Global asset return in pension funds: a dynamical risk analysis

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Abstract. The aim of the paper is to develop a technique for rebalancing pension fund portfolios in function of their pointwise level of risk. The performance of pension funds is often measured by their global asset returns because of the latter's influence on periodic contributions and/or future benefits. However, in periods of market crisis attention is focused on the risk level given their social security (and not speculative) function. We describe the process of the global asset return by a multifractional Brownian motion using the function $H(t)$ to detect high or low volatility phases. A procedure is carried out to balance the asset composition when the established local degree of risk is exceeded. The application is carried out on portfolios obtained in accordance with Italian regulations regarding investment limits.

Keywords. Pension funds, risk control, multifractional Brownian motion.

M.S.C. classification. 28A80, 91B28, 91B30.
J.E.L. classification. C22; G11; G23.

1 Introduction

Pension funds mainly have a social security function of reimbursing workers' savings in the form of a life annuity. This involves an accurate and prudent asset allocation scheme administration.

It has been debated that post retirement benefit plans should have limitations on their asset allocation, based on the risk profile of the different financial instruments available on the financial market.

The performance of pension funds is usually measured in terms of returns rather than risk. Risks are taken into account especially during market crises, when losses in the portfolio of financial instruments of the fund could lead to depreciations in the accrued contributions.

Some studies (see [24]) show that post 2001 bankruptcies of US pension funds had their roots in the actuarial evaluation techniques rather than in asset losses, if long-term stock return is considered. According to [5] and [17], post retirement benefit plans, pertaining to the 'first pillar' of a pension system, should not invest in high-risk financial instruments because this would lead to problems related to moral hazard and to the evaluation of 'superfluous risk'.

Trudda [27] proposes an application to the pension funds of Italian professional Orders, in which marginal increments in global asset return appear to strongly reduce the default probability. He also shows that there is an incentive to take superfluous risks in the case of a slackening of regulations.

More recently, Otranto and Trudda [18] have supported the idea that there is a need for a classification of the various degrees of risk for pension funds. They propose a cluster analysis based on the GARCH volatility of the rates of return. In [19] another methodology is carried out distinguishing between two kinds of risk for pensions funds: constant risk and time-varying risk. Although the method provides a satisfactory ex post risk analysis, the large lag necessary to get reliable estimates weakens its employment in practical applications when a timely response is required.

Bikker, Broaders and Drew [8] study the impact of stock market performance on the investment policy of Dutch pension funds and show that their investment policies are partially driven by the cyclical performance of the stock market. In addition they point out that pension funds respond asymmetrically to stock market shocks: rebalancing is much stronger after negative equity returns.

Stewart [26] analyzes the increasing tendency of pension funds to invest in hedge funds. He observes that in many cases the real risk is not correctly perceived. This is due to an inefficient regulating system and, in several countries, the absence of risk monitoring instrument.

In many cases the rules on pension funds investments are derived from the same laws that regulate investment companies, considering their speculative function. These regulations often indicate a qualitative restriction without limiting the quantitative measurement of the risk.

In Italy the regulating system for pension funds establishes non restrictive rules in the investment portfolio composition. Pension funds can invest in liquid assets, stocks, share of common investment funds. There are some restrictions about investments in equity and bonds traded in the over the counter markets and/or in non OECD countries.

In this paper we concentrate on the investment risk: a dynamical analysis of pension fund's portfolios is performed by estimating the pointwise regularity of the return series, assuming that these can be modeled by a multifractional Gaussian process, using the function $H(t)$ to detect high or low volatility phases. In this framework, the estimator we use quantifies the pointwise degree of the observations' departure from independence. In a more general way we estimate the local smoothness of a signal representing the portfolio quote. The intuition is that this can well synthesize the local degree of risk of a given asset or portfolio. Provided that the window of estimation is sufficiently small, it should be possible to build a warning system for monitoring the risk in pension funds. In this way, we can monitor the risk evolution after a short time using few daily data, thanks to the good rate of convergence of the estimator. In the paper we develop the system and describe the appropriate techniques for the automatic composition of the fund's portfolio in case of infringement of the given risk thresholds.

In the application three investment portfolios are simulated respecting Italian financial laws to show how the levels of risk obtained can be very different.

The paper is organized as follows: in section 2 we recall the main properties of the model we assume to generate the price dynamics. In section 3 the estimator of the pointwise regularity of the process is discussed. Section 4 concerns the analytical relationship between the portfolio's $H(t)$ and the $H(t)$'s of its individual assets. In section 5 we develop an analytical approach through the analysis of the variable $H(t)$. The purpose is the control of risk by a continuous monitoring and rebalancing policy using optimal portfolio definition. An application of three simulated portfolios with different risk degrees is carried out in section 6; we use the introduced approach to evaluate the the levels of risk over the time and to develop a rebalancing technique, through the analysis of the variable $H(t)$. Finally some conclusion are discussed in section 7.

