A stochastic model for the analysis of demographic risk in pay-as-you-go pension funds*

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Abstract. This research presents an analysis of the demographic risk related to future membership patterns in pension funds with restricted entrance, financed under a pay-as-you-go scheme. The paper, therefore, proposes a stochastic model for investigating the behaviour of the demographic variable 'new entrants' and the influence it exerts on the financial dynamics of such funds. Further information on pension funds of Italian professional categories and an application to the *Cassa Nazionale di Previdenza e Assistenza dei Dottori Commercialisti* are then provided.

Keywords. Pension funds, demographic risk, new entrants, Markov chain, professional categories.

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J.E.L. classification. C15, C32, C53, G23, J11.

1 Introduction

The financial sustainability of a pension scheme depends not only on the time-length of the benefit to be paid, subject to longevity risk, but also on the correct quantification of the contributions to be received. The uncertainty related to future contributions primarily affects the retirement plans based on a pay-as-you-go financing system (PAYG), where pensions are directly funded by current employees' salary deductions. Thus, an "intergenerational pact" compels young generations (composed by current and future contributors) to sustain the older

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age groups. For the financial self-sufficiency of such pension schemes, it is essential that in the long term there should be equilibrium between the number of pensioners and the number of workers. Due to the inversion of the production cycle, delivering financial equilibrium in the short-medium term via an increase in the number of new members can be harmful in the long run, when contributors in the preceding period will become pensioners. If the ratio contributors/pensioners was to decrease in the future, due to an increase of an older population, there would be *ceteris paribus* an increase of financial burden for the pension system; thus creating a disequilibrium in the pension scheme and the risk of financial difficulties.

Public PAYG pension schemes are generally opened to different professional groups, in order to prevent financial unbalances due to a decline in a specific profession (and the consequent decline in its contributors). Therefore, variations in the number of contributors are mostly influenced by changes in the age structure of the population. On the other hand, private PAYG pension schemes can choose to admit only a homogeneous class of people (e.g. employees of a specific firm, workers with given professional qualification, etc.). For such "closed" pension funds, the demographic risk related to the variable 'new entrants' is relevant because changes in the job market may influence the number of contributors. This is the case of the self-administered PAYG pension funds of Italian professional categories.

Until the early nineties, social security and pension disbursement in Italy were publicly funded and administered. Nearly all citizens, regardless of their income, were entitled to a pension that allowed them to approximately maintain the same socio-economic status they enjoyed while employed. The right to a pension was generously guaranteed as prescribed by the article 38 of the Italian Constitution. In 1994 the Legislative Decree No. 509 was passed, calling for the privatisation of certain sectors of social security and pension administration. Any professional group organized as an Order/Board (such as lawyers, accountants, engineers, doctors, pharmacists, etc.) was to create and administer its own retirement fund. Members who work autonomously would deposit portions of their incomes during their working years, and receive pensions upon retirement; meanwhile, members who work as employees would still be entitled to public pension cover. These new financial institutions, called *Casse di Previdenza e Assistenza dei Liberi Professionisti*, have been no longer dependent on governmental assistance. The change meant that, should a given fund reach a negative balance, there would be no more financial backup from the public finances; thus retirees would have no pensions available to them.

In this perspective, the present study addresses the evaluation of demographic risk related to the variable 'new entrants' in PAYG pension systems. It starts with a brief review of recent literature on pension fund risk management (Sect. 2) and continues with the mathematical formalization of the problem (Sect. 3) and an application to the *Cassa Nazionale di Previdenza e Assistenza dei Dottori Commercialisti* (Sects. 4 and 5). Finally, conclusions are drawn (Sect. 6).

2 Risk management in pension funds: state of the art

Over the last few years a vast literature regarding management and regulation of risk in pension systems has developed. The main topics on quantitative research have been the development of stochastic models for longevity risk and global asset return. Recent financial scandals have also improved research on governmental regulations for life insurance institutions.

An introduction to longevity risk with a comprehensive literary review can be found in [38] and [39]. Rigorous analyses of mortality projections have been conducted by Lee and Carter in [23], Benjamin and Pollard in [5], Benjamin and Soliman in [6], Haberman and Renshaw in [19], Lee in [24], Olivieri in [29], Thatcher et al. in [45] and Olivieri and Pitacco in [34]. Joint analyses of both financial and longevity risks have been proposed by Olivieri and Pitacco in [31] and by Coppola et al. in [13]. The securitisation of mortality risk has been analysed by Lin and Cox in [25] and by Cairns et al. in [10].

Several stochastic models for global asset return in pension funds have been proposed; see for example Parker in [37], Cairns and Parker in [11], Blake et al. in [8] and [9]. Mandl and Mazurova in [26] use spectral decomposition of stationary random sequences for assessing defined benefit pension schemes under randomly fluctuating rates of return and numbers of entrants. Haberman in [17] identifies a ‘contribution rate risk’ and considers as stochastic components both rate of return and contribution rate. Gerrard et al. in [16] analyse the financial risk faced by members of defined contribution schemes both during the service period and after retirement.

Stochastic analyses of new entrants in private pension schemes have been proposed by Janssen and Manca in [20] and by Colombo and Haberman in [12]. Sinn in [42], [43] and Abio et al. in [1] consider the age structure of future national population as a prime risk factor in PAYG public pension systems. Angrisani et al. in [3] propose a demographic model for studying the impact on PAYG pension systems of future developments of the population. Bianchi et al. in [7] conduce joint demographic and behavioural analyses via dynamic microsimulation to test the economic effects of pension reforms.

A vast literature on risk management policies has been developed following defaults on life insurance sector; see for example Plantin and Rochet in [40]. A debated point is whether competition among pension funds and moral hazard can expose funds to excessive risks that are not compatible with their social not-speculative function; see for example McClurken in [28]. Bader in [4] suggests that pension funds should avoid investing in specific sectors in the stock market. Ryan and Fabozzi in [41] study the defaults of US pension funds due to actuarial losses and not to wrong portfolio investments. Trudda in [47] shows that marginal increments in global asset return appear to strongly reduce the default probability of the pension fund of an Italian professional Order, thus generating an incentive to take superfluous risks in case of lacking of regulations. Otranto and Trudda in [36] urge the need for a risk rating system for pension funds and propose a cluster analysis based on GARCH volatility of their rates of return.

