$$

and, for $1 \leq j \leq r_2$,

$$f(u_{2j} | \ldots) \propto \begin{cases} 
I(u_{2j} < w_{2d_{2j}})^{\delta_{2j}} & \text{if } \delta_{2j} = 1, \\
I(u_{2j} < 1)^{1-\delta_{2j}} & \text{if } \delta_{2j} = 0.
\end{cases}$$

iii) For $(V^{**}, V^{***})$ given $[\Theta, \Sigma, \Lambda, V^*, D, \Delta, Y]$, we need to sample only the elements of $V^{**}$ from the prior distribution of the stick-breaking construction, that is, for each $l = 1, 2$,

$$f(v_{lj} | \ldots) \propto Be(1, \alpha).$$

A.2 Update the mixing parameters $\lambda$

We update the mixing parameters $\lambda_{lj}$ ($l = 1, 2$), where the full conditional posterior distribution of $\lambda_{lj}$ is

$$f(\lambda_{lj} | \ldots) \propto \lambda_{lj}^{-\frac{1}{2}(1-\delta_{lj})} \exp \left\{ -\frac{1}{2} \frac{1}{\lambda_{lj}} \beta_{lj}^2 + \frac{\tau_{0j}}{2} \lambda_{lj} \right\} \lambda_{lj}^{(\gamma_0 - 1)(1-\delta_{lj})} \times 
\lambda_{lj}^{-\frac{1}{2}\delta_{lj}} \exp \left\{ -\frac{1}{2} \frac{1}{\lambda_{lj}} \left( \beta_{lj} - \mu_{ld_{lj}} \right)^2 \delta_{lj} \right\} \lambda_{lj}^{(\gamma_{ld_{lj}} - 1)\delta_{lj}} \exp \left\{ -\frac{\tau_{ld_{lj}}}{2} \lambda_{lj} \right\} \delta_{lj} \times 
\lambda_{lj}^{C_{lj} - 1} \exp \left\{ -\frac{1}{2} \left[ A_{lj} \lambda_{lj} + \frac{B_{lj}}{\lambda_{lj}} \right] \right\} \propto GiG(A_{lj}, B_{lj}, C_{lj}),$$

where $GiG$ stays for Generalize Inverse Gaussian of parameters $A_{lj} > 0$, $B_{lj} > 0$ and $C_{lj}$ a real parameter (see Halphen (1941), Hoermann and Leydold (2013), Devroye (2014), Dagpunar (1988) and Dagpunar (1989)), which, in our case, are defined as

$$A_{lj} = \left[ (1-\delta_{lj}) \tau_{0j} + \delta_{lj} \gamma_{ld_{lj}} \right], \quad B_{lj} = \left[ (1-\delta_{lj}) \beta_{lj}^2 + \delta_{lj} (\beta_{lj} - \mu_{ld_{lj}})^2 \right], \quad C_{lj} = \left[ (1-\delta_{lj}) \gamma_0 + \gamma_{ld_{lj}} \delta_{lj} - \frac{1}{2} \right].$$

We use the $\lambda_{lj}$ just drawn to construct the matrix $\Lambda_l = \text{diag}\{\lambda_l\}$, where diag{$\lambda_l$} returns a diagonal matrix with the elements of $\lambda_l = (\lambda_{l1}, \ldots, \lambda_{lr_l})'$ on the main diagonal. In practice we have two different matrix, $\Lambda_1 = \text{diag}\{\lambda_{11}, \ldots, \lambda_{1r_1}\}$ and $\Lambda_2 = \text{diag}\{\lambda_{21}, \ldots, \lambda_{2r_2}\}$.  