2 The model

In the following we will assume the log price dynamics to be described by a versatile process: the multifractional Brownian motion (mBm). A convenient way to introduce the mBm is recalling its very well-known special case: the fractional Brownian motion (fBm). Defined in a celebrated paper by Mandelbrot and Van Ness [16], the fBm is characterized by a slowly decaying autocorrelation function depending on the parameter $H \in (0,1]$, named *Hurst* exponent. Following the definition that can be found in [9], the process has moving average representation

$$B_H(t) = C\{\pi K(2H)\}^{1/2} \int_{\mathbb{R}} f_t(s) dB(s) \tag{1}$$

with

$$f_t(s) = \frac{1}{\Gamma\left(H+\frac{1}{2}\right)} \left\{ |t-s|^{H-\frac{1}{2}} 1_{]-\infty,t]}(s) - |s|^{H-\frac{1}{2}} 1_{]-\infty,0]}(s) \right\}$$

where $B(\cdot)$ stands for the ordinary Brownian motion, C is a positive constant and K is the function defined on $]0,2[$ as $K(\alpha) = \Gamma(\alpha+1)\frac{\sin\frac{\alpha\pi}{2}}{\pi}$. The process is self-similar[3] of parameter H and has stationary increments. Its covariance function reads as

[3] We recall that the process $\{X(t),\, t \in T\}$ is said *self-similar* with parameter H if for any $\alpha > 0$ $\{X(\alpha t)\} \stackrel{d}{=} \{\alpha^H X(t)\}$, where the equality holds for the finite-dimensional distributions of the process (see e.g. [25]).

$$E\left(B_H(t)B_H(s)\right) = \frac{c^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \tag{2}$$

The fBm can be generalized by allowing H to vary over time. This extension − known as *multifractional Brownian motion* (mBm) (see [20], [21], [1]) − has the following representation

$$M_{H(t)}(t) = C\{\pi K(2H(t))\}^{1/2} \int_{\mathbb{R}} f_t(s)dB(s) \tag{3}$$

with

$$f_t(s) = \frac{1}{\Gamma\left(H(t)+\frac{1}{2}\right)}\left\{|t-s|^{H(t)-\frac{1}{2}}\,1_{]-\infty,t]}(s) - |s|^{H(t)-\frac{1}{2}}\,1_{]-\infty,0]}(s)\right\}$$

where $H : [0,\infty) \to (0,1]$ is required to be a Hölder function of order $0 < \eta \leq 1$ to ensure the continuity of the motion.

Notice that since $H(t)$ is the punctual Hölder exponent of the mBm at point t, the process is locally asymptotically self-similar with index $H(t)$ (see, e.g., [6]) in the sense that, denoted by $Z(t, au) := M_{H(t+au)}(t+au) - M_{H(t)}(t)$ the increment process of the mBm at time t and lag au, it holds

$$\lim_{a\to 0^+} a^{-H(t)} Z(t, au) \stackrel{d}{=} B_{H(t)}(u),\; u \in \mathbb{R}. \tag{4}$$

The above distributional equality indicates that at any point t there exists an fBm with parameter $H(t)$ tangent to the mBm. Moreover, since $B_{H(t)}(u) \sim \mathcal{N}(0, C^2u^{2H(t)})$, the infinitesimal increment of the mBm at time t, normalized by $a^{H(t)}$, normally distributes with mean 0 and variance $C^2u^{2H(t)}$ $(u \in \mathbb{R}\,, a \to 0^+)$.

The increments of the mBm are no longer stationary nor self-similar; despite this, the process is extremely versatile since the time dependency of H is useful to model phenomena whose punctual regularity is time changing.

From a financial viewpoint one can think of $H(t)$ in a suggestive way as a "memory" function, i.e. as the degree of confidence the investors nourish in the past. High values of $H(t)$ correspond to trends (or low volatility phases), i.e. to periods in which the past information weighs in the investors' trading decisions; low values of $H(t)$ are associated to high volatility periods, in which prices display an antipersistent or mean reverting behaviour because of the quick buy-and-sell activity that is typically induced by uncertainty. Standard financial theory is recovered when $H = \frac{1}{2}$, case in which the mBm reduces to the Brownian motion. The level of risk coupled with a financial time series is therefore framed into a dynamical perspective in which it can change from point to point, even in a strong way. What makes the difference here is not much and not only the type of investment (bond, stock, derivatives) but the time.

3 Pointwise estimation of the Hölderian regularity of the mBm

Given a sample path of the mBm, one of the main problems is estimating the function $H(t)$ from actual data. To deal with this problem one could think at adapting the traditional estimators of H available in literature in order to shadow the dynamics of $H(t)$. The weakness of this approach resides in the fact that very large samples are needed to get reliable estimates and in over a long time-span H is likely to change even widely. So, more efficient estimators are needed in the case of the mBm. An answer to this problem is provided by Bianchi [7], who develops the work of Péltier ad Lévy Véhel ([20]) and defines a family of "moving-window" estimators of $H(t)$ based on the k-th absolute moment of a Gaussian random variable of mean zero and given variance V_H (the variance of the unit lag increment of a mBm). Given a series of length N and a window of length δ, the estimator has the form