3 A model for the evaluation of new entrants to a pension fund with restricted entrance

The Population-Education-Profession (P-E-P) model, that we propose here, is a discrete-time stochastic model for the estimation of the new entrants in a pension fund with restricted entrance, such as that of a professional category. The model is based on the study of variables related to the demographic evolution of the population, the development of university instruction and the attraction of the profession. It can be used, with appropriate simplifications, to forecast the entrants in any kind of pension scheme. To the best of our knowledge, it is the first stochastic model specifically designed for the estimation of new members of a professional category (and, subsequently, of its pension fund). An early version of the model and a deterministic application are proposed in [15] and [47].

The intergenerational patterns of employment in a given professional group depend on different specific variables, both demographic (trend in population, trend in study choices, etc.) and economic (appeal of the profession, appeal of the firm, expected income, etc.). Thus, for a correct estimation of the future contributors to such a "closed" pension fund we should address the following questions:

- What will the demographic evolution of the reference population be?
- What are the trends in the choices of study regarding the specific profession?
- What is the attraction of the profession (or of the firm, in case of corporate pension fund)?
- How is the admission to the pension fund regulated (e.g. elective/compulsory entrance)?

The stages that a potential contributor has to leave behind before entering in the pension fund of a professional category have been represented in the Markov chain in Fig. 1. Accordingly, we propose a model based on subsequent estimations of the population in different stages, as described in Table 1. The Markov chain is composed by the following states:

1. Belonging to the cohort of reference (e.g. Italian population aged 18-25);
2. Having a high school diploma;
3. Being enrolled in the required course of study (e.g. Bachelor degree in Law);
4. Being graduated in the specific degree;
5. Starting the training period required for taking the admission exam;
6. Becoming a member of the professional category (e.g. becoming a Lawyer);
7. Joining the pension fund of the professional category.

The $p_{ij}(t)$ in the Markov chain represents the probability of transition from state i to state j at time t; h and k are the expected lengths for successfully completing, respectively, the course of study and the professional training period. At time t, each potential future contributor can only be in one state. An individual can move to a greater state exclusively after fixed time periods (0, h or k time units) depending on the state itself.

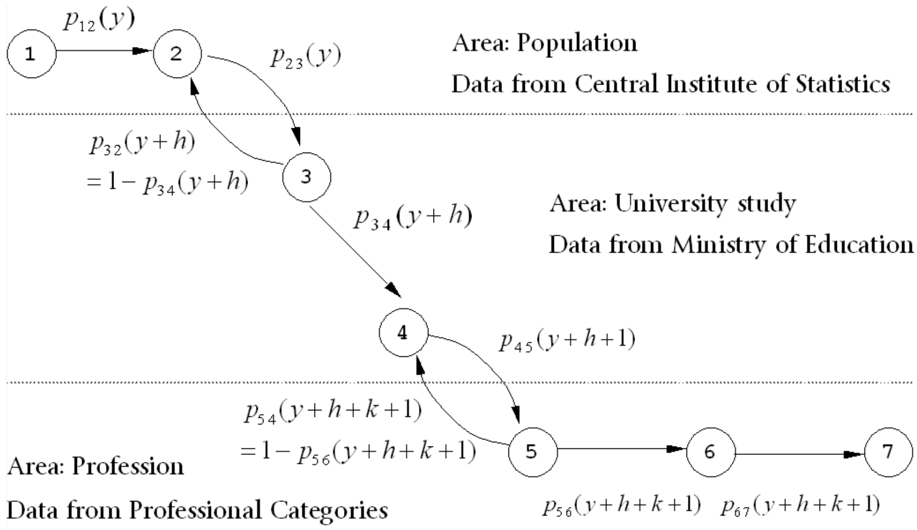

Fig. 1. Markov chain for the estimation of future entrants to the pension fund of a professional category

Let us introduce the stochastic process $POP(t)$ representing the population in state 1 at time t, which is, in other words, the starting population of the Markov chain, defined as:

$$POP(t) = \sum_{x=m}^{M} \overline{POP}_x(t) \ + \ \sigma_{pop}\,\epsilon \tag{1}$$

where $\overline{POP}_x(t)$ is the expected value of the stochastic process $POP_x(t)$ representing the national population of age x at time t; σ_{pop} is the standard deviation of $\sum POP_x(t)$; m and M are integer numbers indicating, respectively, the minimum and maximum age of the cohorts considered in the potential population; ϵ is a standard normal variable. Equation 1 holds for $\sum POP_x(t) \geq -\sigma_{pop}\,\epsilon$ because population must be equal to a non-negative number.

Let $P_{ij}(t)$ be a stochastic process defined as:

$$P_{ij}(t) = \begin{cases} 0 & if \ \ \epsilon < -\frac{\overline{p}_{ij}(t)}{\sigma_{ij}} \\ \overline{p}_{ij}(t) + \sigma_{ij}\,\epsilon & if \ \ \epsilon \geq -\frac{\overline{p}_{ij}(t)}{\sigma_{ij}} \end{cases} \tag{2}$$

where $\overline{p}_{ij}(t)$ is the expected value of the probability of transition $p_{ij}(t)$, σ_{ij} is its standard deviation, and ϵ is a standard normal variable. The $P_{ij}(t)$ can be seen as an approximation of the transition probability $p_{ij}(t)$, but not as a probability in itself because it can assume values higher than one.

