Evolution of risk preferences

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Abstract. This paper presents a simulation, made with a genetic algorithm (GA), in order to shape risk attitudes in a simple environment of lotteries such as the Machina triangle. An overlapping generations case is used where only bankrupted agents exit the market at every generation. The main result is that imposing the simple condition of the bankruptcy benchmark, with instantaneous wealth as fitness measure, is enough to induce stochastic dominance. The resulting indifference curves are shaped graphically, measuring how far they are from the ideal risk neutral case.

Keywords. Risk preferences, genetic algorithm.

M.S.C. classification. 60–04

J.E.L. classification. C61, D81, D83.

1 Introduction

After the contribution of *Expected Utility Theory* by von Neumann and Morgenstern (1944) attitudes towards risk have almost always been directly related to utilities. In that framework, if an agent can measure her utilities and beliefs, and if she can compute weighted averages, she is able to assign a utility to random events. The assumptions has been criticized by some paradoxes, as Allais' (1953) famous one, by empirical results from financial and insurance markets, and by the outcomes of experimental economics.[1] The model remains nevertheless the most robust and widely used one to extend utilities to the world of uncertainty.

We will try to discover the resulting shape of the utility functions over lotteries, adopting only ordinal utilities on a basic set of lotteries with three outcomes.[2]

[1] Starmer (2000) is an extensive survey on the empirical critics to Expected Utility Theory, Bergman (2004) is instead an example of a theoretical approach in proposing alternatives, two recent interesting examples on experiments are List (2004) and Pennings and Smidts (2003).

[2] The assumption of *ordinality* is actually the choice to consider the class of equivalence of utilities under monotonic transformation, as is usually done in the theoretical *micro–foundation* of economics.

Our preferences will be strict, every utility on a given finite set is then equivalent to a permutation of its elements.
The set of lotteries will be a finite grid on the *Machina triangle*, a simple geometrical device to show the simplex of all possible probability distributions on three distinct events. It is well known that the shape of the *iso–utility* curves on the triangle depends directly on the risk attitudes of the agent, among which local risk aversion, that can also easily be measured.
The tool for our analysis will be a *genetic algorithm* (GA) with operators adapted to the particular task. GA is not only a successful technique for heuristic optimization, from the very beginning it has been related to artificial intelligence in a far more intuitive way than *black boxes* as neural networks. Its operators have clear analogies to social and economical phenomena: *selection* and *cross over* replicate persistence of good strategies and learning from successful agents, while *mutation* is the simplest way to implement experimentation.[3]
Another way to think of GAs, when there is reciprocal interaction, is through *evolutionary game theory*. This approach allows us to study heterogeneous results as equilibria of strategies, which are just the genetic codes.[4]

We will imagine a fixed population of N agents, homogeneous in initial wealth M_0 but heterogenous in preferences on a fixed set of L lotteries[5], with different probabilities on three symmetric outcomes: win a fixed sum $S < M_0$; loose it; nothing changes. The genetic code is hence just a permutation of the numbers from 1 to L, the strict preference on the lotteries. At every step a random Pareto optimal distribution of the lotteries is made among the agents and lotteries are played, as explained in next Section.
The process is repeated iteratively. The GA is an *overlapping generations* one, in which the only agents that exit the market are the bankrupted ones, whose place is taken by *new–born* agents that will form their preferences according to the GA rules, receiving M_0 as initial amount of wealth. As *fitness measure* (the factor that determines the likelihood of a genetic code to be transmitted to new–entrants) we have tried the instantaneous *monetary* success, but also *financial* stability and *longevity* in the market.
The reciprocal interaction in this GA comes from the Pareto distribution of lotteries, so that the expected fitness measure will depend in the long run not only on one agent's preferences (genetic code) but also on all the others'.