$$H_{\delta,N}^{k}(t)=\frac{\log\left(2^{k/2}\Gamma\left(\frac{k+1}{2}\right)V_H^{k/2}\right)-\log\left(\frac{\sqrt{\pi}}{\delta}\sum_{j=t-\delta}^{t-1}\left|X_{j+1,N}-X_{j,N}\right|^{k}\right)}{k\log\left(N-1\right)} \tag{5}$$

for $j=t-\delta,...,t-1;\quad t=\delta+1,...,N;\quad k\geq 1$.
Thanks to its good rate of convergence $O\left(\delta^{-\frac{1}{2}}\left(\log N\right)^{-1}\right)$, (5) allows reliable estimates even for very short $\delta's$. The family of estimators (5) was proved to be correct and normally distributed as

$$H_{\delta,N}^{k}(t)\sim\mathcal{N}\left(H(t),\frac{\pi}{\delta k^{2}\ln^{2}\left(N-1\right)2^{k}\left(\Gamma\left(\frac{k+1}{2}\right)\right)^{2}}\sigma^{2}\right) \tag{6}$$

σ^2 being the variance of a Gaussian random variable defined as a proper rescaled sum. Toilsome computations show that when $H=\frac{1}{2}$ the variance of the estimator reduces to

$$Var(H_{\delta,N}^{k}(t))=\frac{\sqrt{\pi}}{\delta k^{2}\ln^{2}\left(N-1\right)\left[\Gamma\left(\frac{k+1}{2}\right)\right]^{2}}\cdot\left(\Gamma\left(\frac{2k+1}{2}\right)-\frac{1}{\sqrt{\pi}}\left[\Gamma\left(\frac{k+1}{2}\right)\right]^{2}\right) \tag{7}$$

and the optimal value of k is deduced by minimizing the last relation. So one finds that the minimum of (7) takes place when $k=2$, value which will be used in the empirical application discussed below. An idea of the way the estimator (5) work is provided by Figure 1. Panel (a) shows a sample path generated by a mBm with sinusoidal functional parameter (four periods were considered, with $H(t)$ ranging in the interval $[0.2,0.8]$); panel (b) shows the variations of the signal (notice the bursts of variance corresponding to low values of $H(t)$); finally, in panel (c) the continuous line is the functional parameter and the zigzagged line is the functional parameter estimated by filtering the original signal through (5), setting $\delta=30$.

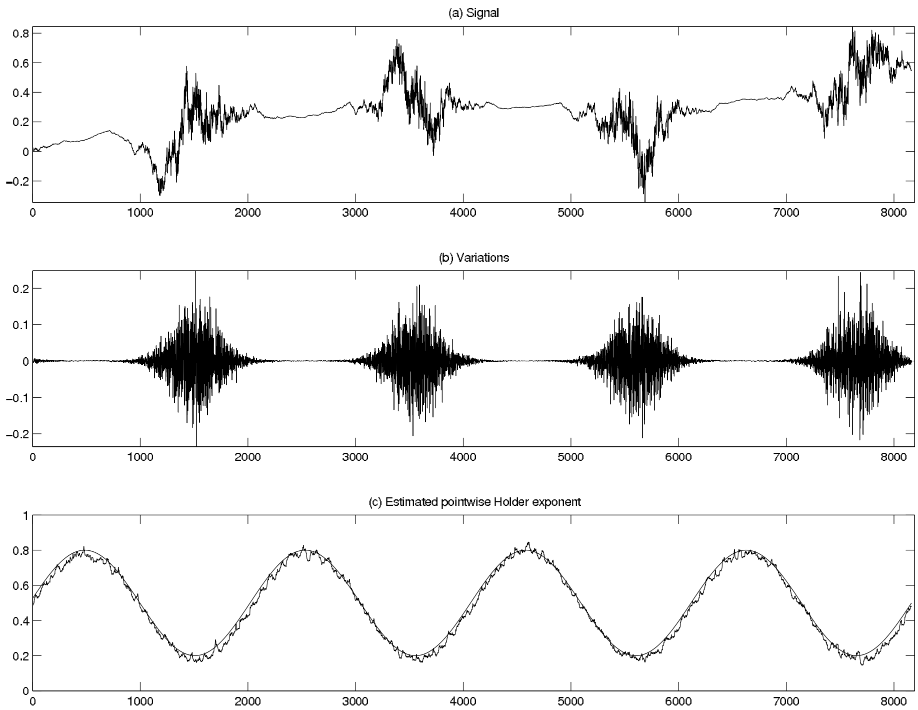

Fig. 1. Estimation of the Holderian function of a simulated mBm

4 The portfolio's $H(t)$

In this section we derive the portfolio's $H(t)$ and write it as a function of the $H(t)$'s of individual assets.
As usual, let

$$\Pi_j = \sum_{s=1}^{N} \alpha_s X_{s,j} \tag{8}$$

denote a portfolio of N assets, each characterized by its own functional parameter ${}_sH^k_{\delta,q,n}(t)$ $(s = 1, ..., n)$ and with unit variance at time n.

