On the efficient application of the repeated Richardson extrapolation technique to option pricing

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Abstract. Richardson extrapolation (RE) is a commonly used technique in financial applications for accelerating the convergence of numerical methods. Particularly in option pricing, it is possible to refine the results of several approaches by applying RE, in order to avoid the difficulties of employing slowly converging schemes. But the effectiveness of such a technique is fully achieved when its repeated version (RRE) is applied. Nevertheless, repeated RE received little attention in the financial literature; this is probably due to the necessity of paying special attention to the numerical aspects of its implementation, such as the choice of both the sequence of the stepsizes and the order of the method. In this contribution, we consider different numerical schemes for the valuation of American options and investigate the possibility of an appropriate application of RRE. As a result, we find that, in the analyzed approaches in which the convergence is monotonic, RRE can be used as an effective tool for improving the accuracy of the approximations.

Keywords. Richardson extrapolation, repeated Richardson extrapolation, American options, randomization technique, flexible binomial method.

M.S.C. classification. 65B05.
J.E.L. classification. C15, C63, G13.

1 Introduction

In finance one has frequently to deal with approximate results that are obtained by iterative methods or computational procedures depending on some parameter (e.g. the time-step). As well known, often the convergence of numerical schemes is slow and this may be a serious problem in many practical situations; consequently convergence acceleration techniques, such as Richardson extrapolation, have been studied and applied in the literature.

In this contribution, we focus on *repeated* Richardson extrapolation (hereafter RRE), which has not been fully exploited in financial applications, probably due to the fact that one has to pay special attention to the numerical aspects of its implementation, such as the choice of both the sequence of the stepsizes and the order of the method.

Although the RRE technique is generally applicable, in order to illustrate how it works, we will focus on valuing a standard American put option. We consider several numerical schemes for the valuation problem and investigate the possibility of an appropriate application of RRE. As a result, when the convergence of the method is monotonic, RRE can be used as an effective tool for improving significantly the accuracy.

In particular, we apply RRE to the randomization approach proposed by Carr [6], the binomial approach of Cox *et al.* [8], the Black-Scholes-binomial method (BBS) of Broadie and Detemple [3], and the flexible binomial method proposed by Tian [21].The accuracy of the randomization method is improved when RRE is applied by choosing a particular sequence of stepsizes. As well known, it is not convenient to use Richardson extrapolation (RE) in the binomial model: this is due to the non-uniform convergence of the method[1]. Numerical results highlight that RE and RRE cannot be recommended for the classical binomial approach. RE has been successfully applied by [3] in their hybrid binomial-Black-Scholes model, but according to our numerical experiments RRE cannot be efficiently applied to the BBS method. When implemented within the flexible binomial setting introduced by [21], RRE based on Romberg sequence of stepsizes gives very accurate and fairly robust results, because of the smoother nature of the convergence of the method.

An outline of the paper is the following. In section 2 we briefly review the financial literature on the Richardson extrapolation technique applied to option pricing problems. Subsection 2.1 explains the RE and RRE techniques and introduces the choice of different stepsize sequences. A wide experimental analysis is carried out in order to test the method; the main results are presented and discussed in section 3. We will provide some insights on how RRE can effectively be used in practice. Section 4 presents some concluding remarks.

2 Richardson extrapolation and its applications in finance

Richardson extrapolation has been applied to accelerate valuation schemes for American options and exotic options. Geske and Johnson [12] first applied Richardson extrapolation in a financial context to speed up and simplify their compound option valuation model. They obtain a more accurate computational formula for

[1] We refer to [13] for a discussion on the rate of convergence of lattice methods. Here non-uniform convergence is understood in the sense that the solution in the binomial setting has not the same rate of convergence at all nodes of the tree.

the price of an American put option using the values of Bermuda options. Geske and Johnson's approach was subsequently developed and improved by [4], and [14]. More recently, Chang *et al.* [7] proposed a modified Geske-Johnson formula based on the repeated Richardson extrapolation.

Richardson extrapolation techniques were also employed to enhance efficiency of lattice methods (see [2]). It is common opinion that it is not convenient to extrapolate on the number of time steps in the binomial model due to the oscillatory nature of the convergence (see [19]). Broadie and Detemple [3] successfully use Richardson extrapolation to accelerate a hybrid of the binomial and the Black-Scholes models. Tian [21], Heston and Zhou [13] and Gaudenzi and Pressacco [11], to mention a few, also apply Richardson extrapolation to binomial and multinomial approaches.

Carr [6] proposes a randomization approach for the valuation of the American put option and uses a repeated instance of Richardson extrapolation to obtain accurate estimates of both the price and the exercise boundary of an American put option. Leisen [18] shows that randomizing the length of the time steps in the binomial model allows the successful use of extrapolation. Huang *et al.* [15] and Ju [16] use extrapolation methods to accelerate the integral representation of the early exercise premium.

