

Calibration of a multiscale stochastic volatility model using European option prices^{*}

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Abstract. In this paper we consider an explicitly solvable multiscale stochastic volatility model that generalizes the Heston model, and propose a new model calibration based on a nonlinear optimization problem. The model considered was introduced previously by the authors in [8] to describe the dynamics of an asset price and of its two stochastic variances. The risk neutral measure associated with the model and the risk premium parameters are introduced and the corresponding formulae to price call and put European vanilla options are derived. These formulae are given as one dimensional integrals of explicitly known integrands. We use these formulae to calibrate the multiscale model using European option prices as data, that is, to determine the values of the model parameters, of the correlation coefficients of the Wiener processes appearing in the model and of the initial stochastic variances implied by the “observed” option prices. The results obtained by solving the calibration problem are used to forecast future option prices. The calibration problem is translated into a suitable constrained nonlinear least squares problem. The proposed formulation of the calibration problem is applied to S&P 500 index data on the prices of European vanilla options in November 2005. This analysis points out some interesting facts.

Keywords. Multiscale stochastic volatility models, calibration model, option pricing.

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1 Introduction

The use of stochastic volatility models to describe asset price dynamics originates from empirical evidence that the price dynamics of assets are driven by processes with nonconstant volatility. In fact, it is well known that difficulties arise when models with constant volatility such as the Black and Scholes model [2] are used to describe asset price dynamics. Examples of such difficulties are the so called volatility “smile” that appears in the volatilities implied by the observed option prices, and the presence of skewness and kurtosis in the approximate asset price probability density function deduced from empirical data. Several alternative models have been proposed to overcome these shortcomings, including “mean reverting” stochastic volatility models such as the Heston model [12]. The Heston model provides a satisfactory description of the price dynamics of several relevant assets, as shown in [13] and [16]. Moreover the Heston model is explicitly solvable in the sense that the joint probability density function associated with the asset price and its stochastic variance can be written as a one-dimensional integral of an explicitly known integrand. The Heston model is a one-factor stochastic volatility model since the stochastic variance of the asset price is modeled as a real stochastic process. However several empirical studies of option price data have shown that the term structure of the implied volatility of the prices of several assets (e.g., market indices and commodities) seems to be driven by two factors, one fluctuating on a fast time scale and another fluctuating on a longer time scale (see, for example, Alizadeh, Brandt, and Diethelm [1]). We can conclude that, in some circumstances, one-factor stochastic volatility models are unable to fully capture the volatility smile and volatility dynamics (see [9], [6], [3], [5]). To overcome this inadequacy, several models that go beyond one-factor stochastic volatility models have been proposed; we group these models into two classes: multiscale stochastic volatility models [8], [11], [17] and jump models [15], [4]. Here we will concentrate on the multiscale stochastic volatility models. Several authors have devoted attention to such models, for example, Fouque, Papanicolaou, Sircar and Solna [10] developed a multiscale stochastic volatility model starting from ideas introduced in [1]. In [10], the proposed model is calibrated on real data to capture the volatility smile, and an explicit series expansion of the formula for the price of a European vanilla call option in the model considered in [10] is given. More recently, Wong and Chan [17] proposed a different multiscale stochastic volatility model and used it to price a long term financial product called dynamic fund protection. Moreover they reported a series expansion for the formula for the price of lookback options. Finally, the authors proposed a new multiscale stochastic volatility model that generalizes the Heston model and describes the dynamics of an asset price and of its two

stochastic variances using a system of three Ito stochastic differential equations [8]. The two stochastic variances vary on two different time scales. Under some hypotheses, the proposed model provides “explicitly” solvable and “easy to use” formulae to price in the model call and put European vanilla options can be deduced.

In this paper, we consider the multiscale model proposed in [8] and we use the formulae to price call and put European vanilla options deduced in [8] to solve a calibration problem, and a forecasting problem. The calibration problem can be stated as follows: *given the observed prices of call and put European vanilla options on a given asset traded on a given day, determine the model parameters, the correlation coefficients of the Wiener processes appearing in the model and the initial stochastic variances implied by the observed prices.* We use these implied values to forecast the prices of the options in the days following the day whose option prices have been used as data in the calibration. We translate the calibration problem into a constrained least squares optimization problem that is a generalization of the optimization problem considered in [8]. In fact, in [8], the objective function is defined using only the observed prices of the European call options. Here the objective function is defined using the observed prices of both call and put options. The new optimization problem formulated using both call and put prices has some interesting consequences, discussed in Section 3, compared to the optimization problem considered in [8]. In the numerical experiment (see Section 3) we consider European vanilla options on the S&P 500 index and we use the results of the calibration to forecast option prices. The forecasted option prices are compared to the observed prices; the results of this comparison are very satisfactory. A more detailed discussion of the multiscale model and a more extended analysis of the 2005 data relative to the S&P 500 and to its options in the year 2005 can be found in [8]. The website: <http://www.econ.univpm.it/recchioni/finance/w7> contains auxiliary material including animations that assists in the understanding of this paper. More general information on the work of the authors and of their coauthors in mathematical finance can be found on the website <http://www.econ.univpm.it/recchioni/finance>.