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A.3 Update $\Theta$

We consider two different cases: the sparse one, where the parameters are $(\mu_0, \gamma_0, \tau_0)$, and the nonsparse case, where the parameters are $(\mu_k, \gamma_k, \tau_k)$, with $k \geq 1$. Since the prior for $\mu_0$ has unit probability mass at 0, the full conditional distribution of $\mu_0$ is $f(\mu_0|\ldots) = \delta_{(0)}(\mu_0)$. The full conditional distribution of the shape and scale parameters $(\gamma_0, \tau_0)$ is:

\[
f((\gamma_0, \tau_0)|\ldots) \propto g(\gamma_0, \tau_0|\mu_0, p_0, s_0, n_0) \prod_{j=1}^{r_1} \left( \frac{(\tau_0/2)^{\gamma_0}}{\Gamma(\gamma_0)} \lambda_{1j}^{\gamma_0-1} \exp \left\{ -\frac{\tau_0}{2} \lambda_{1j} \right\} \right) \times \prod_{j=1}^{r_2} \left( \frac{(\tau_0/2)^{\gamma_0}}{\Gamma(\gamma_0)} \lambda_{2j}^{\gamma_0-1} \exp \left\{ -\frac{\tau_0}{2} \lambda_{2j} \right\} \right), \tag{A.1}
\]

where we assume that:

\[
\begin{align*}
r_{1,0} &= \sum_{j=1}^{r_1} (1 - \delta_{1j}) = r_1 - r_{1,1}, & r_{1,1} &= \sum_{j=1}^{r_1} \delta_{1j}, \\
r_{2,0} &= \sum_{j=1}^{r_2} (1 - \delta_{2j}) = r_2 - r_{2,1}, & r_{2,1} &= \sum_{j=1}^{r_2} \delta_{2j}.
\end{align*}
\]

The distribution in (A.1) has the same kernel of the prior distribution $g(\gamma_0, \tau_0|\ldots)$ given in (10), that is:

\[
f((\gamma_0, \tau_0)|\ldots) \propto \tau_0^{\gamma_0 \gamma_0 - 1} p_0^{\gamma_0 - 1} \exp \left\{ -s_0 \tau_0 \right\} \frac{1}{\Gamma(\gamma_0)^{r_0}} \times \left( \prod_{j=1}^{r_1,0} \lambda_{1j} \right)^{\gamma_0 - 1} \exp \left\{ -\frac{\tau_0}{2} \sum_{j=1}^{r_{1,1}} \lambda_{1j} \right\} \times \left( \prod_{j=1}^{r_2,0} \lambda_{2j} \right)^{\gamma_0 - 1} \exp \left\{ -\frac{\tau_0}{2} \sum_{j=1}^{r_{2,1}} \lambda_{2j} \right\} \times \left( \mu_0, \tau_0 \prod_{j=1}^{r_1,0} \lambda_{1j} \prod_{j=1}^{r_2,0} \lambda_{2j} \right) \exp \left\{ -s_0 + \frac{1}{2} \sum_{j=1}^{r_{1,1}} \lambda_{1j} + \frac{1}{2} \sum_{j=1}^{r_{2,1}} \lambda_{2j}, \right.
\]

In order to draw samples from $g$ we apply here a collapsed Gibbs sampler. Samples from $f(\gamma)$ are obtained by a Metropolis-Hastings
(MH) algorithm with the prior as proposal, we start with a value of \( \gamma^* \sim \mathcal{G}(1/2, 2) \), we remind \( q(\gamma) \) is the probability density function of \( \gamma \) and is distributed as a \( \mathcal{G}(1/2, 2) \). The acceptance probability of the MH step is:

\[
\alpha(\gamma^*, \gamma_{\text{old}}) = \min \left\{ 1, \frac{f(\gamma^*)q(\gamma_{\text{old}})}{f(\gamma_{\text{old}})q(\gamma^*)} \right\}.
\]  
(A.2)

The MH chain updates as follows:

\[
\gamma_{\text{new}} = \begin{cases} 
\gamma_{\text{old}} & \text{if } u > \alpha(\gamma^*, \gamma_{\text{old}}), \\
\gamma^* & \text{if } u \leq \alpha(\gamma^*, \gamma_{\text{old}}),
\end{cases}
\]

where \( u \) is a random number from a standard uniform. Samples from the conditional \( f(\tau|\gamma) \) are easily obtained since \( f(\tau|\gamma) \) is a Gamma distribution.

