Bond immunization and arbitrage in the semi-deterministic setting⋆

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Abstract. Immunization is a widely used tool in bond portfolio management, capable of hedging interest rate risk. Its goal, the construction of a portfolio whose value is not negatively affected by a change in the term structure, can contradict no-arbitrage condition. This paper investigates the existence and functional form of shocks that do not lead to arbitrage opportunities.

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1 Introduction and motivation

Duration and immunization are important topics in bond portfolio analysis from both a theoretical and a practical point of view. Immunization yields the construction of a portfolio of assets whose value, in case of unexpected changes, usually called 'shifts' or 'shocks', in the interest rates term structure, is always greater than the value of a given liability. Immunization is, then, a hedging technique, useful for bondholders and portfolio managers.

The basic methodology underlying immunization is to assume some functional form for the shift and to derive conditions a portfolio has to fulfill in order to prevent it from losing too much value when the shock occurs. This explains why the aim of most financial literature covering this topic is to provide lower bounds for the value of an immunized portfolio in case the term structure is affected by a shock.

Early important results were provided by Macaulay [20], Redington [26] and Fisher and Weil [14]. Redington proved that a portfolio of bonds can be protected against an infinitesimal parallel change in the market term structure.

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Later, Fisher and Weil developed immunization conditions for a portfolio of assets against a liability made of a single cash flow when the term structure shifts in a parallel fashion. More recently, Fong and Vasicek [15] provided a lower bound for the post-shift value of a bond when the first derivative of a differentiable shift function is bounded while Shiu ([27] and [28]) extended Redington and Fisher and Weil results providing immunization conditions when, respectively, the shock has arbitrary magnitude and when it is convex. Montrucchio and Peccati [22] and Uberti [29] further expanded these results by considering '$\alpha-$convex' and 'convex$-\beta$' shock functions. Finally, Hürlimann [16] rewrote immunization results by means of convex ordering and Courtois and Denuit [8] used '$s-$convex' and '$s-$concave' functions to achieve more general immunization conditions. For detailed analyses and reviews of immunization theory the reader can refer to De Felice and Moriconi [11], chapter three in Panjer [25], de La Grandville [12], and chapter two, four and five in Nawalkha et al. [23].

Some scholars, for example Boyle [6], Ingersoll et al. [17], Milgrom [21], and De Felice and Moriconi, noted that immunization contradicts the no-arbitrage principle, whose vast exploitation provided many of the most important achievements in modern financial theory. As said above, if an immunized portfolio is better off right after a shock, its value happens to face a sure positive change with no chance of loss. This creates wealth out of nothing: an obvious financial contradiction that seems to lead immunization into a lost cause. Recently, a relevant argument against this conclusion has been raised by Barber and Copper [2]. In their article the authors show that general no-arbitrage based stochastic affine term structure models are not incompatible with immunization strategies, the main reason being that cost of immunization over time is positive.

On the empirical side, many articles, among them the ones by Litterman and Scheinkman [19], D'Ecclesia and Zenios [9], Barber and Copper [1], and Bliss [5], reported that changes in the term structure can be almost entirely explained by a limited number of factors, while Nelson and Siegel [24] proposed a parsimonious functional form for modeling the instantaneous forward rate of return. Changes in the market term structure should, therefore, be captured and represented by relatively 'simple' shift functions.

This being said, the leading idea of this paper is an attempt to reconcile immunization and no-arbitrage by detecting and investigating plausible shapes in the changes of the term structure. The article provides, in a semi-deterministic setting, a general condition for the existence of 'arbitrage-free' shift functions, identifies an appropriate class of such functions and analyzes its impact on the original term structure. The paper is organized as follows: Section two deploys framework and notation so that key notions such as moments, immunization and an applicable no-arbitrage definition are introduced. Section three reminds some immunization results that will be then used in Section four where a sufficient condition to determine a family of no-arbitrage additive shifts is found. Finally, Section five concludes.

2 Notation and basic definitions

2.1 Notation

Consider a market in which P bonds $\mathbf{x}_i$, $i = 1, ..., P$, are traded. Let $0 < t_1 < ... < t_N$ be the maturities in which at least one bond produces a cash flow. Each bond is represented by a column vector $\mathbf{x}_i \in \mathbb{R}^N$ of fixed and paid for sure cash flows $x_{i,j} \geq 0$ at time t_j, $j = 1, ..., N$. The entire market is encompassed into a $(P \times N)$ matrix $X = \left[\mathbf{x}_i^T\right]$ that is assumed to contain no redundant bond, i.e. $rank(X) = \min(P, N)$.

The market instantaneous forward rate

$$\delta_*(s) = \delta_t(s) + \epsilon(s)\, \mathbf{1}_{\{s \geq t_*\}}, \quad s \geq t$$

is composed of an observable, at time t, component $\delta_t(s)$ implied by X and by the market prices of the bonds and of an additive shock $\epsilon(s)$ (Bierwag [3] and [4]) that occurs in $t_* > t$ and perturbs the original term structure. For sake of simplicity, and without losing generality, it is assumed that only one shock can occur and, should this be the case, $t_* < t_1$. It is also assumed that both $\delta_t(s)$ and $\epsilon(s)$ are Riemann-integrable functions for all $s \geq t$. Interest rate risk is led into the market by the functional form of $\epsilon(s)$ and t_* as they both cannot be predicted in advance.

In this paper, market is said to be deterministic (i.e. there is no interest rate risk) if $\epsilon(s) = 0$ for all $s \geq t$ or if $t_* > t_N$. If, instead, $t_* < t_N$ and $\epsilon(s) \neq 0$ for some $s \leq t_N$, following De Felice [10], market is referred to as semi-deterministic.