2.1 Richardson extrapolation techniques

The basic idea of extrapolation can be summarized as follows (see [9]). Consider the problem of calculating a quantity of interest for which an analytical formula is not provided. In the following, we restrict our attention to the problem of valuing e.g. an American put option. In place of the unknown solution P_0, take a discrete approximation $P(h)$ depending on the *stepsize*[2] $h > 0$, $P(h)$ being a calculable function yielded by some numerical scheme, such that

$$\lim_{h \to 0} P(h) = P(0) = P_0 \,. \tag{1}$$

All extrapolation schemes are based on the existence of an asymptotic expansion. Under the assumption that $P(h)$ is a sufficiently smooth function, we write

$$P(h) = \alpha_0 + \alpha_1 h^{p_1} + \alpha_2 h^{p_2} + \cdots + \alpha_k h^{p_k} + O(h^{p_{k+1}})\,, \tag{2}$$

with $0 < p_1 < p_2 < \ldots$, and unknown parameters $\alpha_0, \alpha_1, \ldots$, where $h \in [0, H]$ for some *basic step* $H > 0$. In particular, we have $\alpha_0 = P_0$.

Compute the function $P(h)$ a number of times with successively smaller stepsizes,

$$h_1 > h_2 > \ldots > 0 \,.$$

In such a way, we obtain a sequence of approximations

$$P(h_1), P(h_2), \ldots$$

[2] h may be the period of time between two exercise dates of the American option. Hence $P(0)$ is the limit of the value of a Bermudan option as h goes to zero.

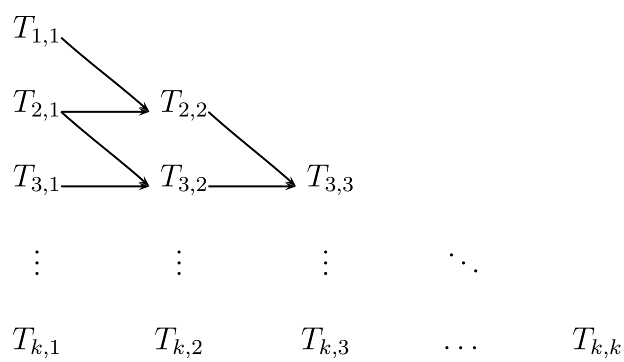
for a given sequence of stepsizes h_i.

We can construct extrapolation schemes of arbitrary order k by considering the following procedure[3]:

1. define $T_{i,1} = P(h_i)$, for $i = 1, 2, \ldots;$
2. for $i \geq 2$ and $j = 2, 3, \ldots, i$, compute

$$T_{i,j} = T_{i,j-1} + \frac{T_{i,j-1} - T_{i-1,j-1}}{\frac{h_{i-j+1}}{h_i} - 1}\,. \tag{3}$$

Recursion (3) is based on polynomial interpolation and an asymptotic h-expansion (taking $p_1, p_2, \ldots$ integers). We can establish the following extrapolation tableau[4] (stopped at k-th order):



The sequence $\{P(h_i)\}$ is taken as the first column in the extrapolation tableau. Each quantity $T_{i,j}$ is computed in terms of two successive approximations. The two point Richardson extrapolation technique can be *repeated*, giving rise to a numerical scheme which is extremely fast and can dramatically improve accuracy[5]. The efficiency of the method relies on the fact that the amount of computation required essentially corresponds to the number of function evaluations.

The idea behind recursion (3) is to provide two mechanisms for enhancing the accuracy: by increasing i one obtains a reduction in the stepsize parameter, while taking j large implies more accurate approximations. Both mechanisms work simultaneously, which indicates that the quantities $T_{k,k}$ are those of most interest. This provides us with the possibility of order control.

[3] Such a procedure is also known as Aitken-Neville algorithm and it is one of the extrapolation schemes which are commonly used.

[4] See [10].

[5] Each repetition requires one additional point. Each entry of the tableau is an approximation to P_0; obviously, more precision is achieved on the diagonal.

The accuracy and efficiency of the method is strictly connected with the choice of the sequence of stepsizes. Define h_i in terms of the basic stepsize H, such that $h_i = H/n_i$ $(i = 1, 2, \ldots)$; any stepsize sequence is characterized by the associated sequence of integers. In our numerical experiments, we considered the following sequences of the stepsize:

- harmonic sequence: $\{1, 2, 3, 4, 5, 6, 7, 8, \ldots, n, \ldots\}$;
- double harmonic (Deuflhard) sequence: $\{2, 4, 6, 8, 10, 12, 14, 16, \ldots, 2n, \ldots\}$;
- Burlisch sequence[6]: $\{2, 4, 6, 8, 12, 16, 24, 32, \ldots, 2n_{k-2}, \ldots\}$ (for $k \geq 4$);
- Romberg sequence: $\{2, 4, 8, 16, 32, 64, 128, \ldots, 2n_{k-1}, \ldots\}$.

All these sequences allow for convergence of the method; this is not always the case (see [5]). The first and the fourth sequence are of common use in the financial literature related to extrapolation combined with option pricing models[7]; though the other sequences are well known in numerical analysis, it seems that their use has not been thoroughly investigated in the financial literature.

3 Efficient implementation of RRE to option pricing: analysis of some numerical experiments

Richardson extrapolation is a well understood technique, which is often applied in finance to enhance precision of results provided by discrete models. Nevertheless, *repeated* Richardson extrapolation has received little attention in the financial literature. In this section, we investigate how to implement efficiently the RRE to option pricing. In order to explain how such a technique applies, we focus on the problem of valuating an American put option.

In particular, we apply the method to valuation models based on different approaches: the randomization technique, the classical lattice method and some extensions of such an approach. The following subsections report the main results of the numerical experiments carried out and provide some insights about the convenient choice of the stepsize sequence when applying RRE within the different valuation frameworks taken into consideration.