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[3] GA was first proposed by Holland (1975), Holland himself was among the authors of one of the first application to economics, the *Santa Fe market*, in Arthur and alii (1997). GA has been theoretically justified, with biological reasons, as the force driving human evolution, examples are the works of Robson (1996, 2001 and 2002). Here we use instead an interpretation of GA as learning, following Arifovic and Eaton (1995) and Riechmann (1998 and 1999), and all the literature on learning in economics, see *e.g.* Cucker and Smale (2001) or, applied to risk–attitudes, Samuelson and Swinkels (2001).
[4] See Weibull (1995) as a survey.
[5] In the simulations we fix $N = L$, but this is not a binding choice and the algorithm can easily be adapted to $N > L$.

The reason for such a model is to find minimal sufficient conditions for stochastic dominance to appear.[6] What comes out is that in all the simulations we get (almost, because of disturbance from the mutation operator) stochastic dominance with the only imposition of a threshold due to bankruptcy. A secondary result is that the three different kinds of fitness measures give asymptotically the same result. The two that require some knowledge of past stages (stability and longevity) are not qualitatively different from the one considering only the amount of *money* in a single time step (instantaneous wealth). Section 2 defines the grid on the Machina triangle and illustrates the genetic algorithm, Section 3 presents the results and Section 4 concludes.

2 The genetic algorithm

Figure 1 shows clearly how there is a one–to–one correspondence from all the points on the Machina triangle and all the distributions $\mathcal{L} = \{p,\ 1-p-r,\ r\}$ on three events partitioning the probability space.

$$\mathcal{L} = \{p,\ 1-p-r,\ r\}$$

$$\begin{cases} win \ \ w.p.\ p \\ draw\ w.p.\ 1-p-r \\ lose \ \ w.p.\ r \end{cases} \qquad \Longleftrightarrow$$

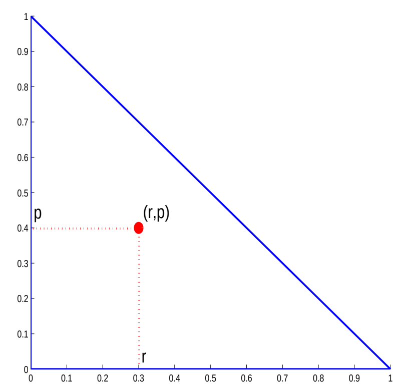

Fig. 1. One–to–one correspondence between all possible lotteries $\mathcal{L}$ on 3 events and the Machina triangle.

The *triangle* is a useful graphical device. Isoutilities curves of an agent between the lotteries (as the ones in Figure 2) give a lot of information about her local and global attitude towards risk. In expected utility theory the curves are all straight parallel lines. Local steepness corresponds to local risk love, neutrality or aversion. Moreover local concavity or convexity correspond to (local) love or aversion to *randomization*.[7]

[6] The research for minimal sufficient conditions is in the same spirit as Gode and Sunder (1993), even if the target here is quite different.

[7] *Randomization*–averse agents are very common in experiments, the concept is different from risk aversion since probability distributions may not change at all, a good survey is still Starmer (2000).

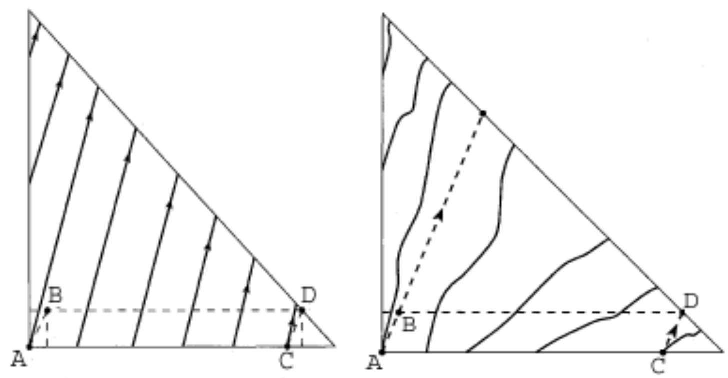

Fig. 2. Two possible *utility functions* (represented with *indifference curves*) on a *Machina triangle*. The first one obeys the *independence axiom*, accordingly to Expected Utility Theory, the second one does not but is perfectly feasible in experimental economics. Allais' paradox' points are present. Figures are taken from Starmer (2000).