The value at time t of one monetary unit due in s is

$$v_*(t, s) = e^{-\int_t^s \delta_*(u)du} = v(t, s)\, f(t, s)$$

where $v(t, s) = e^{-\int_t^s \delta_t(u)du}$ is the discount factor if no shift occurs while

$$f(t, s) = e^{-\int_t^s \epsilon(u)\mathbf{1}_{\{u \geq t_*\}}du} = e^{-\int_{t_*}^s \epsilon(u)du} \tag{1}$$

is the shift (or shock) factor.

The column vector $\mathbf{v}(t) = [v(t, t_j)] \in \mathbb{R}^N$, $t \leq t_1$, is commonly referred to as the market term structure of spot prices. Similarly, column vector $\mathbf{v}_*(t) = [v_*(t, t_j)] \in \mathbb{R}^N$ represents the 'shifted' term structure when a shock occurs. It is handy to denote the 'component-wise' product between two vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^N$ as vector $\langle \mathbf{x}, \mathbf{y} \rangle = [x_j \cdot y_j] \in \mathbb{R}^N$. Accordingly, by letting column vector $\mathbf{f}(t) = [f(t, t_j)] \in \mathbb{R}^N$,

$$\mathbf{v}_*(t) = [v(t, t_j) f(t, t_j)] = \langle \mathbf{v}(t), \mathbf{f}(t) \rangle\,. \tag{2}$$

Bond $\mathbf{x}_i$ 'pre-shift' market price in $t < t_*$ is

$$W(t, \mathbf{x}_i) = \sum_{j=1}^{N} x_{i,j} v(t, t_j) = \mathbf{x}_i^T \mathbf{v}(t) \tag{3}$$

and, if no shift happens, $W(\bar{t}, \mathbf{x}_i) = W(t, \mathbf{x}_i)/v(t, \bar{t})$ for $t \leq \bar{t} < t_1$. Bond $\mathbf{x}_i$ 'post-shift' value in t', $t_* \leq t' < t_1$, is

$$W_*(t', \mathbf{x}_i) = \sum_{j=1}^{N} x_{i,j} v(t', t_j) f(t', t_j) = \mathbf{x}_i^T \mathbf{v}_*(t') = \mathbf{x}_i^T \langle \mathbf{v}(t'), \mathbf{f}(t') \rangle . \tag{4}$$

Such values are collected into two column vectors, both in $\mathbb{R}^P$,

$$\mathbf{w}(t) = X\mathbf{v}(t) \quad \text{and} \quad \mathbf{w}_*(t') = X\mathbf{v}_*(t') . \tag{5}$$

Assume now that $\delta_t(s)$ and $\epsilon(s)$ are not only integrable but, also, differentiable a sufficient number of times for $s \geq t$. Chambers and Carleton [7] propose to measure the impact of a shock on the value of a bond by developing $g(t,s) = \int_t^s \delta_t(u)\, du$ and $g^\epsilon(t,s) = \int_t^s \epsilon(u) \mathbf{1}_{\{u \geq t_*\}} du$ in a Taylor expansion

$$g(t,s) = \sum_{m=0}^{M} \frac{a_m(t)}{m!} (s-t)^m + \frac{a_{m+1}(\theta)}{(m+1)!} (s-t)^{m+1} , \tag{6}$$

and

$$g^\epsilon(t_*, s) = \sum_{m=0}^{M} \frac{a_m^\epsilon(t_*)}{m!} (s-t_*)^m + \frac{a_{m+1}^\epsilon(\theta^\epsilon)}{(m+1)!} (s-t_*)^{m+1} ,$$

where $a_0(t) = g(t,t) = 0$, $a_0^\epsilon(t_*) = g^\epsilon(t_*, t_*) = 0$, $a_1(t) = \delta_t(t)$, $a_1^\epsilon(t_*) = \epsilon(t_*)$, $a_m(t) = \frac{d^{m-1}}{ds^{m-1}} \delta_t(s) \Big|_{s=t}$, $a_m^\epsilon(t_*) = \frac{d^{m-1}}{ds^{m-1}} \epsilon(s) \Big|_{s=t_*}$, $m = 2, ..., M$, $t < \theta < s$, and $t_* < \theta^\epsilon < s$. Letting, now, $\mathbf{a}(t) = [a_m(t)]$ and $\mathbf{a}^\epsilon(t) = [a_m^\epsilon(t)]$, $m = 1, ..., M$, be two column vectors in $\mathbb{R}^M$, values (3) and (4) can be rewritten as

$$W(t, \mathbf{x}_i, \mathbf{a}(t)) = \sum_{j=1}^{N} x_{i,j} e^{-g(t,t_j)}$$

and

$$W_*(t', \mathbf{x}_i, \mathbf{a}(t'), \mathbf{a}^\epsilon(t')) = \sum_{j=1}^{N} x_{i,j} e^{-\left[g(t',t_j) + g^\epsilon(t',t_j)\right]} . \tag{7}$$

The effect of a shift that instantly changes the value $W(t_*, \mathbf{x}_i, \mathbf{a}(t))$ into $W_*(t', \mathbf{x}_i, \mathbf{a}(t'), \mathbf{a}^\epsilon(t'))$ can be captured by developing (7) in a Taylor expansion

$$\begin{aligned} W_*(t_*, \mathbf{x}_i, \mathbf{a}(t_*), \mathbf{a}^\epsilon(t_*)) \simeq & W(t_*, \mathbf{x}_i, \mathbf{a}(t_*)) + \sum_{m=1}^{M} \frac{\partial}{\partial a_m(t_*)} W(s, \mathbf{x}_i, \mathbf{a}(t_*)) \Big|_{s=t_*} \cdot \\ & \cdot a_m^\epsilon(t_*) \end{aligned} \tag{8}$$

where the impact of the shock is represented by $a_m^\epsilon(t_*)$ depends on

$$\frac{\partial}{\partial a_m(t_*)} W(t_*, \mathbf{x}_i, \mathbf{a}(t_*)) = \sum_{j=1}^{N} x_{i,j} v(t_*, t_j) \frac{(t_j - t_*)^m}{m!} .$$