Let $NE(t)$ be a Markov process representing the number of new entrants to the pension fund at time t, defined as:

$$NE(t) = POP(t-h-k)P_{12}(t-h-k)P_{23}(t-h-k)\cdot$$

Table 1. Stages for estimating new entrants to the pension fund of a professional category

Population of potential university students ↓	Regarding trends in population
Number of enrolments at university ↓ Enrolments in the specific field of study ↓ Graduations in the specific field of study ↓	Regarding trends in education
Graduates who start the training period ↓ Trainees who complete the training period ↓ New members of the professional category = Trainees who pass the admission exam ↓	Regarding the appeal of the profession
New entrants to the pension fund	Regarding entrance regulations

$$\cdot P_{34}(t-k)P_{45}(t-k)P_{56}(t)P_{67}(t). \tag{3}$$

To make the Markov chain more compliant with generally available statistics, we simplify it by merging some steps. Thus, we focus on the stochastic processes $P_{13}(t)$, $P_{34}(t)$, $P_{46}(t)$ and $P_{67}(t)$, and we rewrite (3) as:

$$NE(t) = POP(t-h-k)P_{13}(t-h-k)P_{34}(t-k)P_{46}(t)P_{67}(t). \tag{4}$$

Equation (4) represents the fundamental formula of the P-E-P Model. For sufficiently small values of the ratios $\overline{p}_{ij}(t)/\sigma_{ij}$ and assuming the independence of the random processes, it also results:

$$NE(t) \approx [\sum_{x=m}^{M} \overline{POP}_x(t-h-k) + \sigma_{pop}\,\epsilon\,][\overline{p}_{13}(t-h-k) + \sigma_{13}\,\epsilon\,]\cdot$$
$$\cdot\,[\overline{p}_{34}(t-k) + \sigma_{34}\,\epsilon\,][\overline{p}_{46}(t) + \sigma_{46}\,\epsilon\,][\overline{p}_{67}(t) + \sigma_{67}\,\epsilon\,] \tag{5}$$

$$E[NE(t)] \approx \sum_{x=m}^{M} \overline{POP}_x(t-h-k)\;\overline{p}_{13}(t-h-k)\;\overline{p}_{34}(t-k)\;\overline{p}_{46}(t)\;\overline{p}_{67}(t) \tag{6}$$

and:

$$Var[NE(t)] \approx \{[\sum_{x=m}^{M} \overline{POP}_x(t-h-k)]^2 + \sigma_{pop}^2\}[\overline{p}_{13}(t-h-k)^2 + \sigma_{13}^2]\cdot$$
$$\cdot\,[\overline{p}_{34}(t-k)^2 + \sigma_{34}^2][\overline{p}_{46}(t)^2 + \sigma_{46}^2][\overline{p}_{67}(t)^2 + \sigma_{67}^2]. \tag{7}$$

In pension funds of professional categories, the number of future new entrants can be influenced by different variables, related to reference population, education choices, school completion, professional appeal and entrance regulations. The Population-Education-Profession model analyses such variables with five main stochastic processes. Indeed, $POP(t)$ represents the national population who fulfil the university age requirements, $P_{13}(t)$ the rate of diffusion of university studies among the population, $P_{34}(t)$ the success rate in university study, $P_{46}(t)$ the admission rate in the professional category, and $P_{67}(t)$ the inscription rate to the category's pension fund.

All of the parameters of the model can be estimated according to data that is easily available to professional categories, coming from the Central Institute of Statistic, from the Ministry of Education and from Professional Associations. Assuming that the period between $t-1$ and t corresponds to the calendar year y, we can approximate $P_{13}(t)$ as the ratio between university enrolments (in the specific field of study) and the starting population at calendar year y. Accordingly, we can estimate $P_{34}(t)$ as the ratio between graduations at year y and enrolments in previous year $y-h$; $P_{46}(t)$ as the ratio between new members of the professional category at year y and graduates at previous year $y-k$; $P_{67}(t)$ as the ratio between new entrants to the pension fund and new members of the category at year y. Such ratios can sometimes have values higher than 1 (e.g. because of university reforms, of changes in the fund's entrance regulations, etc.) and this is reflected in the definition of $P_{ij}(t)$.

The P-E-P model makes a breakthrough in the evaluation of new entrants to pension funds of professional categories. Classic actuarial techniques are mostly based on the time series analysis of past membership trends; this approach can underestimate the probability of sudden changes in the demographic dynamics, because it only considers the past outcome (new entrants time series) but not how this outcome had been generated. On the contrary, the P-E-P model can evaluate the risk of abrupt changes in the different variables that influence the number of future members (e.g. due to university reforms, decline in the professional appeal, etc.). Accordingly, it can assign different levels of demographic risk in funds that presented similar new entrants time series.

4 A numerical application of the P-E-P model

We used the P-E-P model for estimating the demographic evolution of the *Cassa Nazionale di Previdenza e Assistenza dei Dottori Commercialisti* (CNPADC). The CNPADC is the private self-managed pension fund of Italian Chartered Accountants (*Ordine dei Dottori Commercialisti*) set up by Law n. 335/1995. It is a cash-balance pension fund financed with a pay-as-you-go system, so the money collected during the year from contributors is immediately used for paying the same year's pensions, and only partially saved.

The demographic structure of the CNPADC is that of a “young” retirement fund, meaning that it has still not reached the long-term natural balance between the number of contributors and pensioners. This is a consequence of the huge increase in new memberships that has occurred since the mid-Nineties. In the period 1995-2005, there has been an increase of more than 100% in the total number of members, that have risen from 21,762 to 44,706; the ratio workers/pensioners has increased from 7 to 9.6; instead the ratio between contribution revenues and benefit expenses has been nearly constant, passing from 2.74 to 2.72.

The values of $\overline{POP}_x(t)$ for the years 1998-2006 (thus, influencing the number of new contributors in 2007-2015) have been taken from the official estimates of the Italian population published by the Italian Institute of Statistics (ISTAT). The values of $\overline{POP}_x(t)$ and σ_{pop} for the period 2007-2050 have been estimated on the basis of the National Demographic Forecasting of Italian population published by ISTAT.

