Maximum duration of below par bonds: a closed-form formula

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Abstract. The nature of the relationship between bond's duration and its term to maturity is somewhat complex and, contrary to what our intuition would suggest, there is not always a direct relationship between duration and maturity, in the sense that an increase in maturity does not necessarily entail an increase in duration. For bonds selling below par, an increase in maturity has the following effects: duration first increases, it reaches a maximum and then decreases. For this maximum we provide an explicit formula based on the Lambert W function.

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1 Introduction

There are two main reasons why duration is a basic concept in bond analysis and management: it provides a useful information on the bond's riskiness and it is essential to the procedure of protection against unforeseen changes in interest rates.

A bond's duration is a concept first introduced in 1938 by F. Macaulay (see [7]) to provide more complete summary information about the time structure of a bond than term to maturity. Maturity provides information only about the time of final payment and thus gives an incomplete description of the time pattern of all the cash flows of a security.

For fixed-coupon bonds, duration can be intuitively defined as the average maturity of all bond payments, where each payment is weighted by its present value.

It can be simply shown that, for a given change in interest rates, percentage changes in the bond prices vary proportionately with duration. It follows that the relation between changes in bond prices and term to maturity depends on the relationship between duration and maturity.

Duration is heavily dependent on the coupon rate, the rate of interest, and maturity. The effects of a change in coupon rate or in interest rate on duration are similar, and can be simply analyzed: an increase in the coupon rate or in interest rate will lead to a decrease in duration.

The nature of the relationship between bond's duration and its term to maturity is somewhat complex and contrary to what our intuition would suggest, there is not always a direct relationship between duration and maturity, in the sense that an increase in maturity does not necessarily entail an increase in duration.

At the present time the true nature of the relationship between duration and maturity is well known to financial literature, even if almost every book discussing the dynamics of bond prices report statements as the following " ... for a given change in yields, the fluctuations in market price will be greater the longer the term to maturity" (see for example W.C. Freund [3] at page 51 and G.W. Woodworth [9] at page 191 and more recently H.S. Stoll, R.E. Whaley [8] at page 135).

A precise mathematical examination of the exact form of the nature of the relationship between duration and maturity is carried out in G. Hawawini [4] and [5] where many interesting properties are obtained, through a closed form formula which permits to derive the duration directly.

A more tractable closed form formula for the duration is presented in G.O. Bierwag [1] and in O. de La Grandville [6]; in this case the duration is split into two parts: the first term is the duration of a perpetual bond or consol that promises a regular payment each period forever and the second one is an adjustment factor which can be either negative or positive.

The rest of this note is organized as follows. Section 2 deals with the definition of duration's concept and with some closed form formulas. The relationship between a bond's duration and its term to maturity is analyzed in Section 3. This analysis is summarized by discussing a set of properties. In particular we will see that, for bonds selling below par, an increase in maturity has the following effects: duration first increases, it reaches a maximum and then decreases. For this maximum we provide an explicit formula based on the Lambert W function.

Of course, we cannot claim to have seen every paper written on applications of W function, but of those we have seen, this contribution presents the first application of Lambert function in finance.

2 Duration and its closed formulas

Duration considers a coupon bond as a set of zero coupon bonds with consecutive maturity payments equal to the coupon payments plus a higher payment at the final date. If the evaluation is carried out at the beginning of a coupon period, the duration is defined as

$$D(C_t, i_t, n) = \frac{\sum_{t=1}^{n} \frac{tC_t}{(1+i_t)^t} + \frac{nB_n}{(1+i_n)^n}}{\sum_{t=1}^{n} \frac{C_t}{(1+i_t)^t} + \frac{B_n}{(1+i_n)^n}} \tag{1}$$

where:

- $D(\cdot, \cdot, \cdot)$ is the duration,
- C_t is the amount of coupon payment at time t,
- B_n is the reimbursement value,
- t is the t-th payment,
- i_t is the rate applicable for period t,
- n is the maturity date.