Neglecting term $1/m!$, these derivatives appear in the numerator of the m-th order moment for bond $\mathbf{x}_i$ defined as

$$D_m(t_*, \mathbf{x}_i) = \frac{\sum_{j=1}^{N} x_{i,j}(t_j - t_*)^m v(t_*, t_j)}{\sum_{j=1}^{N} x_{i,j} v(t_*, t_j)} = \frac{\langle \boldsymbol{\tau}_m(t_*), \mathbf{x}_i \rangle^T \mathbf{v}(t_*)}{\mathbf{x}_i^T \mathbf{v}(t_*)},$$

being $\boldsymbol{\tau}_m(t_*) = [(t_j - t_*)^m] \in \mathbb{R}^N$.

The first order moment $D_1(t, \mathbf{x}_i)$ is the duration of bond $\mathbf{x}_i$ while $M_2(t, \mathbf{x}_i) = D_2(t, \mathbf{x}_i) - D_1^2(t, \mathbf{x}_i)$ is its convexity. In the following Section the centered $2\rho-$moment ($\rho \in \mathbb{N}$), namely $M_{2\rho}(t, \mathbf{x}_i) = D_{2\rho}(t, \mathbf{x}_i) - D_1^{2\rho}(t, \mathbf{x}_i)$, will be also used.

The first M moments are collected in a $(P \times M)$ matrix $D(t) = [D_m(t, \mathbf{x}_i)]$ whose $i-$th row $\mathbf{d}^T(t, \mathbf{x}_i) \in \mathbb{R}^M$ contains the moments of bond $\mathbf{x}_i$ and is referred to as 'duration vector' (Chambers and Carleton and de La Grandville [12]). Expression (8) shows that the values of two bonds with the same duration vector react in the same manner to a change in the term structure.

The reason why the term 'moments' is used is clear. Montrucchio and Peccati [22] pointed out that any bond $\mathbf{x}_i$ can be seen as a random variable

$$\chi_i \sim \begin{cases} t_1 - t & \dots & t_N - t \\ \frac{x_{i,1} v(t;t_1)}{W(t,\mathbf{x}_i)} & \dots & \frac{x_{i,N} v(t;t_N)}{W(t,\mathbf{x}_i)} \end{cases} \tag{9}$$

with $\Pr[\chi_i = t_j - t] = \frac{x_{i,j} v(t;t_j)}{W(t,\mathbf{x}_i)} \geq 0$, $j = 1, ..., N$ and $\sum_{j=1}^{N} \frac{x_{i,j} v(t;t_j)}{W(t,\mathbf{x}_i)} = 1$ so that $D_m(t, \mathbf{x}_i) = \mathrm{E}[\chi_i^m]$ and $\mathrm{Var}[\chi_i] = M_2(t, \mathbf{x}_i)$.

To conclude this section, consider a portfolio $\boldsymbol{\lambda} \in \mathbb{R}^P$ where λ_i denotes the units of bond $\mathbf{x}_i$ held. According to (5), the market values of this portfolio are

$$\begin{array}{ll} W(t, \boldsymbol{\lambda}) = [X\mathbf{v}(t)]^T \boldsymbol{\lambda} & \text{for } t < t_* \text{ and} \\ W_*(t', \boldsymbol{\lambda}) = [X\mathbf{v}_*(t')]^T \boldsymbol{\lambda} & \text{for } t_* \leq t' < t_1 \end{array} \tag{10}$$

while, letting $\mathbf{D}^{(m)}(t)$ the m-th column of $D(t)$, the m-th moment of $\boldsymbol{\lambda}$ is

$$D_m(t, \boldsymbol{\lambda}) = \frac{\left\langle \mathbf{D}^{(m)}(t), \mathbf{w}(t) \right\rangle^T \boldsymbol{\lambda}}{W(t, \boldsymbol{\lambda})}.$$

2.2 Immunization

Loosely speaking, a bond or portfolio, is immune of interest rate risk when its value right after a shock is greater or equal to its value right before the shock itself. More formally, portfolio $\boldsymbol{\lambda}$ is immunized at time $t_* < t_1$ if

$$\lim_{t \to t_*^-} W(t, \boldsymbol{\lambda}) = W(t_*, \boldsymbol{\lambda}) \leq W_*(t_*, \boldsymbol{\lambda}). \tag{11}$$

Should this be the case, an investor can buy, in t, one unit of $\boldsymbol{\lambda}$ and, at the same time, go short of the amount $W(t, \boldsymbol{\lambda})$, so that the overall net investment is 0. Recalling that $W(t_*, \boldsymbol{\lambda}) = W(t, \boldsymbol{\lambda})/v(t, t_*)$, the net cash flow of this investment at time t_* is the random variable

$$\mathbf{W} \sim \begin{cases} 0 & W_*(t_*, \boldsymbol{\lambda}) - W(t_*, \boldsymbol{\lambda}) \\ p_1 & 1 - p_1 \end{cases}$$

where $0 \leq p_1 \leq 1$ is the probability that the shock does not happen in t_*.