The remainder of the paper is organized as follows. In Section 2, we describe the multiscale stochastic volatility model considered and, under some hypotheses, we derive an integral representation formula for its transition probability density function and for the price of European vanilla call and put options under the risk-neutral measure. In Section 3, we formulate and solve a calibration problem and a forecasting problem. We use observed option prices on the S&P 500 index to test the solution method of the calibration problem and the forecasting procedure. In Section 4, we present our conclusions.

2 The multiscale stochastic volatility model

Let \mathbb{R} and \mathbb{R}^+ be sets of real and positive real numbers, respectively, and let t be a real variable that denotes time. We consider the (vector valued real) stochastic process $(x_t, v_{1,t}, v_{2,t})$, $t > 0$, solution of the following system of stochastic

differential equations:

$$dx_t = (\hat{\mu} + a_1 v_1 + a_2 v_2)dt + b_1 \sqrt{v_{1,t}} dW_t^{0,1} + b_2 \sqrt{v_{2,t}} dW_t^{0,2}, \quad t > 0, \quad (1)$$

$$dv_{1,t} = \chi_1(\theta_1 - v_{1,t})dt + \varepsilon_1 \sqrt{v_{1,t}} dW_t^1, \quad t > 0, \quad (2)$$

$$dv_{2,t} = \chi_2(\theta_2 - v_{2,t})dt + \varepsilon_2 \sqrt{v_{2,t}} dW_t^2, \quad t > 0, \quad (3)$$

where the quantities $\hat{\mu}$, a_i , b_i , χ_i , ε_i , θ_i , $i = 1, 2$, are real constants. The quantity $\hat{\mu}$ is known as the drift rate. Note that elementary considerations suggest that we must require $\chi_i \geq 0$, $\varepsilon_i \geq 0$, $\theta_i \geq 0$, $i = 1, 2$. Moreover we require $\frac{2\chi_i\theta_i}{\varepsilon_i^2} > 1$, $i = 1, 2$. The condition $\frac{2\chi_i\theta_i}{\varepsilon_i^2} > 1$ guarantees that when $v_{i,t}$ is positive with probability one at time $t = 0$, $v_{i,t}$, the solution of (2), or of (3), remains positive with probability one for $t > 0$, $i = 1, 2$. Finally $W_t^{0,1}$, $W_t^{0,2}$, W_t^1 , W_t^2 , $t > 0$ are standard Wiener processes such that $W_0^{0,1} = W_0^{0,2} = W_0^1 = W_0^2 = 0$, and $dW_t^{0,1}$, $dW_t^{0,2}$, dW_t^1 , dW_t^2 , $t > 0$, are their stochastic differentials, and we assume that:

$$\mathbb{E}(dW_t^1 dW_t^2) = \mathbb{E}(dW_t^{0,1} dW_t^2) = \mathbb{E}(dW_t^{0,2} dW_t^1) = \mathbb{E}(dW_t^{0,1} dW_t^{0,2}) = 0, \quad t > 0, \quad (4)$$

$$\mathbb{E}(dW_t^{0,1} dW_t^1) = \rho_{0,1} dt, \quad \mathbb{E}(dW_t^{0,2} dW_t^2) = \rho_{0,2} dt, \quad t > 0, \quad (5)$$

where $\mathbb{E}(\cdot)$ denotes the expected value of \cdot , and $\rho_{0,1}$, $\rho_{0,2} \in [-1, 1]$ are constants known as correlation coefficients. Note that the autocorrelation coefficients of the stochastic differentials are equal to one (see [8] for further details).

We interpret x_t , $t > 0$, as the log-return of the asset price and $v_{1,t}$, $v_{2,t}$, $t > 0$, as the stochastic variances of x_t , $t > 0$. The fact that $v_{1,t}$, $v_{2,t}$, $t > 0$, are stochastic variances on different time scales is translated in the condition $\chi_1 \ll \chi_2$. With the above interpretation of $(x_t, v_{1,t}, v_{2,t})$, $t > 0$, the assumptions (4)-(5) appear natural.

Equations (1), (2) and (3) must be equipped with an initial condition, that is:

$$x_0 = \tilde{x}_0, \quad v_{1,0} = \tilde{v}_{1,0}, \quad v_{2,0} = \tilde{v}_{2,0}, \quad (6)$$

where \tilde{x}_0 , $\tilde{v}_{i,0}$, $i = 1, 2$ are random variables that we assume to be concentrated in a point with probability one. For simplicity, we identify the random variables \tilde{x}_0 , $\tilde{v}_{i,0}$, $i = 1, 2$, with the points where they are concentrated. Without loss of generality, we can choose $\tilde{x}_0 = 0$. Moreover we assume $\tilde{v}_{i,0} \in \mathbb{R}^+$, $i = 1, 2$. The quantities $\tilde{v}_{i,0}$, $i = 1, 2$ cannot be observed in real markets. Moreover, selecting values of $a_1 = -1/2$, $a_2 = 0$, $b_1 = 1$, $b_2 = 0$ in equations (1), (2), (3) corresponds to the fact that equations (1) and (2) are decoupled from equation (3), and that these values give the Heston model. In this sense the model (1), (2), (3) is a generalized version of the Heston model.