The required degrees for being admitted to the Order of Chartered Accountants have been considered the *Laurea* (*triennale, specialistica* and *vecchio ordinamento*) in Economics, Management and Business Administration. We focus on the period that has followed the main reform of the Italian university system in decades, which has introduced the three-years Bachelor degree. The value of h, the average time for completing university has been considered equal to 5 years. The value of k, the average time for completing the professional training and passing the exam, has been considered equal to 4 years. We assume $m = 18$ and $M = 25$, basing such assumption on the fact that more than 95% of university population in Italy is composed by students aged 18-25, and this ratio has been reasonably constant over time in the last two decades.

The moments of $P_{13}(t)$ and $P_{34}(t)$ have been estimated according to historical data from ISTAT and the Italian Ministry of University (MIUR). The moments of $P_{46}(t)$ and $P_{67}(t)$ have been estimated according to historical data from MIUR, CNPADC and *Fondazione Aristeia*. For the evaluation of $P_{67}(t)$ we have also considered historical data on cancellations from the fund, so that the estimates of new entrants can be considered net of cancellations.

Estimates of new entrants to the CNPADC fund have been made with the P-E-P model and tested with the Monte Carlo method. With 10,000 simulations, we have drawn the probability distribution of the future entrants, divided by sex, for the period 2006-2059. Tables 2 and 3 present, respectively, the parameters of the model and the expected results. Figures 2, 3 and 4 present the percentiles of the frequency distributions for new entrants obtained in the simulations (respectively: total, females and males).

Table 2. CNPADC pension fund: parameters of the P-E-P model

	$\overline{p}_{13}(t)$	σ_{13}	$\overline{p}_{34}(t)$	σ_{34}	$\overline{p}_{46}(t)$	σ_{46}	$\overline{p}_{67}(t)$	σ_{67}
Females	0.0085	0.0007	0.5110	0.1996	0.0811	0.0291	0.6261	0.1088
Males	0.0090	0.0005	0.5110	0.1996	0.0893	0.0320	0.6388	0.1108

Notably, the highest values of new members are in the period 2006-2012. This is an effect of the reform of the university system, started in the academic

Table 3. New members of the CNPADC fund, expected values, years 2006-2059

Year	Mal.	Fem.	Tot.	Year	Mal.	Fem.	Tot.	Year	Mal.	Fem.	Tot.	Year	Mal.	Fem.	Tot.
2006	1,119	915	2,034	2020	595	516	1,111	2034	592	511	1,103	2048	561	447	1,008
2007	1,330	1,088	2,418	2021	592	513	1,105	2035	593	512	1,106	2049	556	443	999
2008	1,509	1,235	2,744	2022	591	511	1,102	2036	593	512	1,105	2050	552	440	992
2009	1,565	1,280	2,845	2023	589	510	1,099	2037	592	511	1,103	2051	549	437	986
2010	1,280	976	2,256	2024	585	506	1,091	2038	590	508	1,098	2052	546	435	981
2011	1,166	904	2,071	2025	582	503	1,085	2039	586	504	1,090	2053	544	434	978
2012	896	724	1,621	2026	578	499	1,077	2040	578	497	1,076	2054	543	432	975
2013	713	573	1,286	2027	576	497	1,072	2041	570	490	1,060	2055	542	432	974
2014	650	561	1,210	2028	574	495	1,068	2042	562	484	1,046	2056	542	432	974
2015	621	541	1,162	2029	574	495	1,068	2043	554	476	1,030	2057	542	432	974
2016	613	533	1,146	2030	576	497	1,072	2044	546	470	1,016	2058	542	432	974
2017	606	526	1,132	2031	579	500	1,078	2045	539	464	1,003	2059	543	433	976
2018	600	521	1,121	2032	584	504	1,088	2046	574	458	1,032				
2019	597	518	1,114	2033	588	508	1,106	2047	568	452	1,020				

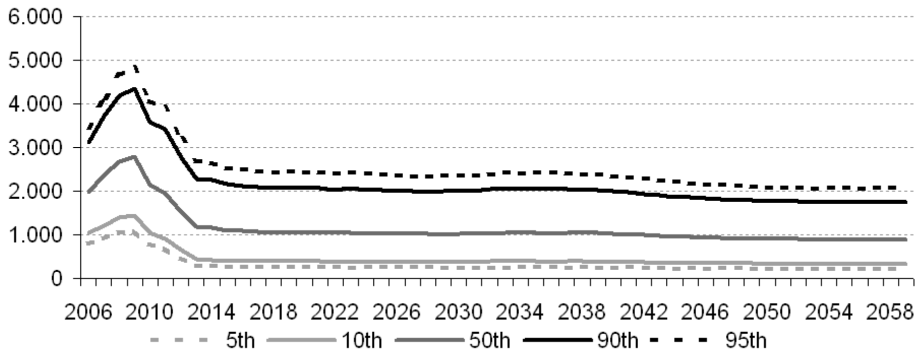

Fig. 2. Total new entrants to the CNPADC pension fund, percentiles of the frequency distributions, years 2006-2059

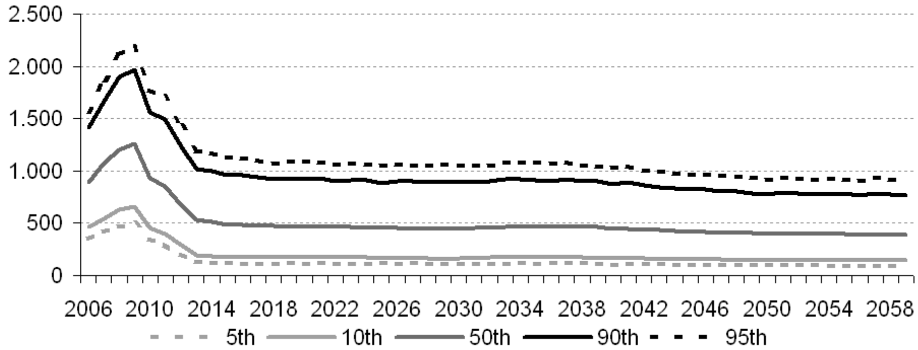

Fig. 3. Female new entrants to the CNPADC pension fund, percentiles of the frequency distributions, years 2006-2059