Portfolio $\boldsymbol{\lambda}$ allows arbitrage opportunities as long as $W_*(t_*, \boldsymbol{\lambda}) > W(t_*, \boldsymbol{\lambda})$ because it ends up with a chance of making a positive sum out of a null investment with no possibility of losing money. The only case in which an immunized portfolio $\boldsymbol{\lambda}$ is compatible with absence of arbitrage is, therefore, when $W(t_*, \boldsymbol{\lambda}) = W_*(t_*, \boldsymbol{\lambda})$. Recalling (2) and (10), this is equivalent to write

$$[X \langle \mathbf{v}(t_*), (\mathbf{f}(t_*) - \mathbf{1}) \rangle]^T \boldsymbol{\lambda} = 0 \tag{12}$$

where the unit vector $\mathbf{1} \in \mathbb{R}^N$.

De La Grandville ([12] and [13]) provides the following general immunization theorem:

Theorem 1. *Consider a portfolio $\boldsymbol{\lambda}$ whose value in t is $W(t, \boldsymbol{\lambda})$. Assume that (6) holds, that a shock occurs in t_*, and let $W(t_*, \boldsymbol{\lambda}) = W(t, \boldsymbol{\lambda})/v(t, t_*)$ be the value of $\boldsymbol{\lambda}$ in $t_* = t + h$, with $h \geq 0$, if no shock happens. To accomplish (11) against any shock of the term structure, a sufficient condition, to be fulfilled in t, is:*

T1) any moment of order m of $\boldsymbol{\lambda}$ is equal to h^m: $D_m(t, \boldsymbol{\lambda}) = h^m$, $m = 1, ..., M$;
T2) the Hessian matrix $\left[\frac{\partial^2}{\partial a_i(t) \partial a_j(t)} W(t_*, \boldsymbol{\lambda})\right]$, $i, j = 0, ..., M$, is positive-definite.

Proof: see de La Grandville [12], chapter 15.

To achieve immunization, theorem 1 applies unconstrained optimization standard methodology to function $W(t_*, \boldsymbol{\lambda})$ where $a_m(t)$ act as variables, being (T1) the necessary condition while (T2) is the sufficient one. If there exists a portfolio $\bar{\boldsymbol{\lambda}}$ that satisfies (T1) and (T2), then $W(t_*, \bar{\boldsymbol{\lambda}})$ is (at least) a (local) minimum with respect to $\mathbf{a}(t)$ so that, in case a shock occurs, $W_*(t_*, \bar{\boldsymbol{\lambda}}) \geq W(t_*, \bar{\boldsymbol{\lambda}})$.

Condition (T1) is fulfilled by solving, with respect to $\boldsymbol{\lambda}$, the following system of linear equations with P variables and $M + 1$ equations

$$\begin{cases} \mathbf{w}^T(t) \boldsymbol{\lambda} = 1 \\ \left\langle D^{(1)}(t), \mathbf{w}(t) \right\rangle^T \boldsymbol{\lambda} = h \\ \dots \dots \dots \\ \left\langle D^{(M)}(t), \mathbf{w}(t) \right\rangle^T \boldsymbol{\lambda} = h^M \end{cases} \tag{13}$$

where, without loss of generality and for sake of compactness, $W(t, \boldsymbol{\lambda}) = 1$. The conservation theorem proposed by De Felice and Moriconi [11] and De Felice states that as moments decreases with respect to time, to keep $\boldsymbol{\lambda}$ immunized the portfolio needs to be readjusted as frequently as possible by solving (13).

As it was assumed that the market does not carry redundant bonds, the coefficient matrix of (13) has linearly independent columns so that its rank is $\min(M+1,P)$. Three cases are possible:

C1 if $M+1>P$, (13) has no solution,[1]
C2 if $M+1=P$, (13), being not homogeneous, has unique solution

$$\boldsymbol{\lambda}_{T1}=\begin{bmatrix}\mathbf{w}^T(t)\\ \left\langle D^{(1)}(t),\mathbf{w}(t)\right\rangle^T\\ \dots\\ \left\langle D^{(M)}(t),\mathbf{w}(t)\right\rangle^T\end{bmatrix}^{-1}\begin{bmatrix}1\\ h\\ \dots\\ h^M\end{bmatrix}\neq\mathbf{0}$$

C3 if $M+1<P$, (13) has an infinite number of solutions.

Under an immunization point of view, the 'best' portfolios are those hedged against the widest set of shocks. This means that, according to (6), M should be chosen as large as possible. On the other hand, such portfolios exist if (13) admits at least one solution. These two conflicting requirements are contemporaneously satisfied when case C2 (i.e. $M=P-1$) applies. This leads to $\boldsymbol{\lambda}_{T1}$ as the unique relevant portfolio. The solutions, with respect to $\mathbf{f}(t_*)$, of equation

$$\left[X\left\langle \mathbf{v}(t_*),(\mathbf{f}(t_*))-\mathbf{1}\right\rangle\right]^T\boldsymbol{\lambda}_{T1}=0\,,$$

obtained by substituting $\boldsymbol{\lambda}_{T1}$ into (12), should be intended as the general condition for a shock factor to allow immunization while not violating no-arbitrage condition.

Having set the basic framework, the following Section presents the results that allow to detect these shifts.

3 Lower and upper bounds for bond values

3.1 Previous results

As seen in Section 2, immunization theory is mainly focused on finding lower bounds for the value of bonds and portfolios in case a shift happens. Most of the results discussed in the Introduction are, unfortunately, not appropriate to tackle the main goal of this article as both a lower and an upper bounds are required to determine post-shift values and to fulfill (12).

Luckily, Shiu [27], Montrucchio and Peccati [22], and Uberti [29] come up with such bounds. Shiu assumes concavity and convexity of the shift factor and considers a portfolio composed of two bonds, one of which is a zero-coupon. Montrucchio and Peccati and Uberti extend this result by analyzing a portfolio made of two generic bonds. To do this, the authors exploit a generalization of the definition of convexity and concavity, namely the notion of $\alpha-$convex and convex$-\beta$ functions.