We will imagine that our agents deal with an imaginary currency, winning and loosing money, so that at time t agent i will have M_t^i units of wealth. At any time step, admissible events for an agent are the possibility of winning a fixed sum S ($M_{t+1}^i = M_t^i + S$); the chance of loosing it ($M_{t+1}^i = M_t^i - S$); and finally the event that makes no changes at all so that $M_{t+1}^i = M_t^i$.
We will consider only a finite square $d \times d$ grid on the triangle, made of $L = d^2$ possible lotteries on the three events, as shown in Figure 3.[8]

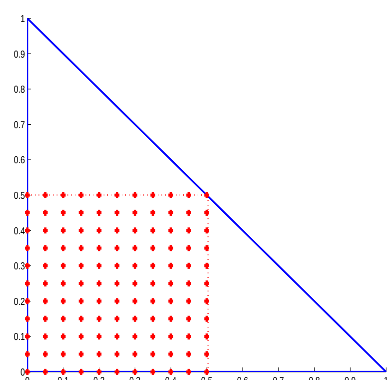

Fig. 3. The finite $d \times d$ grid of $N = d^2$ lotteries we are considering in the GA.

[8] We are excluding half of the triangle (the upper and the right corners) in order to concentrate on the most interesting region, reducing by a factor of 4 the computations.

The genetic code of every agent in the GA will be a strict order of preferences between the N lotteries, as can be graphically seen in Figure 4, where also the corresponding isoutilities are drawn. The set of all possible genetic codes is then finite and equal to $L!$ (factorial of L).

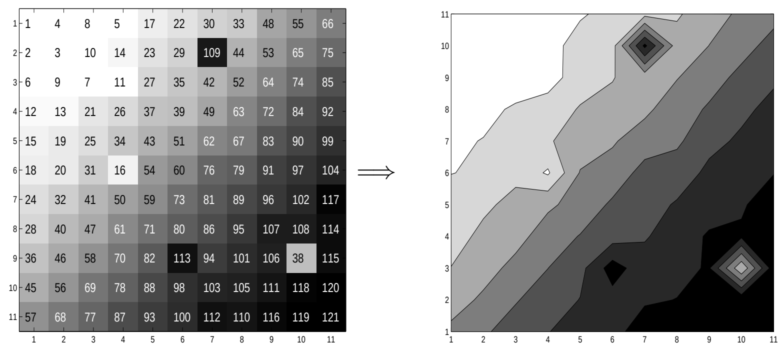

Fig. 4. Genetic code of every agent is a strict order of preferences on the points of the grid on the Machina triangle (tiny numbers indicate the rank, lighter grey is for better ranking), resulting *isoutilities* are shown in the right part.

Since the genetic code of the GA is a permutation, a set in which every element appears only once, custom genetic operators must be slightly adapted, they are however still n–ary functions on a finite set.
Genetic operators consider implicitly a metric on the genetic codes' space $\mathbb{G}$. In our case $\mathbb{G}$ is the of all the possible permutations of the finite set $\{1, 2, \ldots, L\}$. If we call *simple inversion* a shift in the ranking of two adjacent elements (so that, in the set $\{a, b, c, d, e\}$, (a, c, b, d, e) is a simple inversion of (a, b, c, d, e)), the distance between two permutations is the minimum number of simple inversions necessary to obtain one from the other.

Mutation is a unary operator μ that slightly perturbs the value of its argument. In our case mutation may be applied to any agent at the beginning of every stage. It consists of a single shift in the rankings of two random elements. Let us consider as an example permutations on the set $\{a, b, c, d, e\}$: we could mutate the code $\mathcal{G}_t \equiv (d, c, e, b, a)$ by randomly choosing two different numbers from 1 to 5 (say 2 and 4) and inverting the elements in these two positions (*i.e.* c and b), obtaining $\mu_t^{(2,4)}(\mathcal{G}_t) = \mathcal{G}_{t+1} = (d, b, e, c, a)$. The probability of a mutation to happen is set at a low value π.