Let $p_f(x, v_1, v_2, t, x', v'_1, v'_2, t')$, $(x, v_1, v_2), (x', v'_1, v'_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, $t, t' \geq 0$, $t - t' > 0$, be the transition probability density function associated with the stochastic differential system (1), (2), (3), that is, the probability density function of having $x_t = x$, $v_{1,t} = v_1$, $v_{2,t} = v_2$ given the fact that $x_{t'} = x'$, $v_{1,t'} = v'_1$, $v_{2,t'} = v'_2$, when $t - t' > 0$. The transition probability density function $p_f(x, v_1, v_2, t, x', v'_1, v'_2, t')$, $(x, v_1, v_2), (x', v'_1, v'_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, $t, t' \geq 0$, $t - t' >$

0, as a function of the variables (x, v_1, v_2, t) is the solution of the *Fokker Planck* equation with suitable initial and boundary conditions (see [8] for further details) and, as function of the variables (x', v'_1, v'_2, t') , satisfies the following backward equation:

$$\begin{aligned} -\frac{\partial p_f}{\partial t'} &= \frac{1}{2}(b_1^2 v'_1 + b_2^2 v'_2) \frac{\partial^2 p_f}{\partial x'^2} + \frac{1}{2} \varepsilon_1^2 v'_1 \frac{\partial^2 p_f}{\partial v_1'^2} + \frac{1}{2} \varepsilon_2^2 v'_2 \frac{\partial^2 p_f}{\partial v_2'^2} + \varepsilon_1 b_{1\rho_{0,1}} v'_1 \frac{\partial^2 p_f}{\partial x' \partial v_1'} + \\ &\varepsilon_2 b_{2\rho_{0,2}} v'_2 \frac{\partial^2 p_f}{\partial x' \partial v_2'} + \chi_1(\theta_1 - v'_1) \frac{\partial p_f}{\partial v_1'} + \chi_2(\theta_2 - v'_2) \frac{\partial p_f}{\partial v_2'} + (\hat{\mu} + a_1 v'_1 + a_2 v'_2) \frac{\partial p_f}{\partial x'}, \\ &(x', v'_1, v'_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, 0 \leq t' < t, \end{aligned} \quad (7)$$

with the final condition:

$$\begin{aligned} p_f(x, v_1, v_2, t, x', v'_1, v'_2, t') &= \delta(x' - x) \delta(v'_1 - v_1) \delta(v'_2 - v_2), \\ &(x, v_1, v_2), (x', v'_1, v'_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, t \geq 0, \end{aligned} \quad (8)$$

and the appropriate boundary conditions (see [8], Section 2).

Letting $\tau = t - t'$, we assume that the following integral representation formula for $p_f(x, v_1, v_2, t, x', v'_1, v'_2, t')$, $(x, v_1, v_2), (x', v'_1, v'_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, $t' \geq 0, t > t'$, holds:

$$\begin{aligned} p_f(x, v_1, v_2, t, x', v'_1, v'_2, t') &= \\ &\frac{1}{(2\pi)^3} \frac{2}{\varepsilon_1^2} \frac{2}{\varepsilon_2^2} \int_{\mathbb{R}} dk e^{ikx} \int_{\mathbb{R}} dl_1 e^{\frac{2}{\varepsilon_1^2} \iota l_1 v_1} \int_{\mathbb{R}} dl_2 e^{\frac{2}{\varepsilon_2^2} \iota l_2 v_2} f(\tau, k, l_1, l_2, x', v'_1, v'_2), \\ &(x, v_1, v_2), (x', v'_1, v'_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \tau = t - t' > 0, \end{aligned} \quad (9)$$

where ι is the imaginary unit, f is the Fourier transform of the function obtained extending with zero the function p_f defined above as a function of the variables (x, v_1, v_2) when $v_1 \notin \mathbb{R}^+$ and/or $v_2 \notin \mathbb{R}^+$, and k, l_1 , and l_2 are the conjugate variables of x, v_1 , and v_2 respectively (see [8], Section 2, for further details). Using the arguments presented in [14] (pages 602-605), in [8] it was shown that:

$$\begin{aligned} f(\tau, k, l_1, l_2, x', v'_1, v'_2) &= e^{-\iota k x'} e^{A(\tau, k, l_1, l_2)} e^{-\frac{2}{\varepsilon_1^2} v'_1 B_1(\tau, k, l_1)} e^{-\frac{2}{\varepsilon_2^2} v'_2 B_2(\tau, k, l_2)}, \\ &(x', v'_1, v'_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, (k, l_1, l_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \tau > 0, \end{aligned} \quad (10)$$

where the functions A and B_i , $i = 1, 2$ are given by:

$$\begin{aligned} A(\tau, k, l_1, l_2) &= -\iota k \hat{\mu} \tau - \\ &\sum_{i=1}^2 \left[\frac{2\chi_i \theta_i}{\varepsilon_i^2} \left((\nu_i + \zeta_i) \tau + \ln \left(\frac{(\nu_i + \zeta_i - \iota l_i) e^{-2\zeta_i \tau} + (\iota l_i - \nu_i + \zeta_i)}{2\zeta_i} \right) \right) \right], \\ &(k, l_1, l_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \tau > 0, \end{aligned} \quad (11)$$

$$\begin{aligned} B_i(\tau, k, l_i) &= \frac{(\nu_i - \zeta_i)(\nu_i + \zeta_i - \iota l_i) e^{-2\zeta_i \tau} + (\nu_i + \zeta_i)(\iota l_i - \nu_i + \zeta_i)}{(\nu_i + \zeta_i - \iota l_i) e^{-2\zeta_i \tau} + (\iota l_i - \nu_i + \zeta_i)}, \\ &(k, l_i) \in \mathbb{R} \times \mathbb{R}, \tau > 0, i = 1, 2, \end{aligned} \quad (12)$$

where

$$\nu_i = -\frac{1}{2}(\chi_i + \iota k b_i \varepsilon_i \rho_{0,i}), \quad k \in \mathbb{R}, \quad i = 1, 2, \quad (13)$$

$$\zeta_i = \frac{1}{2}(4\nu_i^2 + \varepsilon_i^2(b_i^2 k^2 + 2\iota k a_i))^{1/2}, \quad k \in \mathbb{R}, \quad i = 1, 2. \quad (14)$$

Substituting equations (11) and (12) into equation (9) and integrating with respect to the variables l_1 and l_2 we can derive a representation formula for $p_f(x, v_1, v_2, t, x', v'_1, v'_2, t')$, that is, for the transition probability density function of the stochastic process solution of (1), (2), (3), as a one dimensional integral of an explicitly known integrand (see [8] for further details). The formulae for pricing European vanilla call and put options with strike price $K > 0$ and maturity time T are derived from (9) using the no arbitrage pricing theory, that is, computing the option prices as expected values of a discounted payoff with respect to an equivalent martingale measure, also known as a risk-neutral measure (see for example [7], [15]). However to keep the exposition simple and since we use as data in the formulation of the calibration problem only option prices we can use to compute in the model the option prices the statistical measure associated with the model (1), (2), (3) whose density is given by (9), that is, we can incorporate the risk premium parameters into the parameters χ_i and θ_i , $i = 1, 2$. In fact, in order to consider the risk neutral measure associated with (1), (2), (3), we should simply replace the parameters χ_i , θ_i , $i = 1, 2$ appearing in (1), (2), (3) with the parameters $\tilde{\chi}_i = \chi_i + \lambda_i$, $\tilde{\theta}_i = \chi_i \theta_i / (\chi_i + \lambda_i)$, $i = 1, 2$, where $\lambda_i \in \mathbb{R}$, $i = 1, 2$ are the risk premium parameters (see [8] for more details) and we should impose the constraints $\tilde{\chi}_i \geq 0$, $\tilde{\theta}_i \geq 0$, $i = 1, 2$. Now, writing the transition probability density function p_f as follows $p_f(x, v_1, v_2, t, x', v'_1, v'_2, t') = \tilde{p}_f(x, v_1, v_2, t, x', v'_1, v'_2, t')e^{-2(x-x')}$, $(x, v_1, v_2), (x', v'_1, v'_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, $t > 0$, $t' \geq 0$, $t - t' > 0$, deriving a representation formula for $\tilde{p}_f(x, v_1, v_2, t, x', v'_1, v'_2, t')$, $(x, v_1, v_2), (x', v'_1, v'_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, $t > 0$, $t' \geq 0$, $t - t' > 0$, and using the backward equation (7), we obtain the following formula for the price at time $t = 0$ of a European vanilla call option with time to maturity $\tau = T - t > 0$ (remember that $t = 0$) and strike price K , when at time $t = 0$ the price of the underlying asset is given by S_0 and the stochastic variances are given by $\tilde{v}_{1,0}$, $\tilde{v}_{2,0}$:

$$C(\tau, K, S_0, \tilde{v}_{1,0}, \tilde{v}_{2,0}) = \frac{S_0}{2\pi} e^{-r\tau} e^{2\hat{\mu}\tau} \int_{-\infty}^{+\infty} dk \frac{e^{-\iota k(\log(S_0/K) + \hat{\mu}\tau) - \log(K/S_0)}}{-k^2 - 3\iota k + 2} \cdot \prod_{i=1}^2 \left(e^{-2\chi_i \theta_i (\nu_i^c + \zeta_i^c + \log(s_{i,b}^c / (2\zeta_i^c))) \tau / \varepsilon_i^2} e^{-2\tilde{v}_{i,0}((\zeta_i^c)^2 - (\nu_i^c)^2) s_{i,g}^c / (\varepsilon_i^2 s_{i,b}^c)} \right), \quad (15)$$