[1] To be precise, this case should be divided in two sub-cases: $M>P$ and $M=P$. The first carries no solution at all while the latter can yield either a unique solution or no solution.

Definition 1. *Given $\alpha \in \mathbb{R}$, a real function $\phi(t)$ is said to be $\alpha-$convex on the interval I if $\phi(t) - \frac{1}{2}\alpha t^2$ is convex on I.*

Definition 2. *Given $\beta \in \mathbb{R}$, a real function $\phi(t)$ is said to be convex$-\beta$ on the interval I if $\phi(t) - \frac{1}{2}\beta t^2$ is concave on I.*

Proposition 6 in Montrucchio and Peccati is presented below in a slightly simplified version, more useful for the aim of this paper.

Theorem 2 (Montrucchio and Peccati, 1991). *Given bonds $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $W(t, \mathbf{x}_1) = W(t, \mathbf{x}_2)$, $t < t_1$, if in $t_* > t$ a shift perturbs the term structure so the shift factor $f(t, s)$ is $\alpha-$convex and convex$-\beta$ for $t_* \leq s \leq t_N$, the following bounds for the post-shift value $\Delta W_*(t') = W_*(t', \mathbf{x}_1) - W_*(t', \mathbf{x}_2)$ in t', $t_* \leq t' < t_1$, hold*

$$\Delta D + \frac{\alpha M_2(t, \mathbf{x}_1) - \beta M_2(t, \mathbf{x}_2)}{2} \leq \frac{\Delta W_*(t')}{W(t, \mathbf{x}_1)} \leq \Delta D + \frac{\beta M_2(t, \mathbf{x}_1) - \alpha M_2(t, \mathbf{x}_2)}{2}$$

where $\Delta D = f(t, D_1(t, \mathbf{x}_1)) - f(t, D_1(t, \mathbf{x}_2))$.

Proof: see Montrucchio and Peccati.

3.2 An extension

The goal of this Subsection is to extend theorem 2. To do this a generalization of definitions 1 and 2 is introduced.

Definition 3. *Given $\alpha \in \mathbb{R}$ and $\rho \in \mathbb{N}$, a real function $\sigma(t)$ is $(\alpha, \rho)-$bounded on I if $\sigma(t) - \frac{\alpha t^{2\rho}}{(2\rho)!}$ is convex on I.*

Definition 4. *Given $\beta \in \mathbb{R}$ and $\rho \in \mathbb{N}$, a real function $\sigma(t)$ is bounded$-(\beta, \rho)$ on I if $\sigma(t) - \frac{\beta t^{2\rho}}{(2\rho)!}$ is concave on I.*

Recalling (4) and (9), in case a shock occurs in t_* the value in $t < t_1$ of bond $\mathbf{x}_i$ can be written as

$$W_*(t, \mathbf{x}_i) = W(t, \mathbf{x}_i) \sum_{j=1}^{N} f(t, t_j) \frac{x_{i,j} v(t, t_j)}{W(t, \mathbf{x}_i)} = W(t, \mathbf{x}_i) \operatorname{E}[f(t, \chi_i)] . \tag{14}$$

Letting once again that $W(t, \mathbf{x}_i) = 1$, if $f(t, s)$ is $(\alpha, \rho)-$bounded with respect to s on $t \leq s \leq T_N$, Jensen's inequality allows to write

$$\operatorname{E}\left[f(t, \chi_i) - \frac{\alpha \chi_i^{2\rho}}{(2\rho)!}\right] \geq f(t, \operatorname{E}[\chi_i]) - \frac{\alpha \operatorname{E}^{2\rho}[\chi_i]}{(2\rho)!} = f(t, D_1(t, \mathbf{x}_i)) - \frac{\alpha D_1^{2\rho}(t, \mathbf{x}_i)}{(2\rho)!}$$

so that a lower bound for $\operatorname{E}[f(t, \chi_i)]$ is

$$\begin{aligned} \operatorname{E}[f(t, \chi_i)] \geq & f(t, D_1(t, \mathbf{x}_i)) + \frac{\alpha \left[D_{2\rho}(t, \mathbf{x}_i) - D_1^{2\rho}(t, \mathbf{x}_i)\right]}{(2\rho)!} = \\ = & f(t, D_1(t, \mathbf{x}_i)) + \frac{\alpha M_{2\rho}(t, \mathbf{x}_i)}{(2\rho)!} \end{aligned} \tag{15}$$

If, instead, $f(t,s)$ is bounded$-(\beta,\rho)$ with respect to s on $t \le s \le T_N$, Jensen's inequality yields

$$\mathrm{E}\left[f(t,\chi_i) - \frac{\beta\chi_i^{2\rho}}{(2\rho)!}\right] \le f(t, D_1(t,\mathbf{x}_i)) - \frac{\beta D_1^{2\rho}(t,\mathbf{x}_i)}{(2\rho)!}$$

leading to an upper bound

$$\mathrm{E}\left[f(t,\chi_i)\right] \le f(t, D_1(t,\mathbf{x}_i)) + \frac{\beta M_{2\rho}(t,\mathbf{x}_i)}{(2\rho)!} \tag{16}$$