This formulation can be seen as a weighted average of the terms of payments. Each period is weighted by the present value of the corresponding payment.

In the following we make the simplification that the term structure of interest rates is flat ($i_t = i$ for all t); moreover we suppose that the coupon is constant and the reimbursement value is equal to the face value.

In such hypothesis we can rewrite (1) as

$$D(C_t, i, n) = \sum_{t=1}^{n} t \frac{C_t/B}{(1+i)^t} = \frac{1}{B} \sum_{t=1}^{n} t \cdot C_t (1+i)^{-t} \tag{2}$$

where

$$B = \sum_{t=1}^{n} \frac{C_t}{(1+i)^t} = \sum_{t=1}^{n} C_t (1+i)^{-t} \tag{3}$$

designates the value of the bond; in particular we have: $C_t = C$ for $t = 1, \dots, n-1$, $C_n = C + B_n$, i.e. C_n indicates the global cash flow to be received at maturity n.

Duration has an appealing physical interpretation: it is the center of gravity for the payments in present value.

A main usefulness of duration stems from the equation

$$D = -\frac{1+i}{B} \frac{\partial B}{\partial i} \tag{4}$$

which points out that duration is a measure of the riskiness of a bond.

The use of equation (2) to compute the duration of bonds, especially those having a long maturity, is very tedious; it is often convenient to use a closed form formula, as opposed to an open form which involves sums. Moreover, the analysis of other main properties of duration becomes enormously simplified if we consider the expression of duration in a closed form.

The seminal paper of Macaulay [7] presents an interesting, even if incomplete, sensitivity analysis based on the following closed form

$$D(r, i, n) = 1 + \frac{1}{i} - \frac{(1+i)/r + n(1 + 1/r - (1+i)/r)}{(1+i)^n - 1 - 1/r + (1+i)/r} \tag{5}$$

where $r = C/F$ is the coupon rate. In order to obtain (5) you can consider the ratio

$$D(r,i,n) = \frac{\frac{C}{1+i} + \frac{2C}{(1+i)^2} + \frac{3C}{(1+i)^3} + \ldots + \frac{nC}{(1+i)^n} + \frac{nF}{(1+i)^n}}{\frac{C}{1+i} + \frac{C}{(1+i)^2} + \frac{C}{(1+i)^3} + \ldots + \frac{C}{(1+i)^n} + \frac{F}{(1+i)^n}};$$

summing the terms in the numerator, and in the denominator, of this ratio and substituting C/r for F, we immediately find formula (5).

Hawawini [4] and [5] present another closed form formula which is used to examine analytically the relationship between a bond's duration and its term to maturity.

The Hawawini's expression for duration is

$$D(r,i,n) = \frac{(1+i)a_{\overline{n}|i}r + n(i-r)(1+i)^{-n}}{r + (i-r)(1+i)^{-n}} \tag{6}$$

where $a_{\overline{n}|i}$ is the present value of an n – period annuity at a rate i.

In order to prove (6) you can write duration as

$$D = -\frac{F}{B}\frac{\partial(B/F)}{\partial i}(1+i); \tag{7}$$

then consider the ratio

$$\frac{B}{F} = r\frac{1-(1+i)^{-n}}{i} + \frac{i}{i}(1+i)^{-n} = \frac{r + (i-r)(1+i)^{-n}}{i} \tag{8}$$

and hence

$$\begin{aligned}\frac{\partial \frac{B}{F}}{\partial i} &= -\left[(r + (i-r)(1+i)^{-n})i^{-1} + (i-r)n(1+i)^{-(n+1)} - (1+i)^{-n}\right] i^{-1} \\ &= -\left[ra_{\overline{n}|i}n(i-r)(1+i)^{-(n+1)}\right] i^{-1}.\end{aligned} \tag{9}$$

Substituting (9) and (8) in (7) we get immediately the closed formula (6). We will also find it convenient to rearrange (6) as

$$\frac{D}{n} = \frac{\alpha r + (i-r)(1+i)^{-n}}{r + (i-r)(1+i)^{-n}} \tag{10}$$

with $\alpha = (1+i)a_{\overline{n}|i}/n$.