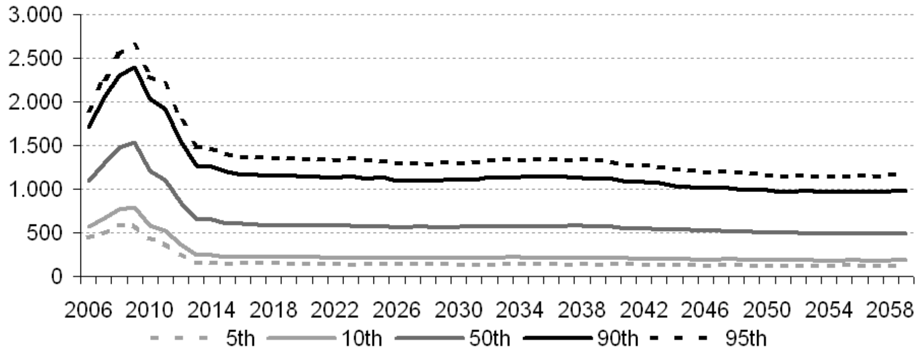

Fig. 4. Male new entrants to the CNPADC pension fund, percentiles of the frequency distributions, years 2006-2059

year 2000/01; students who did not previously complete their *Laurea* (which required from 4 to 6 years, depending on the field of study) have been allowed to be re-enrolled to an equivalent post-reform Bachelor degree (*Laurea triennale*) without having to re-sit for the exams that they had already passed. Thus, a large number of students, who had previously left university without completing their 4-6 years program, have quickly obtained a Bachelor degree. This phenomenon is reflected in the estimates of $P_{13}(t)$ and $P_{34}(t)$, which indicate that it will cease its effect from 2008 (thus, affecting the number of new contributors until 2012).

5 An analysis of the financial dynamics of the CNPADC fund

5.1 The model

The aim of this section is to estimate the effects of the variable 'new entrants' on the financial dynamics of the CNPADC retirement fund in the period 2006-2046.

The fund value V_t, corresponding to the value of the net assets belonging to the fund at time t, has been modelled with the following recursive equation:

$$V_t = V_{t-1} \cdot (1 + r_t) + C_t + R_t - B_t - E_t \tag{8}$$

where r_t is the nominal annual rate of return, C_t, B_t and E_t represent respectively the amounts of contribution income, pension disbursement and administrative expenses generated in the period $[t-1,\ t]$. All of the cash flows are assumed to take place at the end of each period as this is more consistent with the fund's regulations.

5.2 Contribution income

The annual contribution income at time t has been estimated with the following equation:

$$C_t = \sum_{g=1}^{G} \sum_{s=1}^{S} \sum_{x=\alpha+1}^{\pi} \sum_{a=1}^{A} c_{g\,s\,x\,a}(t) \cdot N_{s\,x\,a}(t)\,,$$
$$\forall (x,\, a) \in N \times N - \{x > \hat{x}_{b\,s\,t} \wedge a \geq \hat{a}_{b\,s\,t}\} \tag{9}$$

where $c_{g\,s\,x\,a}(t)$ is the average contribution of type g paid at time t by an individual of sex s, age x and seniority a, and $N_{s\,x\,a}(t)$ is the number of members of the fund alive at time t of sex s, age x and seniority a. The term G represents the number of types of contributions, S the number of sex categories, α and π the minimum and maximum potential age of contributors, and A the maximum potential seniority. The terms $\hat{x}_{b\,s\,t}$ and $\hat{a}_{b\,s\,t}$ represent the retirement requirements of age and seniority, in force at time t, for members of sex s to be entitled to a benefit of type b; thus, the $N_{s\,x\,a}(t)$ considered in equation (9) are cohorts of active members.

It also results:

$$c_{g\,s\,x\,a}(t) = \gamma_{g\,x\,a\,t} \cdot R_{g\,s\,x\,a}(t) \tag{10}$$

where $\gamma_{g\,x\,a\,t}$ and $R_{g\,s\,x\,a}(t)$ represent respectively the contribution rate and the expected income amount for the determination of the contribution of type g due at time t by individuals of sex s, age x and seniority a.

In the application we consider two main types of contributions of the fund: the *soggettivo* and the *integrativo*, determined annually as shares of, respectively, professional income and sales subject to Value Added Tax.

5.3 Pension disbursement

The annual pension disbursement at time t has been estimated with the following equation:

$$B_t = \sum_{d=1}^{D} \sum_{s=1}^{S} \sum_{x=\beta+1}^{\omega} \sum_{a=1}^{A} b_{d\,s\,x\,a}(t) \cdot N_{s\,x\,a}(t)\,,$$
$$\forall (x,\, a) \in N \times N : \{x > \hat{x}_{b\,s\,t} \wedge a \geq \hat{a}_{b\,s\,t}\} \tag{11}$$

where $b_{d\,s\,x\,a}(t)$ is the average contribution of type d received at time t by a pensioner of sex s, age x and seniority a, and $N_{s\,x\,a}(t)$ is the number of members of the fund alive at time t of sex s, age x and seniority a. The term D represents the number of types of benefits, β and ω the minimum and maximum potential age of pensioners. The terms $\hat{x}_{b\,s\,t}$ and $\hat{a}_{b\,s\,t}$ represent the retirement requirements of age and seniority in force at time t; thus, the $N_{s\,x\,a}(t)$ considered in equation (11) are cohorts of retired members.

In the application we consider the two main types of benefits of the fund: the *pensione di vecchiaia* and the *pensione unica contributiva*, computed with a pro-rata mechanism in accordance with CNPADC regulations as amended by the statutory reform of 2004.