3.1 Carr's randomization approach

The model proposed by [6] for the valuation of American put options is based on a particular technique, called *randomization*. This technique, also known as *Canadization*, has been recently applied and generalized by [17] and [1].

According to Carr's definition, randomization is a three-step procedure for solving a valuation problem, which can be summarized as follows: let the value of one of the model parameters be "randomized" by assuming a plausible probability distribution for it; calculate the expected value of the dependent variable

[6] See [5]. Note that some authors report the following sequence: $\{2, 3, 4, 6, 8, 12, \ldots\}$.

[7] Usually two or three point Richardson extrapolation is applied.

(which is unknown in the fixed parameter model) in this random parameter setting; let the variance of the distribution governing the parameter approach zero, holding the mean of the distribution constant at the fixed parameter value. For example, if we consider standard options, we could randomize the initial stock price, the strike price, the initial time, or the maturity date. Carr randomizes the maturity date of an American put option and determines the solution for its value and the optimal exercise boundary.

Let the maturity of the randomized American put be determined by the waiting time to a certain number of jumps of a standard Poisson process, which is assumed to be independent of the underlying stock price process and uncorrelated with any market factor. The value of the random maturity American option approximates the value of its fixed maturity version.

When the randomized American option is supposed to mature at the first arrival of a Poisson process with intensity $1/T$, the maturity $\tilde{T}$ is exponentially distributed with expectation T. Due to the memoryless property of the exponential distribution, it turns out that the early exercise boundary is independent of time and the option value suffers no time decay. As a result, the search for a time-dependent boundary is reduced to the search for a single critical stock price. The fair value of a randomized American put with an exponential distributed maturity is the then solution of the following problem

$$P_0 = \sup_H \mathbb{E}_S[e^{-rt_H}(X - S_{t_H})^+] , \tag{4}$$

for an initial underlying price $S > H^*$, where H^* is the unknown optimal exercise boundary, and t_H is the first passage time of the underlying price process $(S_t)_{t\geq 0}$ through H, r is the risk-free continuous interest rate and X is the option strike price. The expectation in equation (4) can be evaluated in closed form, and the result can be maximized over constant barriers (for more details see [6]).

The assumption of an exponentially distributed maturity leads to simple approximations, which entail too much errors to be used for practical purposes. To obtain more accurate approximations, assume that the time to maturity is subdivided into n independent exponential subperiods. Therefore the randomized American option matures at the n-th jump of a standard Poisson process (with intensity n/T). As a result, the maturity $\tilde{T}$ is gamma distributed, with expectation T and variance T^2/n.

In Carr's n-step setting the randomized American put value and the initial critical stock price are determined by a dynamic programming algorithm. The resulting expression for the randomized option value is a triple sum, which does not require the evaluation of special functions. As the number of subperiods becomes large, the variance of the random maturity approaches zero. So increasing the number of periods improves the accuracy of the solution (of course at the expense of a greater computational cost).

Richardson extrapolation can be used to improve the method. Let $P_0^{(n)}$ denote the randomized option price at time $t = 0$ determined assuming n random subperiods. The N-point Richardson extrapolation is the following weighted average of N approximate values

$$\hat{P}_0^N = \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} P_0^{(n)}. \tag{5}$$

By using Richardson extrapolation, accurate option values can be obtained just with a few random time steps. The method proved robust and quite accurate, moreover the convergence of the results is monotonic, allowing us to consider extrapolations of higher order.

Carr applies extrapolation as defined by (5), which is based on the harmonic sequence[8]. We compared numerical results obtained with different sequences of the steps, and assessed the method on a large set of option valuation problems, considering different values of moneyness, maturity, volatility and risk-free interest rate.

In order to test the goodness of the employment of the RRE within Carr's randomization framework, we make a comparison between the price obtained for an American put option in the 25 000-steps binomial model and the extrapolated prices in the random maturity model. The simulation analysis takes into consideration 3 500 randomly generated option valuation problems. The parameter ranges are: $r \in [0.01, 0.12]$, $\sigma \in [0.1, 0.5]$, $X/S_0 \in [0.7, 1.3]$ (r, σ and X/S_0 are sampled from a uniform distribution on a given interval), with $S_0 = 100$ and $T = 1$. The moneyness interval has been partitioned into seven subsets: $X \in [70, 80]$, $[80, 90]$, $[90, 100]$, $\{100\}$, $[100, 110]$, $[110, 120]$, $[120, 130]$, and we have randomly generated 500 instances from each subset.

We have considered different sequences of the stepsize h_i when applying repeated Richardson extrapolation. The results of the simulation experiments carried out for the option price can be summarized by computing the mean absolute error (MAE) and the root mean square error ($RMSE$)[9] of the simulation results with respect to the binomial price as moneyness varies.

[8] When we consider the harmonic sequence and recurrence (3), we can directly compute the quantities $T_{k,k}$ in the extrapolation tableau using the formula

$$T_{k,k} = \sum_{i=1}^{k} \frac{(-1)^{k-i}\, i^k}{(k-i)!\, i\,!}\, P(h_i)\,,$$

which corresponds to a k-point Richardson extrapolation as in formula (5).