$$\tau > 0, S_0 > 0, \tilde{v}_{1,0}, \tilde{v}_{2,0} > 0,$$

where r is the discount rate and the quantities ν_i^c , ζ_i^c , $s_{i,b}^c$, $s_{i,g}^c$, $i = 1, 2$ are given by:

$$\nu_i^c = -\frac{1}{2}(\chi_i + \iota k b_i \varepsilon_i \rho_{0,i} - 2b_i \rho_{0,i} \varepsilon_i), \quad k \in \mathbb{R}, \quad i = 1, 2, \quad (16)$$

$$\zeta_i^c = \frac{1}{2} \left(4(\nu_i^c)^2 + \varepsilon_i^2 (b_i^2 k^2 + 2\iota k a_i + 4\iota k b_i^2 - 4(a_i + b_i^2)) \right)^{1/2}, \quad k \in \mathbb{R}, \quad i = 1, 2, \quad (17)$$

$$s_{i,g}^c = 1 - e^{-2\zeta_i^c \tau}, \quad s_{i,b}^c = \zeta_i^c - \nu_i^c + (\zeta_i^c + \nu_i^c) e^{-2\zeta_i^c \tau}, \quad \tau > 0, \quad i = 1, 2. \quad (18)$$

Note that because we work with the risk neutral measure, the discount rate r must be chosen equal to $\hat{\mu}$ (see formula (14) in [12]). We note that the relation between the log-return x_t , $t > 0$ and the price S_t , $t > 0$, of the underlying asset is $x_t = \log S_t/S_0$, $t > 0$.

The formula for the price at time $t = 0$ of a European vanilla put option with time to maturity $\tau = T - t > 0$, (remember that $t = 0$) and strike price K , when at time $t = 0$ the price of the underlying asset is S_0 and the stochastic variances are $\tilde{v}_{1,0}$, $\tilde{v}_{2,0}$ is given by:

$$P(\tau, K, S_0, \tilde{v}_{1,0}, \tilde{v}_{2,0}) = \frac{K}{2\pi} e^{-r\tau} e^{-\hat{\mu}\tau} \int_{-\infty}^{+\infty} dk \frac{e^{-\iota k (\log(S_0/K) + \hat{\mu}\tau) - \log(S_0/K)}}{-k^2 + 3\iota k + 2} \cdot \prod_{i=1}^2 \left(e^{-2\chi_i \theta_i (\nu_i^p + \zeta_i^p + \log(s_{i,b}^p / (2\zeta_i^p))) \tau / \varepsilon_i^2} e^{-2\tilde{v}_{i,0} ((\zeta_i^p)^2 - (\nu_i^p)^2) s_{i,g}^p / (\varepsilon_i^2 s_{i,b}^p)} \right), \quad \tau > 0, S_0 > 0, \tilde{v}_{1,0}, \tilde{v}_{2,0} > 0, \quad (19)$$

where the quantities ν_i^p , ζ_i^p , $s_{i,g}^p$, and $s_{i,b}^p$, are given by:

$$\nu_i^p = -\frac{1}{2} (\chi_i + \iota k b_i \varepsilon_i \rho_{0,i} + b_i \rho_{0,i} \varepsilon_i), \quad k \in \mathbb{R}, \quad i = 1, 2, \quad (20)$$

$$\zeta_i^p = \frac{1}{2} \left(4(\nu_i^p)^2 + \varepsilon_i^2 (b_i^2 k^2 + 2\iota k a_i - 2\iota k b_i^2 - 2(a_i + b_i^2)) \right)^{1/2}, \quad k \in \mathbb{R}, \quad i = 1, 2, \quad (21)$$

$$s_{i,g}^p = 1 - e^{-2\zeta_i^p \tau}, \quad s_{i,b}^p = \zeta_i^p - \nu_i^p + (\zeta_i^p + \nu_i^p) e^{-2\zeta_i^p \tau}, \quad \tau > 0, \quad i = 1, 2. \quad (22)$$

Note that the price of European call and put options at time t , $0 < t < T$ can be deduced from (15) and (19) with some obvious changes.

3 Calibration and forecasting problems: some experiments with real data

In this section, we conduct experiments on real data, using values of $a_1 = a_2 = -\frac{1}{2}$ and $b_1 = b_2 = 1$ in equations (1), (2), (3). Let $\underline{\Theta} = (\epsilon_1, \theta_1, \rho_{0,1}, \chi_1, \tilde{v}_{0,1}, \hat{\mu}, \epsilon_2, \theta_2, \rho_{0,2}, \chi_2, \tilde{v}_{0,2})$ be a vector comprised of the parameters of the multiscale model (note that the risk premium parameters can be included in the parameters χ_i, θ_i , $i = 1, 2$), of the correlation coefficients, and of the initial stochastic variances. Let m_c, m_p be two positive integers; we denote the data of the calibration problem with $C^t(\tilde{S}_t, T_i, K_i)$, $i = 1, 2, \dots, m_c$, and, $P^t(\tilde{S}_t, T_i, K_i)$, $i = 1, 2, \dots, m_p$, that is, the observed prices at time t of European vanilla call and put options, respectively, having maturity time T_i and strike price K_i , $i = 1, 2, \dots, \max(m_c, m_p)$, when the price of the underlying asset at time t is \tilde{S}_t . Moreover, we denote the