A third closed form formula for a bond's duration is

$$D(n,r,i) = 1 + \frac{1}{i} + \frac{n(i-r) - (1+i)}{r\left[(1+i)^n - 1\right] + i}. \tag{11}$$

This formula is presented and analyzed both in [1] and in [6].

To prove (11) we start with the ratio

$$\frac{B}{F} = \frac{r}{i}\left[1-(1+i)^{-n}\right] + (1+i)^{-n} = \frac{1}{i}\left\{r\left[1-(1+i)^{-n}\right] + i(1+i)^{-n}\right\}. \tag{12}$$

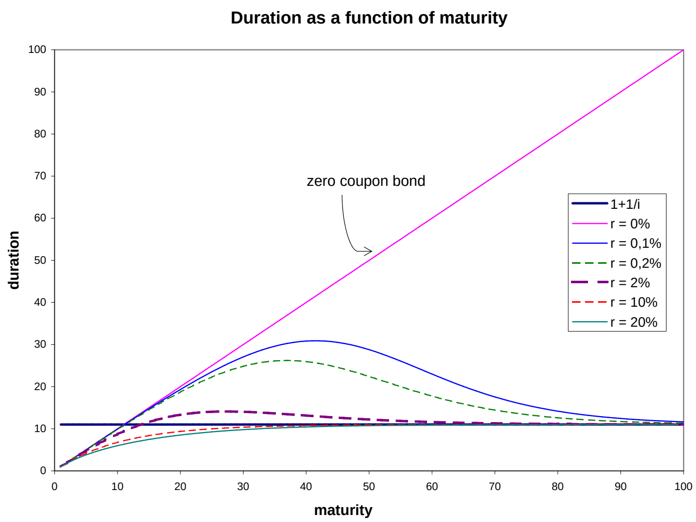

Fig. 1. Duration of a bond as a function of its maturity for $i = 10\,\%$ and coupon rates $r = 0, 0.1, 0.2, 2, 10, 20\,\%$.

Taking the logarithm, we have

$$\log\frac{B}{F} = -\log\, i + \log\left\{r\left[1-(1+i)^{-n}\right] + i(1+i)^{-n}\right\}. \tag{13}$$

The derivative with respect to i is

$$\begin{aligned}\frac{\partial \log(B/F)}{\partial i} &= \frac{\partial \log\, B}{\partial i} = \frac{1}{B}\frac{\partial B}{\partial i} = \\ &= -\frac{1}{i} + \frac{r\,n(1+i)^{-n-1} + (1+i)^{-n} + i(-n)(1+i)^{-n-1}}{r[1-(1+i^{-n})] + i(1+i)^{-n}}.\end{aligned} \tag{14}$$

Since $D = -\frac{(1+i)}{B}\frac{\partial B}{\partial i}$, multiplying the derivative just above by $-(1+i)$, we obtain (11) after simple simplifications.

3 Duration and maturity

The closed form formulas (5), (6) and (11) are more tractable than the definition (2) and can be used to examine the complex relationship between the bond's duration ant its term to maturity. In the following this analysis will be summarized through a set of properties.

Property 1. $D \leq n$. For zero coupon bonds or for bonds with one-period coupon bearing, duration is equal to maturity. In this special case duration increases with maturity. For all other bonds ($r > 0$, $n > 1$), duration is strictly shorter than term to maturity. This property can be immediately proved by examining (10).

Property 2. $\forall r, D(r,i,n) \to 1 + \frac{1}{i}$ as $n \to \infty$. This property is quite astonishing and states that the duration of a perpetual bond is equal to $1 + i^{-1}$ irrespective of its coupon rate. Indeed, referring to the closed form formula (11) and in particular to the ratio (adjustment factor)

$$\frac{n(i-r)-(1+i)}{r\left[(1+i)^n-1\right]+i}, \tag{15}$$

we can see that the numerator is an affine function in n, while the denominator is an exponential one.