Consider now bonds $\mathbf{x}_1$ and $\mathbf{x}_2$ so that $W(t,\mathbf{x}_1) = W(t,\mathbf{x}_2)$. If $f(t,s)$ is both $(\alpha,\rho)-$bounded and bounded$-(\beta,\rho)$ with respect to s when $t \le s \le t_N$ then, according to (14), (15), and (16), the post-shift value in t_* of $\Delta W_*(t_*) = W_*(t_*,\mathbf{x}_1) - W_*(t_*,\mathbf{x}_2)$ can be written as

$$\begin{aligned}\frac{\Delta W_*(t_*)}{W(t,\mathbf{x}_1)} =& W(t_*,\mathbf{x}_1)\,\mathrm{E}\left[f(t,\chi_1)\right] - W(t_*,\mathbf{x}_2)\,\mathrm{E}\left[f(t,\chi_2)\right] = \\ =& W(t_*,\mathbf{x}_1)\left\{\mathrm{E}\left[f(t,\chi_1)\right] - \mathrm{E}\left[f(t,\chi_2)\right]\right\}\end{aligned}$$

so that, mimicking the proof of theorem 2, $\frac{\Delta W_*(t_*)}{W(t,\mathbf{x}_1)}$ has bounds

$$\begin{aligned}\Delta D + \frac{\alpha M_{2\rho}(t,\mathbf{x}_1) - \beta M_{2\rho}(t,\mathbf{x}_2)}{(2\rho)!} \le \frac{\Delta W_*(t_*)}{W(t,\mathbf{x}_1)} \le \\ \le \Delta D + \frac{\beta M_{2\rho}(t,\mathbf{x}_1) - \alpha M_{2\rho}(t,\mathbf{x}_2)}{(2\rho)!}\end{aligned} \tag{17}$$

where, again, $\Delta D = f(t, D_1(t,\mathbf{x}_1)) - f(t, D_1(t,\mathbf{x}_2))$.

While theorem 2 determines bounds for the post-shift bonds value in terms of their first two moments, definitions 3 and 4 allows to identify bounds expressed in terms of duration and even moments of the bonds.

Armed with these bounds, the following Section finally presents a family of no-arbitrage compatible shifts.

4 A sufficient condition for immunization and no-arbitrage

Consider two portfolios $\boldsymbol{\lambda}^1$ and $\boldsymbol{\lambda}^2$ composed so that their values are the same $\left(W\left(t,\boldsymbol{\lambda}^1\right) = W\left(t,\boldsymbol{\lambda}^2\right)\right)$ and their durations match $\left(D_1\left(t,\boldsymbol{\lambda}^1\right) = D_1\left(t,\boldsymbol{\lambda}^2\right)\right)$. According to (17), if the shift factor is both $(\alpha,\rho)-$bounded and bounded$-(\beta,\rho)$ the post-shift value $\Delta W_*(t_*) = W_*\left(t_*,\boldsymbol{\lambda}^1\right) - W_*\left(t_*,\boldsymbol{\lambda}^2\right)$ has bounds

$$\frac{\alpha M_{2\rho}(t,\boldsymbol{\lambda}^1) - \beta M_{2\rho}(t,\boldsymbol{\lambda}^2)}{(2\rho)!} \le \Delta W_*(t_*) \le \frac{\beta M_{2\rho}(t,\boldsymbol{\lambda}^1) - \alpha M_{2\rho}(t,\boldsymbol{\lambda}^2)}{(2\rho)!}.$$

As $\varDelta W(t) = W\left(t, \boldsymbol{\lambda}^1\right) - W\left(t, \boldsymbol{\lambda}^2\right) = 0$, $\varDelta W(t_*) = 0$ as well and (12) states that a shock carries no arbitrage opportunity if $\varDelta W_*(t_*) = 0$. This occurs when

$$\alpha M_{2\rho}(t, \boldsymbol{\lambda}^1) - \beta M_{2\rho}(t, \boldsymbol{\lambda}^2) = \beta M_{2\rho}(t, \boldsymbol{\lambda}^1) - \alpha M_{2\rho}(t, \boldsymbol{\lambda}^2) = 0\,.$$

If $M_{2\rho}(t, \boldsymbol{\lambda}^1) = M_{2\rho}(t, \boldsymbol{\lambda}^2)$, $\varDelta W_*(t_*) = 0$ when $\alpha = \beta = q$, $q \in \mathbb{R}\backslash\{0\}$. If, instead, $M_{2\rho}(t, \boldsymbol{\lambda}^1) \neq M_{2\rho}(t, \boldsymbol{\lambda}^2)$, $\varDelta W_*(t_*) = 0$ only when $\alpha = 0$ and $\beta = 0$.

For the first case, definitions 3 and 4 say that a shift factor is contemporaneously $(q, \rho)-$bounded and bounded$-(q, \rho)$, with respect to s for all $s \geq t$, when $f(t, s) - \frac{q(s-t_*)^{2\rho}}{(2\rho)!}$ is both concave and convex or, in other words, affine. This means that

$$f(t, s) = \left[\frac{q(s-t)^{2\rho}}{(2\rho)!} + c_1(s-t) + c_2\right] \mathbf{1}_{\{s \geq t_*\}}$$

with $c_1, c_2 \in \mathbb{R}$ so that $\frac{q(s-t)^{2\rho}}{(2\rho)!} + c_1(s-t) + c_2 > 0$ for $t_* \leq s \leq t_N$. Recalling (1), a family of no-arbitrage shifts is, then, the class of hyperbolic functions

$$\epsilon_1(s) = -\frac{\frac{q(s-t)^{2\rho-1}}{(2\rho-1)!} + c_1}{\frac{q(s-t)^{2\rho}}{2\rho!} + c_1(s-t) + c_2} \cdot \mathbf{1}_{\{s \geq t_*\}}$$

that, multiplying both numerator and denominator by $((2\rho)!)/q$ and letting $d_1 = \left[c_1(2\rho)!\right]/q$ and $d_2 = \left[c_2(2\rho)!\right]/q$, $\epsilon_1(s)$ can be written as

$$\epsilon_1(s) = -\frac{2\rho(s-t)^{2\rho-1} + d_1}{(s-t)^{2\rho} + d_1(s-t) + d_2} \cdot \mathbf{1}_{\{s \geq t_*\}}\,. \tag{18}$$

In the second case ($M_{2\rho}(t, \boldsymbol{\lambda}^1) \neq M_{2\rho}(t, \boldsymbol{\lambda}^2)$), instead, $\varDelta W_*(t_*) = 0$ only when $\alpha = 0$ and $\beta = 0$ so that the shift is ($c_1 \neq 0$)

$$\epsilon_2(s) = -\frac{c_1}{c_1(s-t) + c_2} \cdot \mathbf{1}_{\{s \geq t_*\}}\,,$$

The initial amplitude of both shifts in t_* is $\epsilon_1(t_*) = \epsilon_2(t_*) = -c_1/c_2$, $c_2 \neq 0$.