We set $k = 2$ because it is easy to show that this value minimizes the estimator's variance when $H = 1/2$. In this way the estimator of the portfolio's $H(t)$ can be written as a function of ${}_sH^k_{\delta,q,n}(t)$ and it reads as:

$$
{}_{\Pi}H^2_{\delta,q,n}(t) = -\frac{\ln\left(\sum_{s=1}^{N}\alpha_s^2\left(\frac{n-1}{q}\right)^{-2\,{}_sH^2_{\delta,q,n}(t)} + \right.}{2\ln\left(\frac{n-1}{q}\right)}
\frac{\left.+2\sum_{p=1}^{N-1}\sum_{r=p+1}^{N}\alpha_p\alpha_r\left(\frac{n-1}{q}\right)^{-\left({}_pH^2_{\delta,q,n}(t)+{}_rH^2_{\delta,q,n}(t)\right)}\rho_{p,r,\delta}\right)}{2\ln\left(\frac{n-1}{q}\right)} \tag{9}
$$

with $\rho_{p,r,\delta} := \frac{\sum_{j=t-\delta}^{t-1}|dX_{p,j,q}||dX_{r,j,q}|}{\sqrt{\sum_{j=t-\delta}^{t-1}|dX_{p,j,q}|^2\sum_{j=t-\delta}^{t-1}|dX_{r,j,q}|^2}}$

In fact, let $d\Pi_{j,q} = \Pi_{j+q} - \Pi_j = \sum_{s=1}^{N}\alpha_s dX_{s,j,q}$ denote the portfolio's increments, where $dX_{s,j,q} := X_{s,j+q} - X_{s,j}$. One has

$$
{}_{\Pi}H^2_{\delta,q,n}(t) = -\frac{\ln\frac{\sum_{j=t-\delta}^{t-1}\left|\sum_{s=1}^{N}\alpha_s dX_{s,j,q}\right|^2}{K^2(\delta-q+1)}}{2\ln\left(\frac{n-1}{q}\right)} =
$$

$$
= -\frac{\ln\frac{\sum_{s=1}^{N}\alpha_s^2\sum_{j=t-\delta}^{t-1}dX^2_{s,j,q}+2\sum_{j=t-\delta}^{t-1}\sum_{p=1}^{N-1}\sum_{r=p+1}^{N}\alpha_p\alpha_r|dX_{p,j,q}||dX_{r,j,q}|}{K^2(\delta-q+1)}}{2\ln\left(\frac{n-1}{q}\right)}
$$

From (5) it readily follows that

$$
\frac{\sum_{j=t-\delta}^{t-1}dX_j^2}{K^2(\delta-q+1)} = \left(\frac{n-1}{q}\right)^{-2H^2_{\delta,q,n}(t)}
$$

and therefore

$$
{}_{\Pi}H^2_{\delta,q,n}(t) = -\frac{\ln\left(\sum_{s=1}^{N}\alpha_s^2\left(\frac{n-1}{q}\right)^{-2\,{}_sH^2_{\delta,q,n}(t)} + \frac{2\sum_{j=t-\delta}^{t-1}\sum_{p=1}^{N-1}\sum_{r=p+1}^{N}\alpha_p\alpha_r\left|dX_{p,j,q}\right|\left|dX_{r,j,q}\right|}{K^2(\delta-q+1)}\right)}{2\ln\left(\frac{n-1}{q}\right)}
$$

$$
{}_{\Pi}H^2_{\delta,q,n}(t) = -\frac{ln\left(\sum_{s=1}^{N}\alpha_s^2\left(\frac{n-1}{q}\right)^{-2\,{}_sH^2_{\delta,q,n}(t)} +\right.}{2\ln\left(\frac{n-1}{q}\right)}
$$

$$
\frac{\left. + \frac{2\left(\sum_{j=t-\delta}^{t-1}\alpha_1\alpha_2|dX_{1,j,q}||dX_{2,j,q}|+\ldots+\sum_{j=t-\delta}^{t-1}\alpha_{N-1}\alpha_N|dX_{N-1,j,q}||dX_{N,j,q}|\right)}{K^2(\delta-q+1)}\right)}{2\ln\left(\frac{n-1}{q}\right)} \tag{10}
$$

A more insightful way of writing relation (10) again exploits (5), from which it is easy

$$
-{}_pH^2_{\delta,q,n}(t) = \frac{\ln\frac{\sum_{j=t-\delta}^{t-1}|dX_{p,j,q}|^2}{K^2(\delta-q+1)}}{2\ln\left(\frac{n-1}{q}\right)} \quad \text{and} \quad -_r\,H^2_{\delta,q,n}(t) = \frac{\ln\frac{\sum_{j=t-\delta}^{t-1}|dX_{r,j,q}|^2}{K^2(\delta-q+1)}}{2\ln\left(\frac{n-1}{q}\right)}
$$