Cross–over is a binary operation[9] χ that computes a somehow defined *average* of the original elements. In our case the result of cross–over from two genetic codes (permutations) is made averaging the original rankings and ordering the result; in case of equal ranking the choice is made randomly. Let's consider an example, again on $\{a, b, c, d, e\}$, where the two *parents* are $\mathcal{G}_1 \equiv (a, c, d, b, e)$ and

[9] Multiple cross–over has been tested in the simulations, with no change in the results.

$\mathcal{G}_2 \equiv (d, a, e, c, b)$. a is on average ranked in position 1.5, d second, c third, e fourth and finally b in position 4.5. The result is $\chi(\mathcal{G}_1, \mathcal{G}_2) = (a, d, c, e, b)$.
Figure 5 gives a visual idea of what happens when permutations are applied on preferences over our Machina triangle grid.

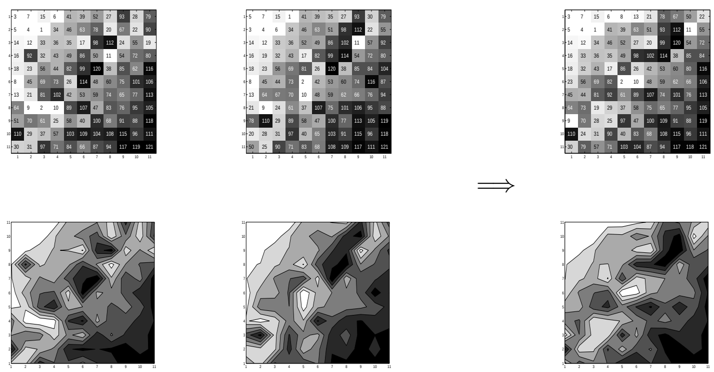

Fig. 5. The two left–side preferences (permutations) on the square grid over the Machina triangle (isoutilities are also shown), give as result of cross–over the right–side one. Isoutilities suggest how the *offspring* is an *average* of her *parents*.

At each time step the lotteries are randomly distributed, then played, and finally bankrupted agents are replaced, through *selection* (discussed later), cross–over and mutation, by new–entrants. The assignment is random but is however *Pareto optimal*, *i.e.* an assignment where it is impossible that two agents can change the lottery they were given and be both better off. A random permutation of the set $\{1, 2, \ldots, N\}$ is made so to have a list of the agents: The first individual will choose her favourite lottery, *i.e.* the first one of her genetic code (which is her preference and also just a permutation of L elements), the second one will do the same, if her first choice is not already taken, in which case she will pick her second choice. Going on like that, every agent will choose, at her turn, her favourite point on the grid among all the available left. The last agent will have no choice and pick the only lottery remaining.[10]
This assignment is Pareto optimal, moreover a hypothetical *rational* agent would optimize her expected utility giving a truthful list of her preference among all the possibilities. It may seem that this mechanism of distribution is too complicated: if every agent could pick her favourite choice the genetic algorithm would apparently converge much faster to an equilibrium. This is however not true, because the algorithm would make a selection only on the favourite lottery, and

[10] See Note 5 at page 66 for the choice to set $N = L$, it would be easy to change the Pareto assignment for any N that is multiple of L, just replicating the lotteries in the initial bundle.

not on the complete list characterizing the genetic code.[11]
After the assignment all the lotteries are played independently and the relative changes are made to the endowments of the agents.

As anticipated the algorithm is an overlapping generations one, it means that (most of the) agents of a previous stage are still present, with exactly the same genetic code, in the next one. Also the amount of money is transferred, except from the variations due to the outcomes of the lotteries played.
In literature many ways are proposed to choose which agents to substitute with new–borns, even random selection. Here it simply happens that whenever an agent goes *bankrupt*, *i.e.* has an amount of money less than S and unluckily looses, that agent is definitely out of the game and is replaced by a new agent. The method chosen here seems very natural and meaningful when dealing with economics, it will show moreover to be a sufficient condition for stochastic dominance to emerge.