Before analyzing the impact of the shocks on the term structure it is necessary to establish under which conditions $\epsilon_1(s)$ and $\epsilon_2(s)$ are continuous function for $s \geq t_*$. This is true when their denominators are not equal to 0 for all $s \geq t_*$, i.e. when equation

$$(s-t_*)^{2\rho} + d_1(s-t_*) + d_2 = 0 \tag{19}$$

has either no real root or when all its real roots are strictly negative and when $c_1(s-t_*) + c_2 = 0$ as a strictly negative root.

The following proposition gives conditions on d_1 and d_2 for existence and sign of roots of equation (19).

Proposition 1. *Equation (19) has at most two real roots. Further,*

1 *(19) has no real roots when* $d_2 > (2\rho-1)\left(\frac{d_1}{2\rho}\right)^{\frac{2\rho}{2\rho-1}} \geq 0$,

2 *(19) has a unique real root* $s^* = t_* + \sqrt[2\rho-1]{-\frac{d_1}{2\rho}}$ *when* $d_2 = (2\rho-1)\left(\frac{d_1}{2\rho}\right)^{\frac{2\rho}{2\rho-1}}$. *Further,* $s^* < t_*$ *when* $d_1 > 0$,

3a *(19) has two strictly negative roots when* $d_2 < (2\rho-1)\left(\frac{d_1}{2\rho}\right)^{\frac{2\rho}{2\rho-1}}$, $d_1 > 0$, *and* $d_2 > 0$,

3b *(19) has a strictly positive and a strictly negative roots when* $d_2 < (2\rho - 1)\left(\frac{d_1}{2\rho}\right)^{\frac{2\rho}{2\rho-1}}$, $d_1 \neq 0$, *and* $d_2 < 0$,

3c *(19) has two strictly positive roots when* $d_2 < (2\rho-1)\left(\frac{d_1}{2\rho}\right)^{\frac{2\rho}{2\rho-1}}$, $d_1 < 0$, *and* $d_2 > 0$.

Proof: see Appendix.

Equation (19) is not verified for $s \geq t_*$ when cases 1, 2, and 3a of proposition 1 apply while the denominator of $\epsilon_2(s)$ is not equal to 0 for $s \geq t_*$ when $-c_1/c_2 < 0$, $c_2 \neq 0$. The initial amplitude $\epsilon_1(t_*)$ can be any number (case 1 of proposition 1), either positive or negative (case 2 as long as $d_2 \neq 0$) or strictly negative (case 3a). The initial amplitude $\epsilon_2(t_*)$ is, instead, strictly negative.

This being said it is now possible to analyze the effect of $\epsilon_1(s)$ and $\epsilon_2(s)$ on the term structure.

Continuity of $\epsilon_1(s)$ and $\epsilon_2(s)$, coupled with the fact that $\lim_{s\to+\infty} \epsilon_1(s) = \lim_{s\to+\infty} \epsilon_2(s) = 0$, allow to conclude that the effect of the shifts vanishes with respect to time; after a sufficiently long period of time since t_*, the shifted term structure returns back to its pristine shape. No-arbitrage conditions drive, then, the market back to its original status.

The impact of the shift can be ascertained by studying the first derivative of $\epsilon_1(s)$ and $\epsilon_2(s)$. As

$$\begin{aligned}\epsilon_1'(s) =& \frac{2\rho\,(s-t_*)^{4\rho-2} + \left(6\rho - 4\rho^2\right) d_1\,(s-t_*)^{2\rho-1}}{\left[2\rho\,(s-t_*)^{2\rho} + d_1\,(s-t_*) + d_2\right]^2} + \\ &+ \frac{\left(2\rho - 4\rho^2\right) d_2\,(s-t_*)^{2\rho-2} + d_1^2}{\left[2\rho\,(s-t_*)^{2\rho} + d_1\,(s-t_*) + d_2\right]^2},\end{aligned}$$

it is possible to determine analytically if and for which values $\epsilon_1(s)$ has extrema only when $\rho = 1$. The shift function becomes

$$\epsilon_1'(s) = \frac{2\,(s-t_*)^2 + 2d_1\,(s-t_*) + d_1^2 - 2d_2}{\left[2\,(s-t_*)^2 + d_1\,(s-t_*) + d_2\right]^2}.$$

and the necessary condition $\epsilon_1'(s) = 0$ admits real roots $s_1^* = t_* + \frac{-d_1-\sqrt{4d_2-d_1^2}}{2}$ and $s_2^* = t_* + \frac{-d_1+\sqrt{4d_2-d_1^2}}{2}$, $s_1^* < s_2^*$, when $d_2 \geq d_1^2/4$. It is evident that $s_1^* < t_*$ while $s_2^* \geq t_*$ when $d_2 \geq d_1^2/2$ so that, in this case, $\epsilon_1(s)$ has a stationary point at s_2^*. Analysis of $\epsilon_1'(s)$ returns that $\epsilon_1(s_2^*)$ is (at least) a local minimum. Substituting $\rho = 1$ into proposition 1 yields that cases 1 ($d_2 > d_1^2/2$) and 2 ($d_2 = d_1^2/2$ along with $d_1 > 0$) are compatible with the existence of the local minimum in s_2^*. Case 3a, instead imposes $\epsilon_1(s)$ to be monotonic for $s \geq t_*$.