[9] The MAE and the $RMSE$ are computed as follows:

$$MAE = \frac{1}{N} \sum_{n=1}^{N} |\, e_n \,|\,, \qquad RMSE = \sqrt{\frac{1}{N} \sum_{n=1}^{N} e_n^2}\,,$$

where $e_n = \frac{\hat{P}_n - P_n}{P_n}$ are the relative errors, being P_n and $\hat{P}_n$ the "true" and the estimated option values, respectively. We do not consider option prices lower than 0.05 when calculating errors.

The outcomes of the simulation experiments are synthesized in tables 1–8. Both the MAE and the $RMSE$ have been computed[10], but only the $RMSE$s are reported here in detail (as they give the same information of MAEs). In tables 1–3 and 5–8 we show the pricing errors on the diagonal of the extrapolation tableau, which have been computed considering the harmonic, Deuflhard, Burlisch and Romberg sequences, respectively.

Of course, one has not to compare the errors line by line in the tables. Such errors are obtained by applying RRE based on different sequences of the number of random steps. Each sequence is characterized in the implementation by a specific computational amount (see [5]).

The harmonic sequence has been stopped at $n = 10$ (in practice we can apply RRE based on the harmonic sequence up to step 15, and in some cases also up to step 20). Deuflhard sequence has been stopped at $n = 20$. We stopped Romberg sequence at $n = 64$ (in some cases RRE has been stopped at $n = 128$, in some other cases only at $n = 48$), because for higher values of the number of steps the method does not achieve higher accuracy, or round-off errors arise (or, simply, the method is too slow)[11]. Hence, one has to make comparisons in terms of accuracy for a given value of computational effort required.

For example, in tables 1–3 and 5–8 we can compare RRE based on Deuflhard sequence with $n = 16$ and $n = 18$ with RRE based on Burlisch sequence with $n = 24$ and $n = 32$. The resulting errors are of the same order, and it appears that one method does not overperform the other one, but both are preferable to the extrapolation based on the harmonic sequence. Moreover, with Burlisch sequence we can achieve higher accuracy if RE is repeated once or twice more. Note also that $RMSE$s of order 10^{-4} or lower are not obtainable with the other sequences (when considering the case $X \in [70, 80]$).

In order to compare the errors relative to Burlisch sequence with those yielded by applying Romberg sequence, we have to consider the case $n = 48$ for the first one and $n = 64$ for the latter. Burlisch sequence seems preferable in terms of accuracy and speed with respect to the other sequences, and this finding is supported by the results obtained for every level of moneyness (not only, but the method performs better the higher the moneyness). For instance, in the case $X \in [120, 130]$ (see table 8), the $RMSE$ is of order $1.6 \cdot 10^{-5}$ for the Burlisch sequence with $n = 48$, while it is $2.5 \cdot 10^{-5}$ for the Romberg sequence with $n = 64$.

It is also interesting to analyze the error reduction on the diagonal of the extrapolation tableau[12]. Table 4 shows the percent variations of the MAE for all the four sequences. As we expected, the advantage in repeating RE is more evident when we apply Romberg and Burlisch sequences.

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[10] In option pricing, the importance of deriving indications by both error measures is documented e.g. in [20].

[11] In table 8 all the sequences have been stopped at earlier iterations since there was no longer error reduction.

[12] Based on the error reduction along the diagonal, a stopping rule for the RRE may be defined, hence allowing for order control.

3.2 Richardson extrapolation applied in a binomial framework

As an interesting exercise, we investigate the possibility of applying RRE both to the CRR binomial and the BBS approaches. It is well known and well documented in the literature (see e.g. [21], and [11]) that the oscillatory nature of the convergence in the CRR model makes infeasible RE, *a fortiori* RRE should not be used. We will see that the technique is useful only in the at-the-money case.

We have carried out a wide simulation analysis, which takes into consideration 3 500 randomly generated option valuation problems, comparing the price obtained for an American put option in the 25 000-steps binomial model and the extrapolated prices. The parameter ranges are: $r \in [0.01, 0.12]$, $\delta \in [0.0, 0.12]$ (where δ is the continuous dividend yield), volatility $\sigma \in [0.1, 0.5]$, $X/S_0 \in [0.7, 1.3]$, with $S_0 = 100$ and $T = 1$. The moneyness interval has been partitioned into 7 subsets: $X \in [70, 80]$, $[80, 90]$, $[90, 100]$, $\{100\}$, $[100, 110]$, $[110, 120]$, $[120, 130]$, and we have randomly generated 500 instances from each subset. We have considered different sequences of the stepsize and the basic step: in particular, $H = T/100$ and $H = T/200$ are used.

As already observed, RE and RRE work only for at-the-money options, hence we discuss this special issue. Nevertheless, also in this case, the choice of the stepsize sequence is crucial. In table 9, we show the pricing errors relative to at-the-money American put options: each entry in the second, fourth, sixth and eight column of table 9 are the error measures of the price estimates obtained by applying repeatedly RE. Only the errors along the diagonal of the extrapolation tableau are reported. The results refer to a basic step $T/200$; we applied the harmonic, Deuflhard, Burlisch and Romberg sequence. Observe that only RRE based on Romberg sequence yields very accurate and robust results (the method continues to gain precision along the diagonal), while with all other sequences just two-point RE can be applied.

It is interesting to investigate what happens when we consider out-of-the-money American put options. It turns out that RE and RRE no longer work, even when considering Romberg stepsize sequence. Table 10 shows the MAEs and $RMSE$s in the extrapolation tableau for the case $H = T/200$. Note that the pricing errors of the extrapolation are higher than those of the non-extrapolated values (compare the first column of the tableau with the errors reported on the diagonal). It is clear from this discussion that RE should not be applied within the CRR model, except in just one case which is of limited interest.