Summing up side by side we get

$$
-{}_pH^2_{\delta,q,n}(t) -_r\,H^2_{\delta,q,n}(t) = \frac{\ln\frac{\sum_{j=t-\delta}^{t-1}|dX_{p,j,q}|^2\sum_{j=t-\delta}^{t-1}|dX_{r,j,q}|^2}{K^4(\delta-q+1)^2}}{2\ln\left(\frac{n-1}{q}\right)}
$$

and therefore

$$
\left(\frac{n-1}{q}\right)^{-({}_pH^2_{\delta,q,n}(t)+{}_rH^2_{\delta,q,n}(t))} = \frac{\sqrt{\sum_{j=t-\delta}^{t-1}|dX_{p,j,q}|^2\sum_{j=t-\delta}^{t-1}|dX_{r,j,q}|^2}}{K^2(\delta-q+1)}.
$$

from which it follows:

$$
\left(\frac{n-1}{q}\right)^{-({}_pH^2_{\delta,q,n}(t)+{}_rH^2_{\delta,q,n}(t))}\rho_{p,r,\delta} = \frac{\sum_{j=t-\delta}^{t-1}|dX_{p,j,q}|\,|dX_{r,j,q}|}{K^2(\delta-q+1)}
$$

and by substituting in (10) one gets (9), where the factor ρ clearly represents the correlation of the absolute increments of the process.

5 The dynamic optimization problem

In order to cope with the optimization problem we use the relationship between the portfolio's $H(t)$ and the functions $H(t)$ of each asset included in the portfolio itself as developed in the previous Section. The procedure intervenes when the $H(t)$ value decreases under a fixed threshold. This dynamic approach, combined with a control on the level of return, is based on the assumption that it exists an inverse relation between the value of the portfolio's $H(t)$ and its exposure to risk. In other words, since high values of $H(t)$ are indicative of trends, once the procedure excluded that the trend is negative, one can use this information to rebalance the portfolio in order to control its level of risk. Let us denote by L, B and S the liquidity, the bond and the stock components of the portfolio, by Π_X the set of indexes pertaining to the investment of type X ($X =$ L, B or S), by N the number of assets in the portfolio, by $\overline{\alpha}^X$ the (fixed) ratio of assets of type X, by $r_\Pi(t)$ the log price of asset k at time t and Δ the length of the window. Equipped with this notation, the risk-minimizer portfolio manager has to solve the following constrained problem

$$
\begin{aligned}
&\max_{\alpha_k} {}_\Pi H^k_{1,q,n}(t)\\
&\quad s.t.\\
&\quad \textstyle\sum_{k\in\Pi_L}\alpha_k = \overline{\alpha}^L\\
&\quad \textstyle\sum_{k\in\Pi_S}\alpha_k = \overline{\alpha}^S\\
&\quad \textstyle\sum_{k\in\Pi_B}\alpha_k = \overline{\alpha}^B\\
&\quad \alpha_k \geq 0\\
&\quad \overline{\alpha}^L + \overline{\alpha}^S + \overline{\alpha}^B = 1\\
&\quad \textstyle\sum_{h=t-\Delta}^{t}\frac{r_\Pi(h)}{\Delta+1} \geq \phi; \quad \phi \geq 0
\end{aligned}
\tag{11}
$$

which means to determine the vector $(\alpha_1, ..., \alpha_N)$ defining the H-optimal portfolio.

At each time t the algorithm checks whether the portfolio's $H(t)$ is lower than the fixed threshold. If not the portfolio is mantained, otherwise it is rebalanced using the new weights determined by solving the optimization problem. For the constrained optimization we used an extension of primal interior point methods, which applies sequential quadratic programming techniques to a sequence of barrier problems. Trust regions are used to ensure the robustness of the iteration and to allow using the second order derivatives. The software was developed in MatLab environment at the L.I.S.A. [4], using the function fmincon (see [23]) already implemented in the optimization toolbox.

[4] The computer lab for advanced scientific computing operating within D.I.Me.T. The authors aim a special thank to dr. Augusto Pianese and to dr. Alexandre Pantanella for the algorithm implementation.

6 Application

In order to estimate the risk dynamic, we started using three portfolios complying with the Italian laws on pension funds investments. The analyzed portfolios were characterized by strong differences in terms of returns variability and therefore in the risk profile. In the maximum and medium risk portfolios were included investments in stock components traded in over the counter markets and in non OECD countries. We used daily data from 30/09/2003 to 19/04/2007.

Table 1 shows the fixed ratios of the portfolios' components ($\overline{\alpha}^L$, $\overline{\alpha}^B$ and $\overline{\alpha}^S$).

Table 1. Composition of the three portfolios

Portfolio	Liquidity	Bonds	Stocks
Minimum risk	10%	90%	0%
Medium risk	10%	70%	20%
Maximum risk	10%	40%	50%

The portfolio composition respected the investment limits imposed by Italian regulation (over the counter and non OECD assets). The minimum risk portfolio was characterized by a larger investment of 90% in bonds (Bond PE 22.5%, Arca MM 22.5%, Arca RR 22.5%, Arca TE 22.5%), 10% liquidity (Libor 5%, Arca BT 5%), with a very low standard deviation value showing the low risk profile offered.