prices of European vanilla call and put options obtained using (15) and (19) as $C^{t,\underline{\theta}}(\tilde{S}_t, T_i, K_i)$, $i = 1, 2, \dots, m_c$ and $P^{t,\underline{\theta}}(\tilde{S}_t, T_i, K_i)$, $i = 1, 2, \dots, m_p$, respectively, and choose the maturity time $\tau = \tau_i = T_i - t$, $i = 1, 2, \dots, \max(m_c, m_p)$ and the asset price $S_0 = \tilde{S}_t$. Note that usually when a call option is traded for a couple T_i, K_i the corresponding put option is also traded and vice versa; this is implicit in our assumption that it is possible to denote with T_i, K_i , $i = 1, 2, \dots, \max(m_c, m_p)$ the couple maturity time, strike price of the option prices used as data. If necessary, our notation can be easily generalized to handle the data that are actually available.

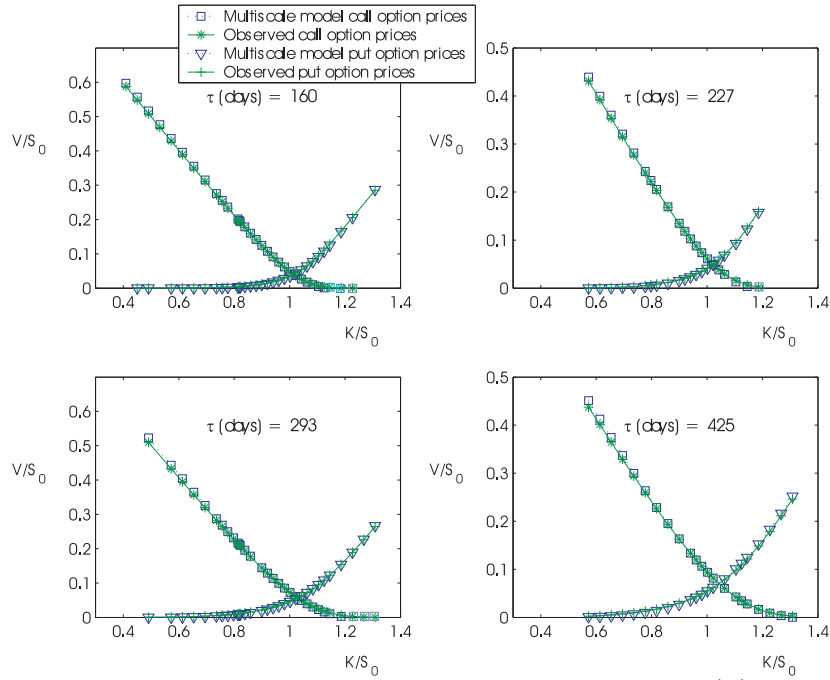


Fig. 1. November 7, 2005: European vanilla call and put option prices (V) on the $S\&P$ 500 index forecasted using the multiscale model and prices observed in the market versus moneyness K/S_0

Let \mathbb{R}^{11} be the 11-dimensional real Euclidean vector space and let \mathcal{M} be the set of admissible vectors $\underline{\theta}$, that is:

$$\mathcal{M} = \{\underline{\theta} = (\epsilon_1, \theta_1, \rho_{0,1}, \chi_1, \tilde{v}_{0,1}, \hat{\mu}, \epsilon_2, \theta_2, \rho_{0,2}, \chi_2, \tilde{v}_{0,2}) \in \mathbb{R}^{11} \mid \epsilon_i \chi_i, \theta_i \geq 0, i = 1, 2, \frac{2\chi_i \theta_i}{\epsilon_i^2} \geq 1, -1 \leq \rho_{0,i} \leq 1, \tilde{v}_{0,i} \geq 0, i = 1, 2\}, \quad (23)$$

at time t , $t \geq 0$. We calibrate the model (1), (3), (2) by solving the following constrained nonlinear least squares problem:

$$\min_{\Theta \in \mathcal{M}} L_t(\Theta), t \geq 0, \quad (24)$$

where the objective function $L_t(\Theta)$, $t \geq 0$, is defined as follows:

$$L_t(\Theta) = \sum_{i=1}^{m_c} \left[C^{t, \Theta}(\tilde{S}_t, T_i, K_i) - C^t(\tilde{S}_t, T_i, K_i) \right]^2 + \sum_{i=1}^{m_p} \left[P^{t, \Theta}(\tilde{S}_t, T_i, K_i) - P^t(\tilde{S}_t, T_i, K_i) \right]^2, t \geq 0. \quad (25)$$

The optimization problem (24) is a translation of the calibration problem for the model (1), (2), (3) stated in Section 1.