Property 3. If $r \geq i$ then $D'(r,i,n) > 0$. The duration of a coupon bond selling at par ($i = r$) or above par ($i < r$) increases monotonically with its term to maturity and approaches the limit $1+i^{-1}$. Indeed, if $r = i$ the adjustment factor becomes equal to $-1/(r(1+r)^{n-1})$ and thus $D'(r,r,n) > 0,\ \forall n$. On the other hand, if $r > i$ the ratio (15) is always negative and tends monotonically to zero.

Table 1. Maximum values in duration for below par bonds with $i = 5\,\%$ and $i = 10\,\%$.

$i = 0.05$			$i = 0.1$		
r	n^*	$D^*(r, 0.05, n^*)$	r	n^*	$D^*(r, 0.1, n^*)$
0.001	71.94	51.02	0.001	41.50	30.90
0.002	64.94	42.64	0.002	36.93	26.22
0.003	60.01	38.18	0.005	31.74	20.67
0.005	55.89	33.07	0.006	30.87	19.67
0.006	54.75	31.33	0.010	28.79	17.07
0.010	52.94	27.19	0.020	27.34	14.10
0.020	57.37	22.87	0.050	32.95	11.45
0.030	73.38	21.38	0.060	38.18	11.18
0.040	125.51	21.01	0.090	120.49	11.000012
0.045	230.50	21.00	0.095	230.49	11.00
0.049	1070.5	21.00	0.099	1110.5	11.00

Property 4. If $r < i$ (i.e. for below par bonds), then the duration first increases; it crosses the right line $1 + 1/i$ and arrives at a maximum; then it decreases toward the limit $1 + 1/i$. In other words, let n^* the maximum point and D^* the maximum value; we have

$$\begin{aligned} D' &> 0 \qquad \text{when} \quad n < n^* \\ D' &< 0 \qquad \text{when} \quad n > n^*. \end{aligned} \tag{16}$$

Looking at the ratio (15) we can easily prove that the duration of a coupon bond selling below par initially increases and for $n = (1+i)/(i-r)$ intersects the asymptote $1 + 1/i$; then it reaches a maximum value D^* for $n = n^*$ and finally decreases toward the limit value $1 + 1/i$.

Property 5.

$$n^* = \frac{b(1+i) + a(1 + Lambert\,W(a \cdot exp[-\frac{a+b(1+i)}{a}]/r))}{a \cdot b} \tag{17}$$

where: $a = i - r$, $b = \log(1+i)$, and $W(\cdot)$ is the $Lambert\,W$ function. This function satisfies the equation

$$Lambert\,W(x) \cdot e^{Lambert\,W(x)} = x$$

and can thus be used to express solutions of transcendental equation involving exponentials or logarithms (for more details on $Lambert\,W$ function, please consult [2]).

For coupon bonds below par, using (17) we can directly compute the abscissa of the maximum value D^*, in opposition to assertions given in [7] at page 50 and in [6] at page 86.

Properties 3, 4 and 5 are geometrically shown in figure 1 which points out the complex relationship between a bond's duration and its term to maturity, as the coupon rate varies.

The rationale underlying properties 3, 4 and 5 is easily explained with some physical interpretation of bond's cash flows. If the coupons are large, duration increases monotonically with maturity, but for small coupons duration initially increases and then decreases. Indeed, adding one payment entails obviously to shift on the right the center of gravity, but also to delay the reimbursement of the principal. The resultant of these two forces which move in opposite direction obviously depends on their intensity.

Table 1 contains other relevant information for the bonds with $i > r$. We can see that as r increases toward i the maximum value of duration D^* decreases monotonically and tends to the limit value $1 + 1/i$, while the ascissa n^* first decreases and then rises greatly. Using the explicit formula (17), this particular behavior of n^* and D^* have been checked for $i = 1, 2, \ldots, 20\,\%$.

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