The medium risk portfolio was composed of 70% bonds (Bond PE 17.5%, Arca MM 17.5%, Arca RR 17.5%, Arca TE 17.5%), 10% liquid assets (Libor 5%, Arca BT 5%) and 20% stocks (Mibtel 2.8%, Ibm 2.8%, Nasdaq 2.8%, DowChem 2.8%, Ibovespa 2%, Shangai 2%, Google 2.8%, Kospi 2%. It included stocks component traded in OECD unregulated markets (2% Kospi) and in non OECD regulated markets (2% Ibovespa, 2% Shangai) This portfolio was characterized by an intermediate risk.

The third portfolio (maximum risk) was composed of 10% liquid assets (Libor 5%, Arca BT 5%), 40% bonds (Bond BDPE 10%, Arca MM 10%, Arca RR 10%, Arca TE 10%), 50% stocks (Mibtel 8.8%, IBM 8.8 %, Nasdaq 8.8%, Down Chem 8.8%, Ibovespa 2%, Shangai 2%, Google 8.8%, Kospi 2%). It included stocks component traded in OECD unregulated markets (2% Kospi) and in non OECD regulated markets (2% Ibovespa) In spite of its strong bonds component, this portfolio presents high returns variability expressed by an high standard deviation value.

Figure 2 displays the global asset return and the estimated $H(t)$ values of the three initial portfolios. As expected, the return increases with the risk of the portfolios (panel (a)) whereas high values of $H(t)$ are associated with a low a priori risk (panel (b)).

In order to define a rebalancing strategy, we developed a procedure working as follows: given a threshold H^*, at each time we test whether the current estimation of $H(t)$ is below the fixed threshold, which means that - under the assumptions of our model - the portfolio is going subject to an excess risk. In this case, we rebalance the portfolio solving the optimization problem (11). Otherwise we maintain the current portfolio. Notice that the last constraint of problem (11) is meant to guarantee a minimum positive return ϕ for the portfolio; the condition is necessary because the sole $H(t)$ does not give information about the direction of the local trend, which can be negative as well as positive. The strat-

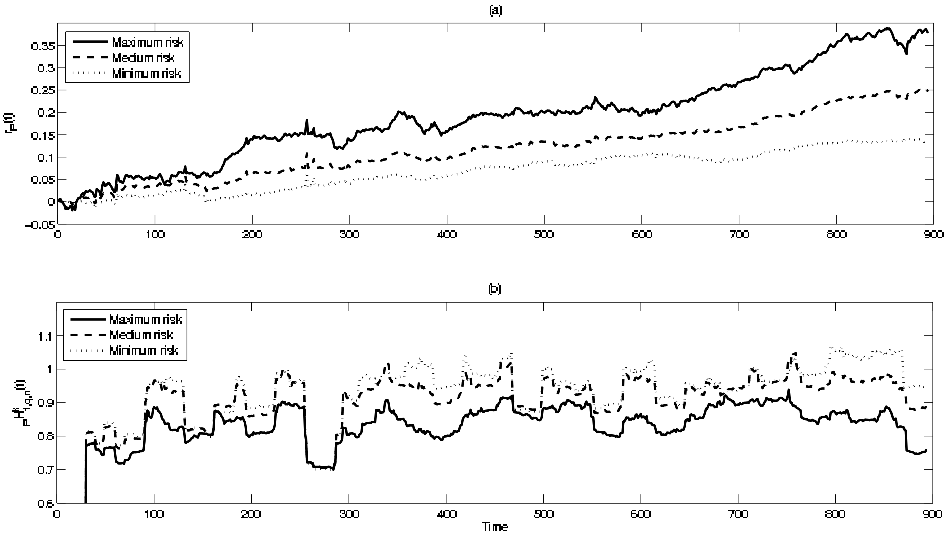

Fig. 2. Global asset return and the estimated $H(t)$ values of the three initial portfolios

egy described above was applied to the three portfolios with different thresholds $0.75 \leq H^* \leq 0.90$ and ϕ given by the daily rate of return equal to the five-years average Gross Domestic Product (GDP), using $\Delta = 0$. Obviously, the number of rebalancings strongly depends on the thresholds H^* and ϕ (they increase with the former and decrease with the latter). An example of the results produced by the strategy is shown in Figures 3-6, obtained setting $H^* = 0.85$. Figure 3 displays the values of $H(t)$ for the rebalanced portfolio (continuous line) and for the initial portfolio (dotted line). The vertical bars below indicate the times in which the rebalancing has occurred. Observe that a convenient choice of the assets heavily modifies the risk profile, even of .2817 (at day 279). It is obvious that the reduction of risk reflects in a lower return, as shown in Figure 4 which displays the global asset return of the initial and the rebalanced portfolios (the maximum difference is under 0.1 on a time horizon of three years). Figure 5 displays the risk-return profile of the portfolios; since differently from the traditional Markowitz's model here we use $H(t)$ as a proxy of the risk level, the risk-dominant rebalanced portfolios are located in the upper right area of the graph. It is apparent the effect of the rebalancing strategy, which forces upward $H(t)$ (solid squares) with respect to the values of the initial portfolio (empty circles). Finally, Figure 6 shows the 's of the rebalanced portfolios. In this regard, observe that the optimization problem (11) does not contain constraints about over the counter and non OECD assets. This means that the new portfolios generally do not comply the limits imposed by Italian regulation concerning the a priori risky markets; nonetheless, they are less risky than the initial portfolio'.