5.4 Mortality rate

Let $q_{s\,x}(t)$ be the probability at time t that an individual of sex s and age x, still alive at time t, will die before time $t+1$, defined as:

$$q_{s\,x}(t) = \begin{cases} 0 & if \quad \epsilon < -\frac{\overline{q}_{s\,x}(t)}{\sigma_{s\,x}} \\ \overline{q}_{s\,x}(t) + \sigma_{s\,x}\,\epsilon & if \quad -\frac{\overline{q}_{s\,x}(t)}{\sigma_{s\,x}} \leq \epsilon \leq \frac{1-\overline{q}_{s\,x}(t)}{\sigma_{s\,x}} \\ 1 & if \quad \epsilon > \frac{1-\overline{q}_{s\,x}(t)}{\sigma_{s\,x}} \end{cases} \tag{12}$$

where $\overline{q}_{s\,x}(t)$ and $\sigma_{s\,x}$ represent respectively the expected value and the standard deviation of $q_{s\,x}(t)$, and ϵ is a standard normal variable.

We model $\overline{q}_{s\,x}(t)$ as:

$$\overline{q}_{s\,x}(t) = (1 + \mu_{s\,x})^{t-t_0} \cdot q_{s\,x}(t_0) \tag{13}$$

where $q_{s\,x}(t_0)$ and $\mu_{s\,x}$ represent respectively an initial known value and the expected annual rate of change of $q_{s\,x}(t)$.

The aim of the mortality model is to evaluate the impact of the accidental component of longevity risk on the financial dynamics of the CNPADC fund. Such component is due to random deviations from the expected mortality values. It is a simple stochastic model for which parameters can easily be estimated from official data published by national institutes of statistics. The discrete time approach has been preferred since the time unit in the application is the year.

5.5 Rate of return

Let r_t be a stochastic process representing the annual nominal rate of return defined as:

$$r_t = \overline{r}_t + X_t \tag{14}$$

where $\overline{r}_t$ is the expected nominal rate of return in the period $[t-1,\, t]$ and X_t is an $AR(1)$ process defined as:

$$X_t = \varphi X_{t-1} + \sigma\,\epsilon \tag{15}$$

with $-1 < \varphi < 1$, where φ and σ are the parameters of the process, and ϵ is a standard normal variable. The proposed model represents a discrete form of the Vasicek model, with properties of normality, stationarity, mean reversion, and finite variance. The prudential asset allocation of the CNPADC fund is compatible with such properties.

The approach based on the Vasicek model seems to be particularily suitable for describing the rate of return in first pillar pension funds such as the CNPADC. It takes into account the possibility of obtaining negative values, which is a desirable feature when modelling the rate of return. Indeed, pension funds of Italian professional Orders can occasionally suffer financial losses although their social not-speculative function. This has happened for example in 2008.

5.6 Technical assumptions

The model has been employed with the following assumptions.

Demographic hypotheses:

- Effective population of pensioners and contributors on the 1st of January 2006, according to data from the CNPADC, divided by sex, age and seniority.
- Future entrants determined according to the P-E-P model as stated in the previous section, age of entry 29.
- Initial mortality rates $q_{s\,x}(t_0)$ equal to rates of the 2006 Italian mortality table published by ISTAT; values of $\mu_{s\,x}$ and $\sigma_{s\,x}$ estimated according to data from ISTAT on Italian mortality in the period 1981-2006.

Financial hypotheses:

- Fund's net assets equal to 2,067,793,989 euros on the 1st of January 2006, according to the 2005 financial report.
- Administrative costs of year 2006 equal to 28,447,830 euros, appreciated in the following years at 5% nominal annual rate according to technical assumptions in [2].
- Parameters for the estimation of the rate of return are $X_0 = 0$, $\varphi = -0.612$ and $\sigma = 0.03667$, according to results obtained by Melis in [27] for Italian fixed income investment funds.
- Inflation rate equal to Italian Government's expectations exposed in the budget *DPEF 2007-2011*, thus equal to 2% in 2006, to 1.7% in 2007, to 2.1% in 2008, to 1.9% in 2009, and to 1.6% in the following years.

Contributions:

- Two types of contributions (*soggettivo* and *integrativo*) determined in accordance with CNPADC regulations. New entrants exercise the statutory right of exemption from contributions for the first 3 years.
- Annual professional incomes and VAT sales equal, for each cohort of same sex and age, to the effective average values registered in 2005, appreciated at inflation rate.
- Subjective contribution rate equal to 10.7%[1] of annual professional income; sums paid under the defined-contribution scheme accrued at 3.4% nominal annual rate.
- Integrative contribution rate equal to 4% in 2006-2010 and successively to 2% of annual professional VAT sales, according to the statutory reform of 2004.

Pensions:

- Benefits paid to pensioners who retired before the 1st of January 2006 equal to the effective average values registered in 2006.

[1] Subjective contribution rate varies electively between 10% and 17% of annual professional income. In 2005 the average rate was 10.71%.

- Two types of benefits (*pensione di vecchiaia* and *unica contributiva*) for pensioners who retire after the 1st of January 2006, determined with a pro-rata mechanism in accordance with CNPADC regulations as amended by the statutory reform of 2004.
- All benefits appreciated annually at the inflation rate. Each cohort of contributors retires immediatly after fulfilling the requirements. We do not consider benefit reversion to survivors.

5.7 Results

The probabilistic structure of the fund value has been estimated with stochastic simulation based on Monte Carlo techniques. This approach generates a range of outcomes which represents a probability distribution, conditional on the assumptions made. The number of outcomes has been 10,000 for each test made.

In the first test, the three main risk factors — mortality, new entrants and rate of return — are considered as stochastic variables. Results are presented in Fig. 5 and indicate that, with 99.9% confidence level, the CNPADC fund will maintain a positive value in the forecasting period. The probability distribution of the fund value is nearly standard, slightly leptokurtic and right-skewed. The values of skewness and kurtosis tend to increase with the passing of time, as demonstrated in Tab. 4.

The probability distribution of the total balance presents a large variation; its median value reaches a peak of 900 millions in 2026 and then decreases, reaching its minimum, 15 millions, in 2042, as demonstrated in Fig. 6. The percentile distribution of the pension balance indicates that this value will turn negative in the period between 2033 and 2037, with 99.9% confidence level, because of the ageing of the population; its probability distribution presents a low dispersion around the median values, as demonstrated in Fig. 7. Returns on investments are expected to partially cover the increase in pension disbursement, thus preventing from abrupt slumps in the total balance.