In the nonsparse case, we generate samples \((\mu_{lk}, \gamma_{lk}, \tau_{lk})_k^{N^*}, l = 1, 2\) by applying a single move Gibbs sampler with full conditional distributions \( f(\mu_{lk} | \ldots) \) and \( f(\gamma_{lk}, \tau_{lk} | \ldots) \). The full conditional

\[
f(\mu_{lk} | \ldots) \propto \mathcal{N}(\mu_{lk}|c, d) \prod_{j|\delta_{lj}=1, d_{lj}=k} \mathcal{N}(\beta_{lj}|\mu_{lk}, \lambda_{lj})
\]

\[
\propto \frac{1}{\sqrt{2\pi d}} \exp \left\{ -\frac{1}{2d} (\mu_{lk} - c)^2 \right\} \prod_{j|\delta_{lj}=1, d_{lj}=k} \frac{1}{\sqrt{2\pi \lambda_{lj}}} \exp \left\{ -\frac{1}{2\lambda_{lj}} (\beta_{lj} - \mu_{lk})^2 \right\}
\]

\[
\propto \exp \left\{ -\frac{1}{2d} (\mu_{lk} - c)^2 - \sum_{j|\delta_{lj}=1, d_{lj}=k} \frac{1}{2\lambda_{lj}} (\beta_{lj} - \mu_{lk})^2 \right\}
\]

is proportional to the normal \( \mathcal{N}(\tilde{E}_k, \tilde{V}_k) \) with parameters \( \tilde{E}_k = \tilde{V}_k \left( \frac{c}{\varphi} + \sum_{j|\delta_{lj}=1, d_{lj}=k} \beta_{lj} \lambda_{lj} \right) \) and \( \tilde{V}_k = \left( \frac{1}{\varphi} + \sum_{j|\delta_{lj}=1, d_{lj}=k} 1/\lambda_{lj} \right)^{-1} \). On the other hand, the joint conditional posterior of \((\gamma_{lk}, \tau_{lk})_k^{N^*}\) is:

\[
f((\gamma_{lk}, \tau_{lk})_k^{N^*}) \propto g(\gamma_{lk}, \tau_{lk}|\nu_1, p_1, s_1, n_1) \prod_{j|\delta_{lj}=1, d_{lj}=k} \left( \frac{(\tau_{lk}/2)^{\nu_1}}{\Gamma(\nu_1)} \lambda_{lj}^{\gamma_{lk} - 1} \exp \left\{ -\frac{\tau_{lk}}{2} / \lambda_{lj} \right\} \right),
\]

(A.3)

where we have defined \( r_{l,1k} = \sum_{j=1}^{n_{lj}} \delta_{lj} (d_{lj} = k) \). Hence (A.3) can be
reduced as

\[
f((\gamma_{lk}, \tau_{lk}) \ldots) \propto \tau_{lk}^{-\gamma_{lk} - 1} p_1^{-\gamma_{lk} - 1} \exp\{-s_1 \tau_{lk}\} \frac{1}{\Gamma(\gamma_{lk})} \times \]

\[
\times \frac{(\tau_{lk}/2)^{1/2}}{\Gamma(\gamma_{lk})^{1/2}} \left( \prod_{j|\delta_j=1,d_j=k} \lambda_{lj} \right)^{-\gamma_{lk} - 1} \exp \left\{ -\frac{\tau_{lk}}{2} \sum_{j|\delta_j=1,d_j=k} \lambda_{lj} \right\}
\]

\[
\propto g \left( \gamma_{lk}, \tau_{lk}|\nu_1 + r_{l,1k}, p_1 \prod_{j|\delta_j=1,d_j=k} \lambda_{lj}, s_1 + \frac{1}{2} \sum_{j|\delta_j=1,d_j=k} \lambda_{lj}, n_1 + r_{l,1k} \right),
\]

for \( k \in D^* \) and from the prior \( G_0 \) for \( k \notin D^* \). As in the sparse case, we apply a MH algorithm, with the acceptance probability as described in (A.2).