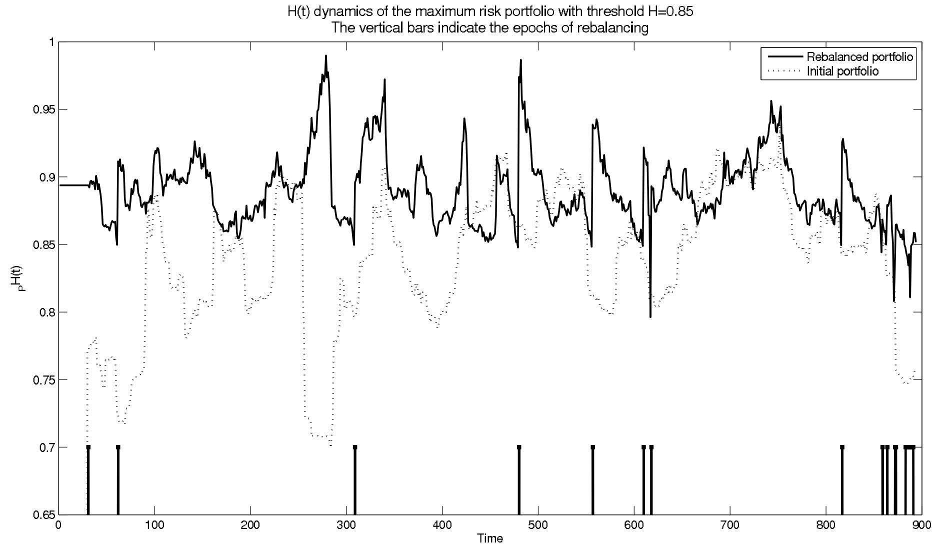

Fig. 3. $H(t)$ dynamics of the maximum risk portfolio with threshold $H = 0.85$

The analyzed portfolios deserve a couple of further comments. First, in all cases the values of $H(t)$ are significantly far from the central value assumed by standard financial theory. This is consistent with a number of works, but here - differently from what occurs in the case of single stocks or indexes - the values are also significantly above . Second, large variations characterize the estimates; for the minimum, the medium and the maximum risk portfolios the ranges are respectively 0.162, 0.164, 0.136. Again, this is inconsistent with the models assuming a constant value of H and strongly suggests a dynamical approach to portfolio management. Looking at things with more detail one realizes that the estimates of $H(t)$ seem to cluster towards low values. This is reflected by the negative skewness of the distributions: -0.61, -0.68 and -0.59 respectively for the minimum, the medium and the maximum risk portfolios.

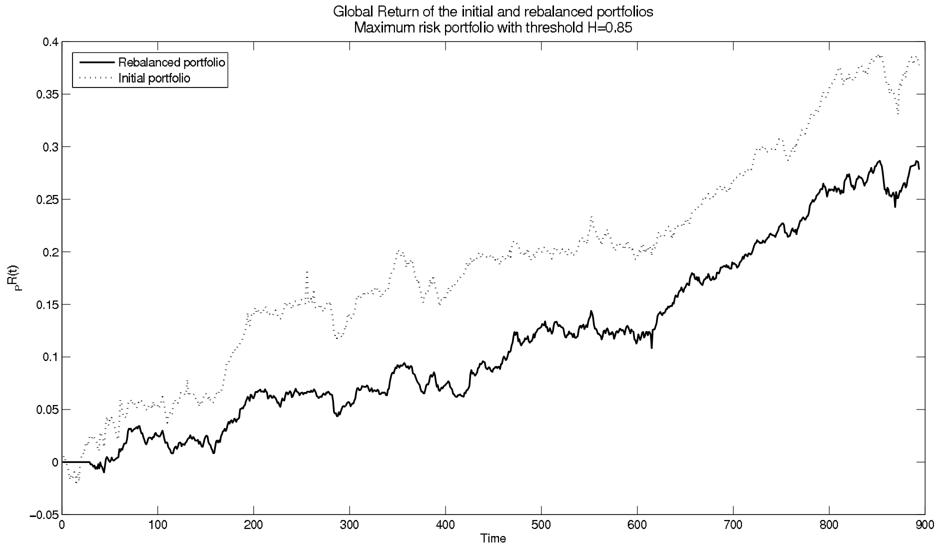

Fig. 4. Global Return of the initial and rebalanced portfolios Maximum risk portfolio with threshold $H = 0.85$