We briefly discuss also the feasibility of RRE within the hybrid binomial-Black-Scholes model proposed by [3]. The convergence of the BBS method is smoother compared to the binomial method, so that one may wonder if RRE could be used. Two-point RE has been applied successfully to the BBS method, but still RRE does not perform well. In our numerical experiments (based on the same set of 3 500 randomly generated options pricing problems considered in the previous experiments), we find that the extrapolated values along the diagonal of the tableau entail higher errors than the approximate values below the diagonal and even with respect to the non-extrapolated values. Hence, RRE should not be applied within the BBS model. Our results (which are not reported here for the sake of brevity) are in accordance with the findings in [7].

3.3 Tian's flexible binomial model

Tian [21] introduces in the CRR binomial model a so called "*tilt factor*" λ, with the effect of modifying the shape and span of the binomial lattice. In this *flexible* binomial (FB) model, the up- and down- factors are defined as follows:

$$u = e^{\sigma\sqrt{\Delta t}+\lambda\sigma^2\Delta t} \qquad d = e^{-\sigma\sqrt{\Delta t}+\lambda\sigma^2\Delta t}, \tag{6}$$

where λ is an arbitrary constant that can be positive, zero, or negative. The parameter $\sigma > 0$ is the volatility, $\Delta t = T/n$ (with n number of steps) is the timestep, and T is the option maturity.

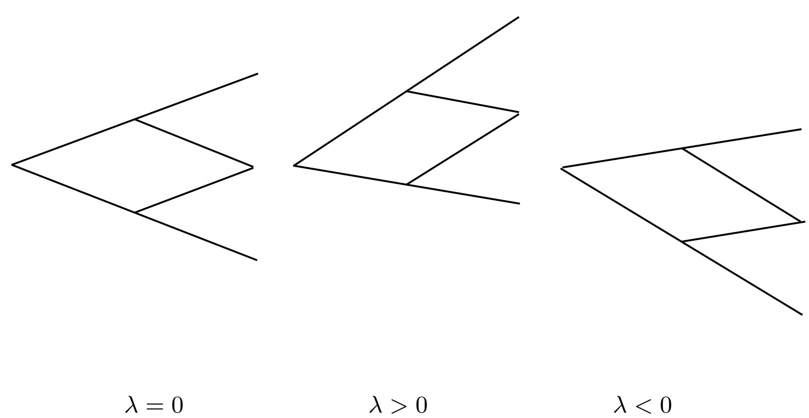

Fig. 1. The flexible binomial lattice for different values of the tilt parameter λ.

Of course, when $\lambda = 0$ one recovers the CRR model with $u_0 = e^{\sigma\sqrt{\Delta t}}$ and $d_0 = u_0^{-1}$. Tian shows that for every choice of the tilt parameter (provided that λ is finite and bounded) the flexible binomial model converges to the continuous-time counterpart.

Figure 1 shows the flexible binomial lattice for $\lambda = 0$, $\lambda > 0$ and $\lambda < 0$. A positive tilt parameter causes an upward transformation of the tree, while the effect of a negative λ is a downward shift.

The introduction of the tilt parameter in the binomial model allows for convenient adjustment to the tree in order to position nodes relative to the strike price (or the barrier) of the option. For the particular choice

$$\lambda = \frac{2\,(\eta - j_0)\,\sqrt{\Delta t}}{\sigma T}\,, \tag{7}$$

where

$$\eta = \frac{\log(X/S_0) - n\log(d_0)}{\log(u_0/d_0)}\,, \qquad j_0 = \left[\frac{\log(X/S_0) - n\log(d_0)}{\log(u_0/d_0)}\right]\,, \tag{8}$$

$u_0 = e^{\sigma\sqrt{\Delta t}}$, $d_0 = u_0^{-1}$, and $[\,\cdot\,]$ denotes the closest integer to its argument, the strike is always located on node $(n,\, j_0)$ at the option maturity. As a result, convergence of the FB model is smoother than in the CRR model, thus allowing the use of Richardson extrapolation.

Tian applies a two-point RE which considers prices obtained with a FB model with $n/2$ and n steps. We will see that RRE can be successfully applied. It is worth noting that not all the stepsize sequences perform equally well; in the numerical trials Romberg sequence overperformed all the other sequences (some results of the simulation are omitted here), yielding very accurate and robust results.

In the numerical experiments, we compare the American put prices obtained in the n-steps FB model when RRE based on Romberg sequence is applied, and those in the 50 000-steps FB model. We randomly generated 3 500 option valuation problems. The parameter ranges are: $r \in [0.01, 0.12]$, $\delta \in [0.0, 0.12]$ (where δ is the continuous dividend yield[13]), $\sigma \in [0.1, 0.5]$, $X/S_0 \in [0.7, 1.3]$, with $S_0 = 100$ and $T = 1$. As in the previous trials, the moneyness interval has been partitioned into seven subsets: $X \in [70, 80]$, $[80, 90]$, $[90, 100]$, $\{100\}$, $[100, 110]$, $[110, 120]$, $[120, 130]$, and we have generated 500 instances from each subset. We have considered a basic step $H = T/100$ and Romberg sequence.