New–entrants select their *parents* through the classical *selection* operator, based on probabilities proportional to the *fitness measure*. When adopting the social interpretation of genetic algorithms as learning and imitation processes, the *fitness* has to be the measure of something strongly related to utility. The most immediate idea is the one of *money* M_t^i, or a monotone function of it, as a candidate. There are however other possible desirable features that come out from our simulations, such as *longevity* (the number of iterations an agents has been present in the game) or *financial stability* (the difference between the highest and lowest values of M^i in the last τ iterations, if longevity is greater than τ).
Individuals with a lot of money, in the first case, with a lot of permanence in the market or with a stable resource level, in the last one, should be good models for new–entrants in the market, the beginners would try to imitate them and learn from their behaviour.
In the simulation four cases have been tested: money, its natural logarithm, longevity and stability. According to the fitness measure chosen, probabilities are normalized, more *good looking* agents will have greater possibilities to be chosen as models.

3 Simulations

During the simulations we decide to measure how far the preferences of our agents are from a hypothetical *risk dominated* one. The risk dominated (by both *first* and *second* order dominance) is one in which all the lotteries are ranked first of all by expected utility and then inversely by variance. We will in particular measure both how far we are from the dominated preference (calling it the *first measure* m_1), and how far from a preference in which only expected utility is considered (*second measure* m_2).

[11] Extreme example: if the strategy of an agent is identified just by her favourite lottery, and the favourite lottery is the same for all the agents, there would be no selection on all the other lotteries.

Permutation ϕ: ***Ideal* state** π:

Fig. 6. ϕ is a given permutation and π is the ideal stochastic dominated one.

Figure 6 is an example where the grid is just a 3×3 one. We are measuring ϕ compared to the ideal case π. The first measure is given by taking the elements in the same positions and summing absolute differences:
$m_1(\phi) = \sum_{i=1}^{d^2} |\phi_i - \pi_i| = 18.$
The second one compares every element on the grid in ϕ with all the elements in the same bottom–right to top–left diagonal of π (the ones with the same expected outcome), and takes the minima of the absolute differences:

$$m_2(\phi) = \sum_{i=1}^{d^2} \underbrace{\min_j\{|\phi_i - \pi_j|\}}_{\mathbb{E}(\mathcal{L}_i)=\mathbb{E}(\mathcal{L}_j)} = 14 \ .$$

Table 1. Chosen variables.

Parameter	Page	Short description	Chosen value
d	68	side of the grid	11
$L\ (= d^2)$	66	lotteries	121
N	66	agents	121
M_0	66	initial endowment	10
S	66	amount played in a single lottery	1
π	69	probability of a mutation	0.01 and 0.005
τ	71	memory for financial stability	1.000 steps
		number of steps	20.000, 15.000 or 2000
		average on last...	4.000, 3.000 or 400 steps

Table 1 illustrates the variables chosen for our simulations, Figure 7 shows the results in one realization, while first column in Table 2 lists the average results over 200 iterations of that same setup.[12]
The fitness measure here is simply instantaneous wealth. The plot shows the average and the variance–spread for the first and second measures, for all the

[12] As will be the case for the other sets of simulations, we show graphical results for only one realization. We tried however 200 iterations for each specification of the parameters, with similar results. The basic statistics are shown in Table 2.