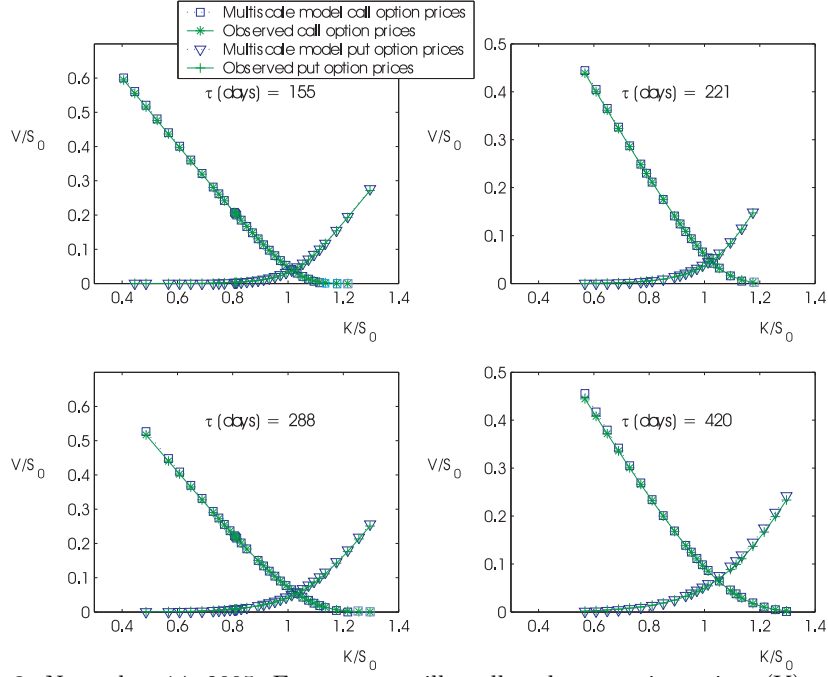


Fig. 2. November 14, 2005: European vanilla call and put option prices (V) on the *S&P* 500 index forecasted using the multiscale model and prices observed in the market versus moneyness K/S_0

We solve the optimization problem (24) using a projected steepest descent method. This method is an iterative scheme that, starting from an initial vector $\Theta^0 \in \mathcal{M}$, generates a sequence $\{\Theta^n\}$, $n = 0, 1, \dots$, of vectors $\Theta^n \in \mathcal{M}$, $n =$

$0, 1, \dots$, moving along a descent direction obtained via a suitable projection on the constraints of the vector given by the negative of the gradient with respect to $\underline{\Theta}$ of L_t . The procedure stops when the vector $\underline{\Theta}^n$ generated satisfies for the first time the following condition:

$$L_t(\underline{\Theta}^n) \leq e_{tol}, \text{ or } n > n_{max}, \quad (26)$$

where e_{tol} , n_{max} are positive constants that will be chosen as outlined below.

Experiments using synthetic and real data that show the adequacy of a formulation of the calibration problem similar to (23), (24), (25) and its ability to capture satisfactorily the “smile” effect can be found in [8].

Let us consider the values of the model parameters, the correlation coefficients, and the initial stochastic variances implied by the observed prices of the European vanilla call and put options on the S&P 500 index and by the value of the S&P 500 index in November 2005. The S&P 500 index is one of the leading indices of the New York Stock Exchange. We solve the calibration problem (24) using the call and put option prices available to us relative to the prices on November 3, 2005, and we have $m_c = 303$ and $m_p = 284$. The implied values obtained by solving the calibration problem using the data from November 3, 2005 are used to forecast the option prices of November 7 ($m_c = 303$, $m_p = 290$), November 14 ($m_c = 305$, $m_p = 295$), and November 28 ($m_c = 292$, $m_p = 265$), 2005. Note that since we use all available prices, the calibration procedure works simultaneously on out of money, at the money and in the money call and put option prices. In the stopping criterion (26), we use $e_{tol} = 0.07$ and $n_{max} = 10000$. In the forecasting of option prices, we assume that the underlying asset price is known on the forecasting day and we forecast the values of the stochastic variances $v_{1,t}$, $v_{2,t}$, $t > t_0 = 0$ on the forecasting day. In particular, starting from the stochastic variances $\tilde{v}_{1,0}$, $\tilde{v}_{2,0}$ at time $t = t_0 = 0$ obtained from the calibration procedure, we forecast the stochastic variances using the mean values $\hat{v}_{1,t|\underline{\Theta}}$, $\hat{v}_{2,t|\underline{\Theta}}$, $t > t_0 = 0$, of the random variables $v_{1,t}$, $v_{2,t}$, $t > 0$, that is, we use the formulae (see [8] for further details):

$$\hat{v}_{1,t|\underline{\Theta}} = \mathbb{E}(v_{1,t}|\underline{\Theta}) = \theta_1(1 - e^{-\chi_1(t-t_0)}) + e^{-\chi_1(t-t_0)}\tilde{v}_{1,0}, \quad t > t_0 = 0, \quad (27)$$

$$\hat{v}_{2,t|\underline{\Theta}} = \mathbb{E}(v_{2,t}|\underline{\Theta}) = \theta_2(1 - e^{-\chi_2(t-t_0)}) + e^{-\chi_2(t-t_0)}\tilde{v}_{2,0}, \quad t > t_0 = 0. \quad (28)$$