The results of the simulation experiments carried out are summarized in tables 11 and 12. Both the MAE and the $RMSE$ have been computed but are not reported here in detail. Only the errors along the diagonal of the extrapolation tableau are presented. In table 11 the percent variations of the MAE are shown. Unlike the case of the CRR model, RRE performs well for all option moneyness, and not only in the at-the-money case. The pricing errors decrease monotonically as we consider a larger number of steps and higher order of extrapolation.

4 Concluding remarks

Richardson extrapolation and repeated RE are useful techniques in order to enhance accuracy of approximate solutions yielded by numerical schemes in problems that arise in finance. Nevertheless, such techniques should no longer be applied when convergence is non-uniform. Provided smooth convergence, RRE can improve accuracy and efficiency of the results. We have also found that the choices of the basic step and the stepsize sequence are critical.

[13] Note that Tian considers only the case $\delta = 0$.

In particular, we implemented the RRE within Carr's randomization approach with a different choice of the stepsizes sequence, obtaining more efficient results. Numerical experiments carried out suggest that it is not convenient to apply RRE to the CRR and the BBS methods, due to the non-monotonic behavior of the pricing errors, while in the flexible binomial approach, where a simple two-point RE has been used sofar, we employed successfully the RRE.

Finally, it seems interesting to investigate the possibility of applying repeated Richardson extrapolation to other models, and in particular to Monte Carlo simulation methods for valuing American options. It is worth noting that one should be careful when employing extrapolation techniques combined with these latter approaches, because of the difficulty of determining the accuracy of the approximations, which sometimes are also biased. To this regard, *ad hoc* smoothing procedures and discrete monitoring corrections may be required. This interesting task is left for future research.

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Table 1. RMS relative errors of randomized American put option prices when Richardson extrapolation is applied repeatedly in Carr's randomization approach ($S = 100$, $X \in [70,\ 80]$, $r \in [0.01,\ 0.12]$, $\sigma \in [0.1,\ 0.5]$, $T = 1$).

n_i	$RMSE$	n_i	$RMSE$	n_i	$RMSE$	n_i	$RMSE$
1	0.14220576	2	0.08976625	2	0.08976625	2	0.08976625
2	0.03762973	4	0.01773814	4	0.01773814	4	0.01773814
3	0.01391092	6	0.00549127	6	0.00549127	8	0.00466951
4	0.00590460	8	0.00222155	8	0.00222155	16	0.00147634
5	0.00285097	10	0.00121348	12	0.00114269	32	0.00060459
6	0.00170553	12	0.00078884	16	0.00069349	64	0.00026906
7	0.00118567	14	0.00055291	24	0.00043189		
8	0.00087775	16	0.00040649	32	0.00028092		
9	0.00067173	18	0.00031016	48	0.00018259		
10	0.00052799	20	0.00024374	64	0.00012264		

Table 2. RMS relative errors of randomized American put option prices when Richardson extrapolation is applied repeatedly in Carr's randomization approach ($S = 100$, $X \in [80,\ 90]$, $r \in [0.01,\ 0.12]$, $\sigma \in [0.1,\ 0.5]$, $T = 1$).

n_i	$RMSE$	n_i	$RMSE$	n_i	$RMSE$	n_i	$RMSE$
1	0.17828577	2	0.10833117	2	0.10833117	2	0.10833117
2	0.03911643	4	0.01644720	4	0.01644720	4	0.01644720
3	0.01093506	6	0.00394575	6	0.00394575	8	0.00335649
4	0.00395702	8	0.00160132	8	0.00160132	16	0.00108761
5	0.00200959	10	0.00091801	12	0.00086479	32	0.00045836
6	0.00129208	12	0.00059881	16	0.00052616	64	0.00020470
7	0.00090593	14	0.00041895	24	0.00032749		
8	0.00066608	16	0.00030823	32	0.00021353		
9	0.00050842	18	0.00023556	48	0.00013943		
10	0.00039982	20	0.00018548	64	0.00009427		

Table 3. RMS relative errors of randomized American put option prices when Richardson extrapolation is applied repeatedly in Carr's randomization approach ($S = 100$, $X \in [90,\ 100]$, $r \in [0.01,\ 0.12]$, $\sigma \in [0.1,\ 0.5]$, $T = 1$).

n_i	$RMSE$	n_i	$RMSE$	n_i	$RMSE$	n_i	$RMSE$
1	0.17707630	2	0.10112571	2	0.10112571	2	0.10112571
2	0.02601866	4	0.00979537	4	0.00979537	4	0.00979537
3	0.00525559	6	0.00217374	6	0.00217374	8	0.00188378
4	0.00223239	8	0.00101962	8	0.00101962	16	0.00069572
5	0.00130805	10	0.00059122	12	0.00055669	32	0.00029425
6	0.00083735	12	0.00038419	16	0.00033753	64	0.00013142
7	0.00058107	14	0.00026869	24	0.00021001		
8	0.00042712	16	0.00019764	32	0.00013698		
9	0.00032618	18	0.00015104	48	0.00008966		
10	0.00025645	20	0.00011900	64	0.00006093		

Table 4. Percent variation of mean absolute relative errors along the diagonal of the extrapolation tableau in Carr's randomization approach ($S = 100$, $X \in [90,\ 100]$, $r \in [0.01,\ 0.12]$, $\sigma \in [0.1,\ 0.5]$, $T = 1$).