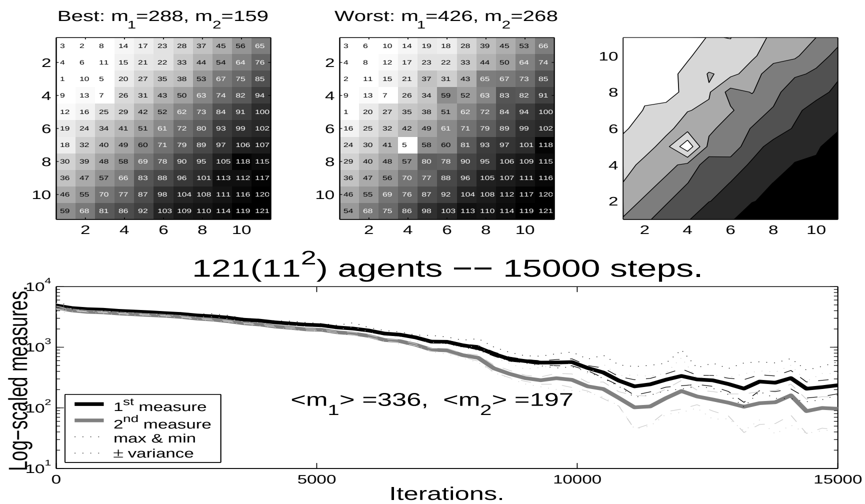

Fig. 7. Realization of a simulation with 121 (11^2) agents, fitness measure is M_t^i. Two preferences (for the most and the less successful agents) are given, for the latter also iso–utilities are shown. The plot has log–measures in the y–axes, the average value for m_1 and m_2 are taken on the last 3000 steps.

agents every 150 steps. The pressure towards zero for these two measures is very hard (the y–axes in the plot is log–scaled), but at the end the values are stable at a level significantly greater than 0. A look at the isoutilities in the figure shows that the preferences on the grid are almost risk dominated, but do not exactly match the ideal order (whose measures would be both 0). Do we have stability strictly above null because the resulting equilibrium is different from the risk dominated preference, or because of the noise given by mutations?
A partial response to the last question is given by the result of two distinct sets of simulations. First of all, comparing the first two columns of Table 2, we see that reducing the probability π of a mutation the final averages of the measures, and their standard deviation, over 200 iterations each, decrease toward risk–dominance. A second reinforcement for this hypothesis comes from the results in Figure 8, and in the third column of Table 2, where all the agents start from the ideal stochastic dominated utility. The resulting final averages are very similar, and statistically compatible, to the ones obtained before. It should be noted that, because of the reciprocal interaction in the Pareto distribution, when all the other agents are risk dominated, a rational agent should at least maintain the expected outcome ordering. We argue that a small departure from complete stochastic dominance is due only to the noise from mutations.

We used the most intuitive fitness measure of instantaneous wealth. To test the robustness of this result we examined other fitness measures.
The logarithm of M is a classical choice used in GAs to smooth the pressure of the selection mechanism, here it is used as a consistency check for the simple M.

Longevity and *stability* are instead fitness measures that require more elaborate agents, that is agents endowed with some knowledge about wealth distribution and identity of the other agents in the past stages of the GA.
Figure 9 and Table 2 show the outcome from Figure 8 compared with simulations that are analogous except for the fitness measure used. The results for the four kinds of fitness measures are almost the same, even if the speed of convergence and the standard deviation vary. The good news is that the most simple one is enough for our purpose. In particular, in order to reach stochastic dominance, a bankruptcy benchmark is sufficient and there is no need for additional assumptions, as could be the case of agents with some memory, in the sense defined above, of the process.