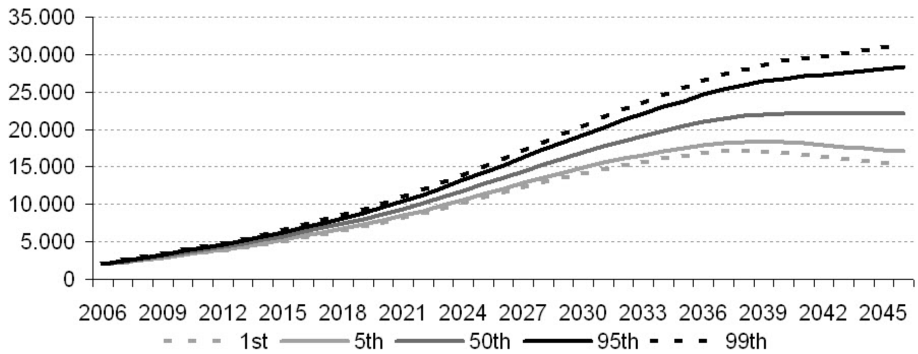

Fig. 5. Value of the CNPADC pension fund in million euros, percentiles of the frequency distributions, years 2006-2046

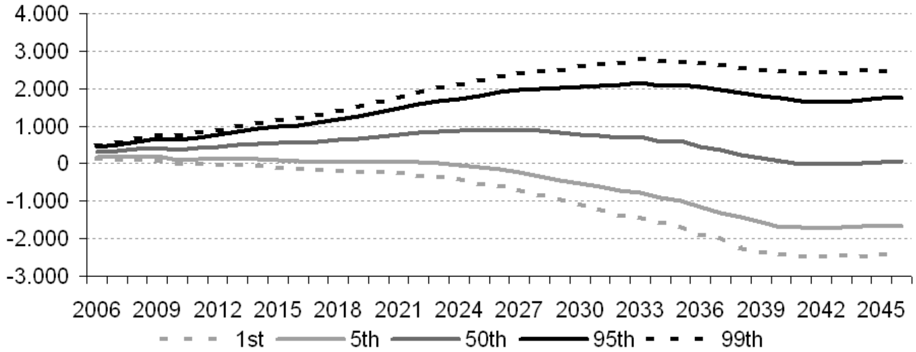

Fig. 6. Total balance of the CNPADC pension fund in million euros, percentiles of the frequency distributions, years 2006-2046

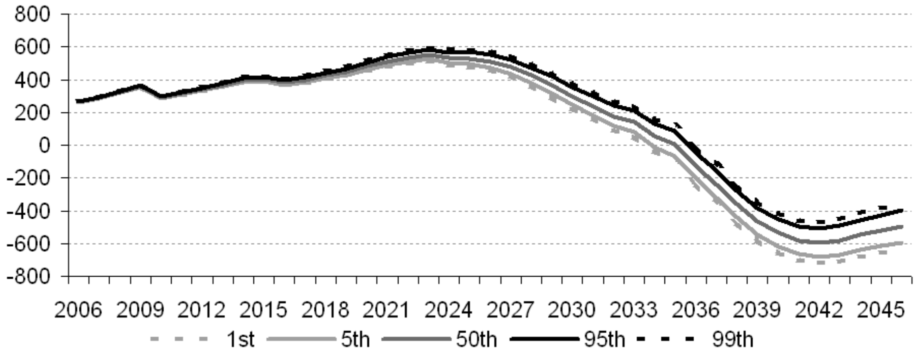

Fig. 7. Pension balance of the CNPADC pension fund in million euros, percentiles of the frequency distributions, years 2006-2046

We have conducted three other tests in which each risk factor — mortality, rate of return and new entrants — is considered as stochastic variable while the others are assumed deterministic. Finally, one last simulation has estimated the expected financial dynamics of the fund considering all variables as deterministic. Results have been used to conduct a sensibility analysis of the fund, with the following conclusions.

The effects on the financial dynamics of the fund of random deviations from given mortality trends (that is, the accidental component of longevity risk) are negligible, and account for less than 1% of dispersion around the median value. Indeed, this is a pooling risk and its effects tend to disappear in large pension funds such as the CNPADC.

Most of the variance in the fund value distribution seems to be described by the stochastic behaviour of the rate of return. This can be inferred by examining the similarities between the percentile distribution obtained in the first test, with all the three risk factors considered as stochastic, and the percentile distribution obtained under the hypotheses of stochastic rate of return and deterministic mortality and new entrants. The values are presented in Figs. 5 and 8.

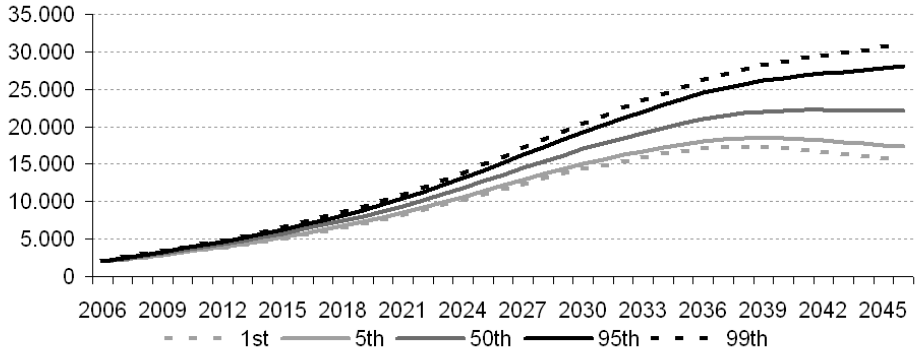

Fig. 8. Value of the CNPADC pension fund in million euros, obtained considering stochastically only the variable 'rate of return', percentiles of the frequency distributions, years 2006-2046

Finally, the CNPADC fund seems to have a relatively low exposure to the risk related to future new entrants. This is suggested by the large impact of the rate of return on the percentile distribution of the fund value. Nonetheless, the stochastic behaviour of new entrants describes almost all of the variation in the pension fund balance. This can be deduced by examining the similarities between the variance in the pension balance distribution, obtained in the first test, and the fund value distribution obtained under the hypotheses of stochastic new entrants and deterministic mortality and rate of return. The values are presented in Figs. 9 and 7.