n_i	$\Delta\%MAE$	n_i	$\Delta\%MAE$	n_i	$\Delta\%MAE$	n_i	$\Delta\%MAE$
1		2		2		2	
2	-86.14	4	-90.92	4	-90.92	4	-90.92
3	-80.84	6	-77.51	6	-77.51	8	-80.44
4	-54.72	8	-51.94	8	-51.94	16	-62.24
5	-40.36	10	-41.88	12	-45.28	32	-57.71
6	-35.92	12	-35.04	16	-39.41	64	-55.49
7	-30.57	14	-30.14	24	-37.89		
8	-26.49	16	-26.55	32	-34.92		
9	-23.69	18	-23.69	48	-34.75		
10	-21.45	20	-21.33	64	-32.34		

Table 5. RMS relative errors of randomized American put option prices when Richardson extrapolation is applied repeatedly in Carr's randomization approach ($S = 100$, $X = 100$, $r \in [0.01,\ 0.12]$, $\sigma \in [0.1,\ 0.5]$, $T = 1$).

n_i	$RMSE$	n_i	$RMSE$	n_i	$RMSE$	n_i	$RMSE$
1	0.14751803	2	0.08140506	2	0.08140506	2	0.08140506
2	0.01542737	4	0.00603751	4	0.00603751	4	0.00603751
3	0.00340090	6	0.00152150	6	0.00152150	8	0.00131711
4	0.00162969	8	0.00070668	8	0.00070668	16	0.00047732
5	0.00090427	10	0.00040283	12	0.00037868	32	0.00019480
6	0.00057281	12	0.00025808	16	0.00022526	64	0.00009117
7	0.00039687	14	0.00017683	24	0.00013570		
8	0.00028852	16	0.00012697	32	0.00008467		
9	0.00021720	18	0.00009441	48	0.00005179		
10	0.00016813	20	0.00007212				

Table 6. RMS relative errors of randomized American put option prices when Richardson extrapolation is applied repeatedly in Carr's randomization approach ($S = 100$, $X = [100,\ 110]$, $r \in [0.01,\ 0.12]$, $\sigma \in [0.1,\ 0.5]$, $T = 1$).

n_i	$RMSE$	n_i	$RMSE$	n_i	$RMSE$	n_i	$RMSE$
1	0.10966076	2	0.06214959	2	0.06214959	2	0.06214959
2	0.01498229	4	0.00555543	4	0.00555543	4	0.00555543
3	0.00289336	6	0.00121969	6	0.00121969	8	0.00105500
4	0.00127029	8	0.00056339	8	0.00056339	16	0.00038286
5	0.00072132	10	0.00032447	12	0.00030535	32	0.00016018
6	0.00046018	12	0.00020985	16	0.00018401	64	0.00008952
7	0.00031927	14	0.00014587	24	0.00011357		
8	0.00023370	16	0.00010670	32	0.00007358		
9	0.00017761	18	0.00008118	48	0.00004795		
10	0.00013900	20	0.00006374				

Table 7. RMS relative errors of randomized American put option prices when Richardson extrapolation is applied repeatedly in Carr's randomization approach ($S = 100$, $X = [110,\ 120]$, $r \in [0.01,\ 0.12]$, $\sigma \in [0.1,\ 0.5]$, $T = 1$).

n_i	$RMSE$	n_i	$RMSE$	n_i	$RMSE$	n_i	$RMSE$
1	0.05546058	2	0.03363912	2	0.03363912	2	0.03363912
2	0.01203470	4	0.00450030	4	0.00450030	4	0.00450030
3	0.00244982	6	0.00081508	6	0.00081508	8	0.00069602
4	0.00075614	8	0.00034123	8	0.00034123	16	0.00023173
5	0.00043381	10	0.00019718	12	0.00018526	32	0.00009561
6	0.00028519	12	0.00012585	16	0.00010994	64	0.00004159
7	0.00019673	14	0.00008656	24	0.00006695		
8	0.00014787	16	0.00006279	32	0.00004294		
9	0.00011957	18	0.00004744	48	0.00002772		
10	0.00008840	20	0.00004952				

Table 8. RMS relative errors of randomized American put option prices when Richardson extrapolation is applied repeatedly in Carr's randomization approach ($S = 100$, $X = [120,\ 130]$, $r \in [0.01,\ 0.12]$, $\sigma \in [0.1,\ 0.5]$, $T = 1$).

n_i	$RMSE$	n_i	$RMSE$	n_i	$RMSE$	n_i	$RMSE$
1	0.02814233	2	0.01816596	2	0.01816596	2	0.01816596
2	0.00828701	4	0.00330275	4	0.00330275	4	0.00330275
3	0.00204261	6	0.00059310	6	0.00059310	8	0.00049622
4	0.00053175	8	0.00020764	8	0.00020764	16	0.00014088
5	0.00025272	10	0.00011997	12	0.00011276	32	0.00005794
6	0.00017203	12	0.00007678	16	0.00006691	64	0.00002507
7	0.00012127	14	0.00005236	24	0.00004033		
8	0.00008863	16	0.00003791	32	0.00002563		
		18	0.00003152	48	0.00001643		

Table 9. Mean absolute and RMS relative errors of put option prices in the CRR framework when Richardson extrapolation is applied repeatedly ($S = 100$, $X = 100$, $r \in [0.01, 0.12]$, $\delta \in [0.01, 0.04]$, $\sigma \in [0.1, 0.5]$, $T = 1$). The results refer to an initial number of steps 200, and the harmonic (second column), Deuflhard (fourth column), Burlisch (sixth column) and Romberg (eighth column) sequences.