### A.4 Update \( \beta \)

The full conditional posterior distribution of \( \beta \) is:

\[
f(\beta|\ldots) \propto \exp \left\{ -\frac{1}{2} \left( \sum_t \beta_t' X_t' \Sigma^{-1} X_t \beta_t + \right. \right.
\]

\[
- 2 \beta_t' \sum_t X_t' \Sigma^{-1} y_t \left. \right) \prod_{j=1}^n \exp \left\{ -\frac{1}{2} \beta_t^2 (1 - \delta_{ij}) - \frac{1}{2} \sum_{j|\delta_j=1,d_j=k} \lambda_{lj} \right\}
\]

\[
\propto \exp \left\{ -\frac{1}{2} \left( \sum_t \beta_t' X_t' \Sigma^{-1} X_t \beta_t + \right. \right.
\]

\[
- 2 \beta_t' \sum_t X_t' \Sigma^{-1} y_t \left. \right) \frac{1}{2} \left( \beta_t' \Lambda_t^{-1} \beta_t - 2 \beta_t' \Lambda_t^{-1} (\mu_t^* \odot \delta_t) \right)
\]

\[
\sim \mathcal{N}_{r_l}(\tilde{\nu}_l, M_l),
\]

where

\[
M_l = \left( \sum_t X_t' \Sigma^{-1} X_t + \Lambda_t^{-1} \right)^{-1},
\]

\[
\tilde{\nu}_l = M_l \left( \sum_t X_t' \Sigma^{-1} y_t + \Lambda_t^{-1} (\mu_t^* \odot \delta_t) \right),
\]

and \( \mu_t^* = (\mu_{ld_{11}}, \ldots, \mu_{ld_{r_1}})' \), \( \delta_t = (\delta_{t1}, \ldots, \delta_{tr_1})' \).
A.5 Update $\Sigma$

Let $S = \{S_1, \ldots, S_{nS}\}$ and $P = \{P_1, \ldots, P_{nP}\}$ be the set of separators and of prime components, respectively, of the graph $G$. So the density of the hyper-inverse Wishart for $\Sigma$ conditional on the graph $G$ is:

$$p(\Sigma) = \prod_{P \in \mathcal{P}} p(\Sigma_P) \left( \prod_{S \in S} p(\Sigma_S) \right)^{-1}, \quad (A.4)$$

where

$$p(\Sigma_P) \propto |\Sigma_P|^{-(b+\text{Card}(P))/2} \exp \left\{ -\frac{1}{2} \text{tr}(\Sigma_P^{-1} L_P) \right\}, \quad (A.5)$$

with $L_P$ is the positive-definite symmetric diagonal block of $L$ corresponding to $\Sigma_P$.

By using the sets $S$ and $P$ and since we are working with the decomposable graph, we know that the likelihood of the graphical gaussian model can be approximated as the ratio between the likelihood in the prime components and the likelihood in the separator components. So the posterior for $\Sigma$ factorizes as follows:

$$p(\Sigma|\ldots) \propto \prod_{t=1}^{T} (2\pi)^{n/2} |\Sigma|^{-1/2} \exp \left( -\frac{1}{2} (y_t - X_t'\beta)' \Sigma^{-1} (y_t - X_t'\beta) \right) p(\Sigma)$$

$$\propto |\Sigma|^{T/2} \exp \left( -\frac{1}{2} \text{tr} \left( \sum_t (y_t - X_t'\beta)' \Sigma^{-1} (y_t - X_t'\beta) \right) \right) p(\Sigma)$$

$$\propto \prod_{P \in \mathcal{P}} |\Sigma_P|^{-T/2} \exp \left( -\frac{1}{2} \text{tr} \left( \sum_t (y_t - X_t'\beta)' \Sigma_P^{-1} (y_t - X_t'\beta) \right) \right) \times$$