In the general case $\rho \geq 2$, existence of stationary points can be inferred by exploiting proposition 1 again. Let $\varepsilon(s)$ be the numerator of $\epsilon_1'(s)$ so that

$$\varepsilon'(s) = \left(4\rho^2 - 2\rho\right)(s - t_*)^{2\rho-3}\left[(s - t_*)^{2\rho} + \frac{3 - 2\rho}{2} d_1 (s - t_*) + (1 - \rho) d_2\right].$$

A stationary point for $\varepsilon(s)$ is $s = t_*$. The term in square brackets could have one (case 3b in proposition 1) or two (case 3c) roots greater than t_* so that the shift can have one or two extrema when $s > t_*$. To have only one critical point greater than t_* case 3b requires

$$(1 - \rho) d_2 < (2\rho - 1)\left(\frac{3 - 2\rho}{4\rho} d_1\right)^{\frac{2\rho}{2\rho-1}} \tag{20}$$

along with $\left[(3 - 2\rho)d_1\right]/2 \neq 0$ (that is equivalent to $d_1 \neq 0$) and $(1 - \rho) d_2 < 0$ (i.e. $d_2 > 0$). To allow for two critical points for $s > t_*$, case 3c imposes condition (20) along with $\left[(3 - 2\rho)d_1\right]/2 < 0$ (i.e. $d_1 > 0$) and, again, $(1 - \rho) d_2 < 0$.

The first derivative of $\epsilon_2(s)$ is

$$\epsilon_2'(s) = \frac{c_1^2}{\left[c_1(s - t_*) + c_2\right]^2} > 0$$

for all s, to that shift $\epsilon_2(s)$ is, unlike $\epsilon_1(s)$, an always increasing function.

To conclude, $\epsilon_2(s)$ resembles the shift proposed by Khang [18]

$$\zeta(s) = \frac{\lambda}{1 + a(s - t_*)}$$

whose aim was to give some immunization condition when short-term rates are more volatile than long-term ones, as can often be the case. Shock $\epsilon_2(s)$ belongs to the class proposed by Khang when $\lambda = -c_1/c_2$ and $a = c_1/c_2$.

5 Conclusions

This work has presented a theoretical framework capable of detecting the existence of shifts consistent with no-arbitrage condition and has determined explicitly a class of such shocks. To achieve this, immunization results that provide lower and upper bounds for post-shift bond and portfolio values have been exploited and extended. What, at first hand, seemed a sheer contradiction between immunization, viewed as a bond portfolio management technique, and the requirement of absence of arbitrage, as the main pillar of quantitative finance, finds here a plausible reconciliation.

This theoretic result can also be seen as a stepping stone for subsequent empirical research answering the following questions: how does a ‘real market’ shift look like? Does the economic forces that drive the market allow ‘non arbitrage-tight’ shifts?

Answers to these questions are left for further research.

Appendix

Proof of proposition 1

Consider $p(x) = x^{2n} + d_1 x + d_2$, $n \in \mathbb{N}$. First of all, $\lim_{x\to+\infty} p(x) = \lim_{x\to-\infty} p(x) = +\infty$. Secondly, as the first order condition $p'(x) = 2nx^{2n-1} + d_1 = 0$ has a unique solution $x^* = \sqrt[2n-1]{-\frac{d_1}{2n}}$ and $p''(x) = (4n^2 - 2n)x^{2n-2} \geq 0$ for all $x \in \mathbb{R}$, x^* is the global minimum for $p(x)$. In fact, if $d_1 \neq 0$ then $x_* \neq 0$ and $p''(x^*) > 0$ while if $d_1 = 0$ then $p^{(2n)}(x^*) = \prod_{s=0}^{2n-1}(2n-s) > 0$. Further

$$p(x^*) = \left(-\frac{d_1}{2n}\right)^{\frac{2n}{2n-1}} + d_1\left(-\frac{d_1}{2n}\right)^{\frac{1}{2n-1}} + d_2 = -\left(\frac{d_1}{2n}\right)^{\frac{2n}{2n-1}}(2n-1) + d_2$$

is the global minimum point. Three cases are possible:

1: $p(x)$ has no real roots when $d_2 > (2n-1)\left(\frac{d_1}{2n}\right)^{\frac{2n}{2n-1}}$ (i.e. when $p(x^*) > 0$),
2: $p(x)$ has one real root $x^* = \sqrt[2n-1]{-\frac{d_1}{2n}}$ when $d_2 = (2n-1)\left(\frac{d_1}{2n}\right)^{\frac{2n}{2n-1}}$ ($p(x^*) = 0$) and
3: $p(x)$ has two real roots when $d_2 < (2n-1)\left(\frac{d_1}{2n}\right)^{\frac{2n}{2n-1}}$ ($p(x^*) < 0$).

If case 2 applies, $x^* < 0$ when $d_1 > 0$ and, from this, it results that $d_2 > 0$. Recalling that $x^* > (<)0$ when $d_1 > (<)0$ and noting that $p(0) = d_2$, case 3 can be divided in three sub-cases:

3a: $p(x)$ has two strictly negative roots when $d_1 > 0$ and $d_2 > 0$,
3b: $p(x)$ has one strictly negative and one strictly positive roots when $d_1 \neq 0$ and $d_2 < 0$,
3c: $p(x)$ has two strictly positive roots when $d_1 < 0$ and $d_2 > 0$.

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