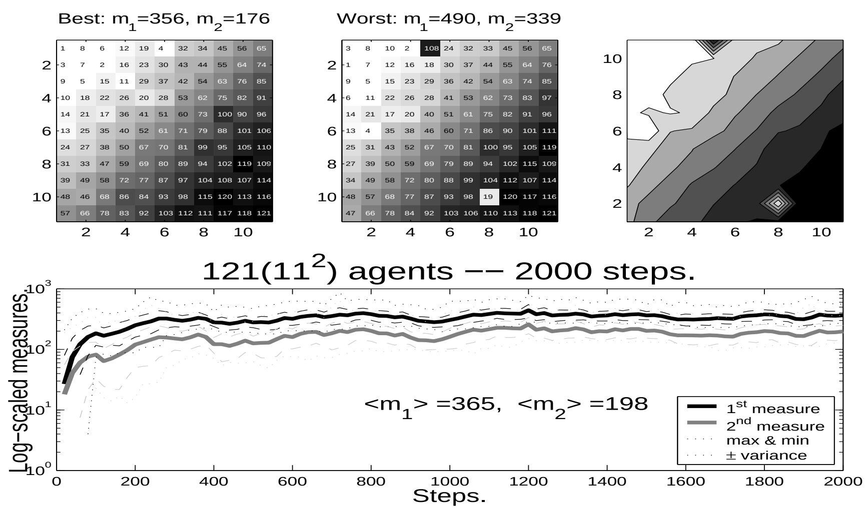

Fig. 8. Realization of a simulation with 121 (11^2) agents (fitness measure is M_t^i), all starting from the ideal risk dominated preferences. Two preferences (for the most and the less successful agents) are given, for the previous also isoutilities are shown. The plot has log–measures in the y–axes, the average value for m_1 and m_2 are taken on the last 400 steps.

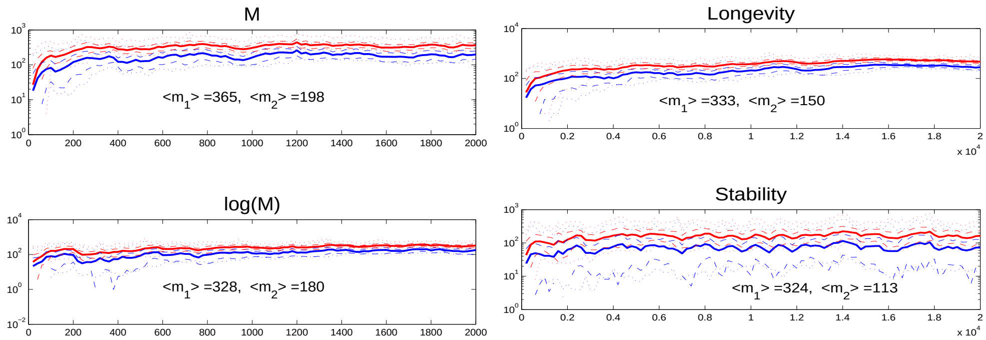

Fig. 9. Realizations of four simulations: (i) the top–left is the one in Figure 8. The other three are with different kinds of fitness measures: ii) the logarithm of instantaneous wealth; iii) longevity in the market; iv) stability in the last τ steps. Iterations for (iii) and (iv) are 10 times as many than in the first two.

Table 2. Results of the simulations: 200 iterations have been run for all the cases considered in the Figures 7 (here also a lower probability of mutation is considered), 8 and 9, maintaining the same parameters (standard deviation in parenthesis).

Starting population:	random		risk dominated			
Fitness measure:	M		M	$\log(M)$	longevity	stability
Figure:	7	–	8	9 (SW)	9 (NE)	9 (SE)
π of mutation:	0.01	0.005	0.01			
$< m_1 >$:	345 (16)	328 (12)	354 (17)	324 (11)	330 (14)	315 (25)
$< m_2 >$:	196 (12)	186 (9)	199 (12)	179 (8)	147 (11)	108 (15)

4 Conclusion

We ran a GA to evolve preferences in the Machina triangle, with the three possible outcomes of a simple lottery. We tried to model the GA in the most intuitive way, using only strict preferences and Pareto optimal lotteries' assignment, in an overlapping generation setup where only bankrupted agents are replaced. We found that the only element forcing the agents towards risk dominance is the unique hard budget constraint: the bankruptcy benchmark itself. There are more than one possible fitness measure with economical meaning in this framework, some of them require the agents to have partial information about previous stages of the GA. Nevertheless the most simple one of instantaneous wealth is enough to induce stochastic dominance, which is not reached completely only because of the noise given by the mutation operator.

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