$$\prod_{S \in S} |\Sigma_S|^{-T/2} \exp \left( -\frac{1}{2} \text{tr} \left( \sum_t (y_t - X_t'\beta)' \Sigma_S^{-1} (y_t - X_t'\beta) \right) \right) \frac{\prod_{P \in \mathcal{P}} |\Sigma_P|^{-(b+\text{Card}(P))/2} \exp \left\{ -\frac{1}{2} \text{tr}(\Sigma_P^{-1} L_P) \right\}}{\prod_{S \in S} |\Sigma_S|^{-(b+\text{Card}(S))/2} \exp \left\{ -\frac{1}{2} \text{tr}(\Sigma_S^{-1} L_S) \right\}} \frac{\prod_{P \in \mathcal{P}} |\Sigma_P|^{-(b+\text{Card}(P)+T)/2}}{\prod_{S \in S} |\Sigma_S|^{-(b+\text{Card}(S)+T)/2}} \frac{\exp \left( -\frac{1}{2} \text{tr} \left( \sum_t (y_t - X_t'\beta)' (y_t - X_t'\beta) + L_P \right) \right)}{\exp \left( -\frac{1}{2} \text{tr} \left( \sum_t (y_t - X_t'\beta)' (y_t - X_t'\beta) + L_S \right) \right)}.$$ 

So we have that the posterior distribution for $\Sigma$ is drawn from:

$$p(\Sigma|\ldots) \propto \mathcal{HIV}_G \left( b + T, L + \sum_{t=1}^{T} (y_t - X_t'\beta)' (y_t - X_t'\beta) \right).$$
A.6 Update Graph G

We apply a Markov chain Monte Carlo for multivariate graphical models for learning the graph structure \( G \) (see Giudici and Green (1999) and Jones et al. (2005)). We see due to the prior independence assumption of the parameters that:

\[
p(y|G) = \int \prod_{t=1}^{T} (2\pi)^{-n/2} |\Sigma|^{-n/2} \exp \left( -\frac{1}{2} (y_t - X'_t \beta)^{\top} \Sigma^{-1} (y_t - X'_t \beta) \right) p(\beta)p(\Sigma|G)d\beta d\Sigma.
\]

This integral is difficult to compute and evaluate analytically and we apply a Candidate's formula along the line of Chib and Greenberg (1995) and Wang (2010). Following Jones et al. (2005) we apply a local-move Metropolis-Hastings based on the conditional posterior \( p(G|...). \) A candidate \( G' \) is sampled from a proposal distribution \( q(G'|G) \) and accepted with probability

\[
\alpha = \min \left\{ 1, \frac{p(G'|y)q(G|G')}{p(G|y)q(G'|G)} \right\}.
\]

We use the add/delete edge move proposal of Jones et al. (2005).

A.7 Update D and \( \Delta \)

The full conditionals of \( D \) are obtained by sampling from the two different cases, when \( \delta_{ij} = 1 \) and \( \delta_{ij} = 0 \) \( (l = 1, 2) \). Starting for \( \delta_{ij} = 1 \), we have

\[
P(d_{ij} = d, \delta_{ij} = 1|\ldots) \propto (1 - \pi_l)N(\beta_{ij}|\mu_{id}, \lambda_{ij})Ga(\lambda_{ij}|\gamma_{id}, \tau_{id}/2)I(u_{ij} < w_{id}) \\
\quad \propto \frac{(1 - \pi_l)N(\beta_{ij}|\mu_{ld}, \lambda_{ij})Ga(\lambda_{ij}|\gamma_{ld}, \tau_{ld}/2)}{\sum_{k \in A_{w_{ij}}(u_{ij})} N(\beta_{ij}|\mu_{ik}, \lambda_{ij})Ga(\lambda_{ij}|\gamma_{ik}, \tau_{ik}/2)} \quad \forall d \in A_{w_{ij}}(u_{ij}),
\]

for \( \delta_{ij} = 1 \), while we have

\[
P(d_{ij} = d, \delta_{ij} = 0|\ldots) \propto \pi_l I(u_{ij} < \tilde{w}_{id}),
\]