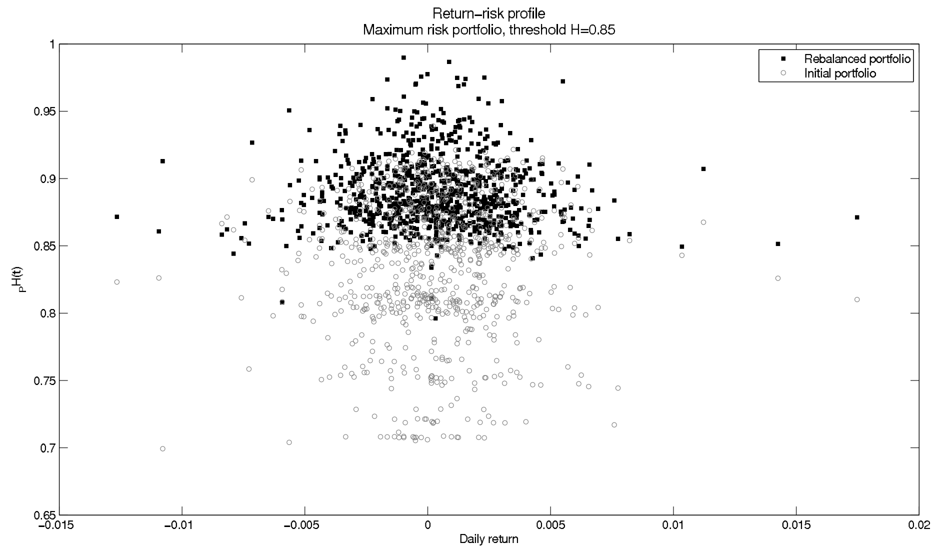

Fig. 5. Return-risk profile. Maximum risk portfolio threshold $H = 0.85$

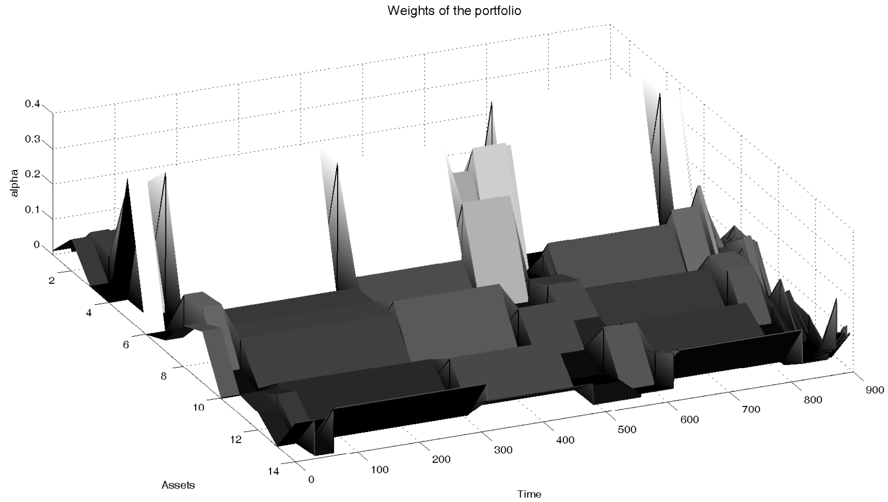

Fig. 6. Weights of the portfolio. The assets are the following: 1−Libor; 2−Arca BT; 3−Bond PE; 4−Arca MM; 5−Arca RR; 6−Arca TE; 7−MibTel; 8−IBM; 9−Nasdaq 10−DowChem; 11−Ibovespa; 12−Shangai; 13−Google; 14−Kospi.

7 Concluding remarks and further developments

The September 2001 market crisis caused the failure of some pension funds in the USA and Europe. A debate about the financial investment limits and the risk structure of pension funds was opened. Several analysis highlights the tendency of the Funds to increase the portfolios risk in order to obtain higher values of the expected global asset return. Some economic theories study phenomena like as a moral hazard problem because of accounting rules which encourage Pension Corporation to assume excessive risk. Many authors emphasize that pension funds have to maintain a prudent profile because the social function (in particular for the first pillar) prevails over the speculative function. In general, financial laws use mutual fund regulations to determine the limits of investments in risky financial instruments. Moreover, regulations are often qualitative and do not use quantitative methods.

In order to investigate the regulation potency, in our applications we use investment portfolios compliant with the Italian laws on pension funds. The results highlights how Italian pension fund regulation permits investments whit very different risk degrees.

A dynamic approach is introduced in order to constantly balance the investment portfolio to control the risk evolution. The risk dynamic is analyzed using a multifractional Brownian motion to describe the log price of the global asset portfolios. We use the function $H(t)$ to evaluate the volatility level in the instant t: when the estimation of $H(t)$ is below the fixed threshold H^*, an optimization problem is applied to rebalance the portfolio over the time in order to control the volatility of global asset return. It's important to note that the procedure respond a volatility changes in a quick time using only the lag data because of the convergence proprieties of $H(t)$. The applications show that using this procedure to control the excess of risk, a cost in term of lower global asset return is payed. An interesting development of this work will be to investigate the relationship between the level of maximum volatility required H^* and the reduction of returns using our strategy.

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