n	MAE	n	MAE	n	MAE	n	MAE
200	0.00085132	400	0.00042292	400	0.00042292	400	0.00042292
400	0.00002516	800	0.00001387	800	0.00001387	800	0.00001387
600	0.00002678	1200	0.00001477	1200	0.00001477	1600	0.00000984
800	0.00004084	1600	0.00002087	1600	0.00002087	3200	0.00000769
				2400	0.00002000	6400	0.00000691
n	$RMSE$	n	$RMSE$	n	$RMSE$	n	$RMSE$
200	0.00087489	400	0.00044335	400	0.00044335	400	0.00044335
400	0.00011797	800	0.00005646	800	0.00005646	800	0.00005646
600	0.00012403	1200	0.00006737	1200	0.00006737	1600	0.00002316
800	0.00017122	1600	0.00011880	1600	0.00011880	3200	0.00001083
				2400	0.00012342	6400	0.00000744

Table 10. Mean absolute (first tableau) and RMS (second tableau) relative errors of put option prices in the CRR framework when Richardson extrapolation is applied repeatedly ($S = 100$, $X \in [90, 100]$, $r \in [0.01, 0.12]$, $\delta \in [0.01, 0.12]$, $\sigma \in [0.1, 0.5]$, $T = 1$). Extrapolation tableau with Romberg stepsize sequence and initial number of steps 200.

	MAE				
n	T_{i1}	T_{i2}	T_{i3}	T_{i4}	T_{i5}
400	0.00044412				
800	0.00022241	0.00045825			
1600	0.00011121	0.00022538	0.00041994		
3200	0.00005540	0.00011510	0.00021350	0.00029184	
6400	0.00002493	0.00005376	0.00010115	0.00014021	0.00016512
	RMSE				
n	T_{i1}	T_{i2}	T_{i3}	T_{i4}	T_{i5}
400	0.00051264				
800	0.00025997	0.00058794			
1600	0.00012898	0.00029641	0.00054829		
3200	0.00006323	0.00015029	0.00027497	0.00037815	
6400	0.00003006	0.00006984	0.00013080	0.00018203	0.00021558

Table 11. Mean absolute relative errors of put option prices in the flexible binomial model when Richardson extrapolation is applied repeatedly ($S = 100$, $X \in [70, 130]$, $r \in [0.01, 0.12]$, $\delta \in [0.01, 0.12]$, $\sigma \in [0.1, 0.5]$, $T = 1$). $\Delta\% MAE$ is the percentage of variation of the mean absolute errors along the diagonal of the extrapolation tableau. Extrapolation is based on Romberg stepsize sequence and initial number of steps 100.

n	$X \in [70, 80]$	$\Delta\% MAE$	$X \in [80, 90]$	$\Delta\% MAE$	$X \in [90, 100]$	$\Delta\% MAE$
200	0.00392174		0.00256274		0.00126017	
400	0.00019418	-95.05	0.00010093	-96.06	0.00003225	-97.44
800	0.00012275	-36.79	0.00006944	-31.20	0.00001982	-38.53
1600	0.00006096	-50.34	0.00003443	-50.41	0.00001027	-48.19
3200	0.00002719	-55.40	0.00001634	-52.54	0.00000587	-42.83
6400	0.00001640	-39.68	0.00001080	-33.92	0.00000508	-13.50

n	$X \in [100, 110]$	$\Delta\% MAE$	$X \in [110, 120]$	$\Delta\% MAE$	$X \in [120, 130]$	$\Delta\% MAE$
200	0.00070133		0.00043071		0.00027674	
400	0.00001673	-97.61	0.00001241	-97.12	0.00001194	-95.68
800	0.00000930	-44.42	0.00000799	-35.58	0.00000784	-34.33
1600	0.00000554	-40.42	0.00000500	-37.43	0.00000410	-47.76
3200	0.00000391	-29.45	0.00000294	-41.30	0.00000192	-53.05
6400	0.00000345	-11.68	0.00000221	-24.77	0.00000137	-28.59

Table 12. RMS relative errors of put option prices in the flexible binomial model when Richardson extrapolation is applied repeatedly ($S = 100$, $X \in [70, 130]$, $r \in [0.01, 0.12]$, $\delta \in [0.01, 0.12]$, $\sigma \in [0.1, 0.5]$, $T = 1$). Extrapolation is based on Romberg stepsize sequence and initial number of steps 100.

n	$X \in [70, 80]$	$X \in [80, 90]$	$X \in [90, 100]$
200	0.00439491	0.00302903	0.00131513
400	0.00027668	0.00015759	0.00004912
800	0.00017227	0.00010980	0.00003038
1600	0.00008599	0.00005182	0.00001514
3200	0.00003762	0.00002714	0.00000743
6400	0.00002193	0.00001496	0.00000580

n	$X \in [100, 110]$	$X \in [110, 120]$	$X \in [120, 130]$
200	0.00073406	0.00047608	0.00033208
400	0.00007285	0.00003379	0.00002647
800	0.00003227	0.00002059	0.00001760
1600	0.00001859	0.00001434	0.00000931
3200	0.00000985	0.00000896	0.00000389
6400	0.00000841	0.00000413	0.00000213