with \( d \in A_{\tilde{w}}(u_{ij}), \) where \( A_{\tilde{w}}(u_{ij}) = \{k : u_{ij} < \tilde{w}_k\} \) which is equal to \{0\}, because \( \tilde{w}_k = 0, \forall k > 0, \)

\[
P(d_{ij} = d, \delta_{ij} = 0|\ldots) \propto \left\{ \begin{array}{ll}
\pi_l I(u_{ij} < 1)N(\beta_{ij}|0, \lambda_{ij})Ga(\lambda_{ij}|\gamma_0, \tau_0/2) & \text{if } d = 0, \\
0 & \text{if } d > 0.
\end{array} \right.
\]

\[
\propto \pi_l N(\beta_{ij}|0, \lambda_{ij})Ga(\lambda_{ij}|\gamma_0, \tau_0/2) \quad \text{if } d = 0.
\]

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A.8 Update \( \pi = (\pi_1, \pi_2) \)

We assume that the prior for \( \pi_l \) is \( \text{Be}(1, \alpha_l) \), so we have that the full conditional for \( \pi_l \) is,

\[
 f(\pi_l | \ldots) \propto \text{Be} \left( r_l + 1 - \sum_{i=1}^{r_l} I(\delta_{li} = 1), \alpha_l + \sum_{i=1}^{r_l} I(\delta_{li} = 1) \right).
\]
B  Simulated and Real Data Results

Figure B.1: Posterior distribution of the number of clusters for different sample sizes. Block matrices of coefficients at the top left panel for $m = 20$, top right panel for $m = 40$ and bottom right panel for $m = 80$, while random element in the matrix of coefficients at the bottom left panel for $m = 80$. 
Figure B.2: Hamming distance between $B$ and its posteriors. Block matrices of coefficients at the top left panel for $m = 20$, top right panel for $m = 40$ and bottom right panel for $m = 80$, while random element in the matrix of coefficients at the bottom left panel for $m = 80$. 
Figure B.3: Posterior mean of the matrix of $\delta$, where white color means $\delta = 0$ and black one means $\delta = 1$. Block matrices of coefficients at the top left panel for $m = 20$, top right panel for $m = 40$ and bottom right panel for $m = 80$, while random element in the matrix of coefficients at the bottom left panel for $m = 80$. 
Figure B.4: Estimated graph structure from the adjacency matrix $\hat{\delta}$. 
Figure B.5: Networks of GDP for OECD countries with respect to different lag in the sparse pattern, (a) $t - 1$, (b) $t - 2$, (c) $t - 3$, (d) $t - 4$. 
Figure B.6: GDP growth rates $Y_{it}$ (histogram), predictive distribution (solid line) and best normal (dashed line) for all the countries of the panel.
Figure B.7: GDP growth rates $Y_{it}$ (histogram), predictive distribution (solid line) and best normal (dashed line) for all the countries of the panel.
Figure B.8: Predictive results for all countries. In each plot: GDP growth rates $Y_{it}$ (black lines); heatmap (grey areas) of the 95% high probability density region of the predictive density functions (darker colors represent higher density values) evaluated at each time point, for $t = 1, \ldots, T$ at the value of the predictors $Y_{it-1}, \ldots, Y_{it-p}$ for $i = 1, \ldots, 25$. 
Figure B.9: Predictive results for all countries. In each plot: GDP growth rates $Y_{it}$ (black lines); heatmap (grey areas) of the 95% high probability density region of the predictive density functions (darker colors represent higher density values) evaluated at each time point, for $t = 1, \ldots, T$ at the value of the predictors $Y_{it-1}, \ldots, Y_{it-p}$ for $i = 1, \ldots, 25$. 

(a) Japan  (b) Luxembourg  (c) Mexico

(d) Netherlands  (e) Norway  (f) Portugal

(g) South Africa  (h) Spain  (i) Sweden

(j) Switzerland  (k) Turkey  (l) United Kingdom

(m) United States