Note that in our experiment, $t = t_0 = 0$ corresponds to November 3, 2005, and that we have assumed that the underlying asset price is known on the forecasting days (November 7, 14, and 28, 2005). The dates November 3, 7, 14, and 28 were chosen to show that the model parameters calibrated using data from the beginning of the month (November 3) can be used to accurately forecast the option prices approximately a week (November 7), two weeks (November 14) and a month (November 28) into the future.

Figures 1, 2, and 3 show the forecast prices of European vanilla call and put options on November 7, 14, and 28, 2005, respectively, compared with the observed prices. The agreement between the observed and forecast prices is very satisfactory for all dates, indicating that the future prices of the options could be

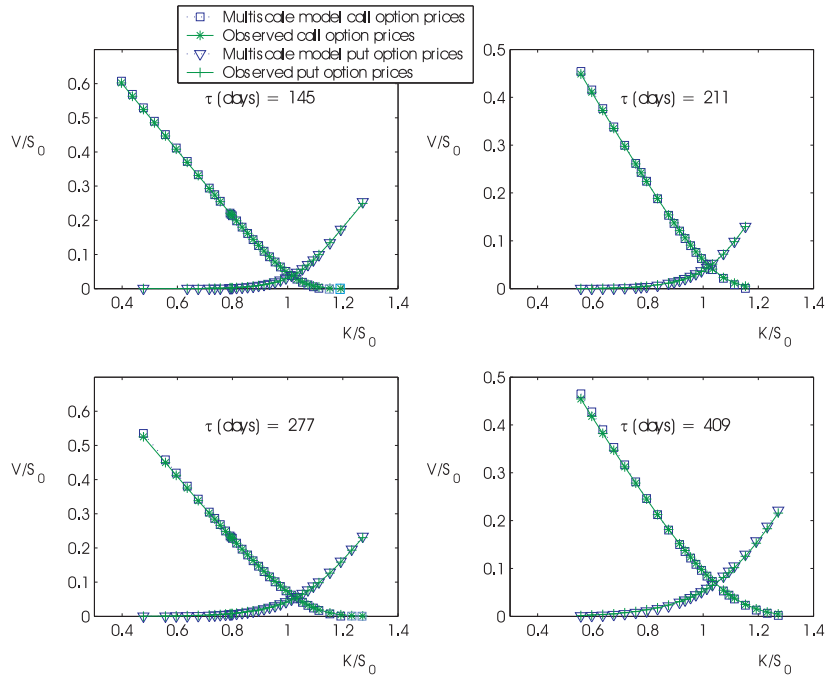


Fig. 3. November 28, 2005: European vanilla call and put option prices (V) on the $S\&P$ 500 index forecasted using the multiscale model and prices observed in the market versus moneyness K/S_0

well forecast using the implied values obtained from the prices of November 3, 2005 in conjunction with formulae (15), (19), (27), (28) and the price of the underlying asset on the forecasting day. Comparison of the calibration procedure employed here, which uses both call and put option prices, and the calibration procedure of [8], which uses only call option prices, discloses that the procedure proposed here provides substantially better forecasts of at the money and out of the money option prices. The superiority of the proposed method becomes more evident as the time to maturity increases.

4 Conclusions

Previously, the authors showed [8] that the two factor model (1), (2), (3) captures the volatility smile better than the one factor model [12] and provides high-quality forecasts of European vanilla option prices when the time to maturity is large. Here we have presented a calibration procedure that yields improved forecasts of at the money and out of the money option prices; this improvement can be attributed to the proposed procedure giving more weight to these prices than the procedure used in [8]. In fact, these prices are small compared with those of in the money options, and hence play a minor role in the minimization

of the objective function (25) especially in the first steps of the minimization procedure. When only the call option prices are used in the objective function (as in [8]), the number of in the money and out of the money options traded in a given day can vary substantially depending on the asset price. When we add the put option prices as data, however, we balance the portfolio of options used as data. In fact, the call and put options traded in a given day usually have the same strike price and maturity time, so that when a put option is in the money the corresponding call option is out of the money and vice versa. Moreover, by adding the put option prices, we increase the number of data on at the money options, that is, we increase the weight of the at the money options data in the calibration. The above considerations account for the observed improvement in the quality of the forecasted prices of the at the money and out of the money options when we substitute the calibration procedure of [8] with the calibration procedure suggested here. These findings suggest that use of a weighted least squares procedure may further improve the model calibration. Finally we point out that the calibration problem studied here can be generalized by considering the parameters $a_i, b_i, i = 1, 2$, as unknowns to be determined in the calibration. The use of these four extra parameters could potentially improve the ability of the model to describe the financial data considered.

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