6 Conclusions

In the present paper we have addressed the issue of the demographic risk related to future membership patterns in retirement funds with restricted entrance, financed under a pay-as-you-go scheme.

We have proposed a discrete-time Markov model for the estimation of new entrants in pension funds of professional categories, that highlights the interactions between demographic, economic and regulatory variables. The model considers the effects of trends in population, trends in education choices, appeal of the profession, and entrance regulations.

Numerical applications of the model have analysed the demographic and financial dynamics of the pension fund of Italian Chartered Accountants. Demographic results have revealed the effects of a main reform in the university

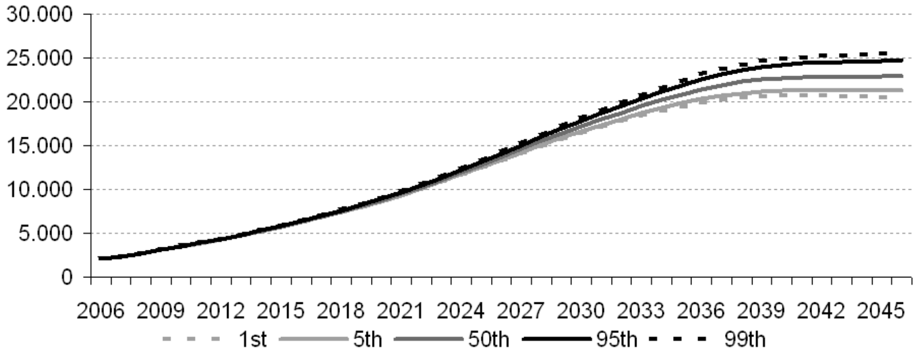

Fig. 9. Value of the CNPADC pension fund in million euros, obtained considering stochastically only the variable 'new entrants', percentiles of the frequency distributions, years 2006-2046

system on the number of new entrants to the pension fund. Financial results suggest that the fund has a relatively low exposure to the risk related to future new entrants, and that its main risk factor is the rate of return. Instead, the risk of random deviations from expected mortality trends generates negligible effects because of the large population of the fund.

Table 4. Value of the CNPADC fund: moments of the frequency distributions obtained in the Monte Carlo simulation

Date:	1st Jan. 2010	1st Jan. 2015	1st Jan. 2020	1st Jan. 2025
Avg. Value:	3,525,456,641	5,768,926,483	8,725,501,938	12,728,412,509
St. Deviation:	143,505,878	310,161,641	542,057,540	873,711,092
Skewness:	0.096	0.130	0.196	0.226
Kurtosis:	0.054	0.052	0.119	0.127
Date:	1st Jan. 2030	1st Jan. 2035	1st Jan. 2040	1st Jan. 2045
Avg. Value:	17,045,796,388	20,532,458,231	22,259,140,016	22,292,121,288
St. Deviation:	1,337,223,372	1,930,828,426	2,597,537,506	3,271,388,475
Skewness:	0.237	0.305	0.358	0.365
Kurtosis:	0.116	0.198	0.266	0.234

Including other risk factors constitutes a main area of interest for further extensions. Specifically, the risk of regular deviations from expected mortality trends (that is, the systematic component of longevity risk) could be included.

Table 5. Expected cash flows of the CNPADC pension fund under deterministic assumptions, values in thousand euros, years 2006-2046

Year	Value at 1st January A	Subjective Contrib. B	Integrative Contrib. C	Pension Disburs. D	Pension Balance E=B+C-D	Investm. Returns F	Admin. Costs G	Total Balance H=E+F-G	Value at 31st Dec. I=A+H
2006	2,067,794	235,721	155,133	126,378	264,476	70,305	28,448	306,333	2,374,127
2007	2,374,127	251,469	165,874	127,616	289,727	80,720	29,870	340,577	2,714,704
2008	2,714,704	273,516	180,726	129,717	324,525	92,300	31,364	385,461	3,100,165
2009	3,100,165	296,535	195,999	132,946	359,588	105,406	32,932	432,062	3,532,227
2010	3,532,227	320,021	105,717	135,779	289,959	120,096	34,578	375,476	3,907,703
2011	3,907,703	344,246	113,646	143,858	314,034	132,862	36,307	410,588	4,318,291
2012	4,318,291	369,640	121,979	151,338	340,281	146,822	38,123	448,980	4,767,272
2013	4,767,272	395,262	130,491	158,308	367,446	162,087	40,029	489,504	5,256,776
2014	5,256,776	420,800	139,148	162,362	397,586	178,730	42,030	534,286	5,791,061
2015	5,791,061	442,274	146,518	183,278	405,514	196,896	44,132	558,278	6,349,340
2016	6,349,340	449,171	149,275	216,779	381,667	215,878	46,338	551,206	6,900,546
2017	6,900,546	465,270	155,035	223,675	396,630	234,619	48,655	582,593	7,483,139
2018	7,483,139	486,811	162,532	224,363	424,980	254,427	51,088	628,319	8,111,457
2019	8,111,457	499,487	167,154	220,442	446,199	275,790	53,643	668,346	8,779,803
2020	8,779,803	519,236	174,026	212,218	481,044	298,513	56,325	723,233	9,503,036
2021	9,503,036	535,622	179,714	204,108	511,228	323,103	59,141	775,190	10,278,226
2022	10,278,226	550,441	184,774	199,133	536,082	349,460	62,098	823,443	11,101,670
2023	11,101,670	562,260	188,751	197,256	553,755	377,457	65,203	866,009	11,967,678
2024	11,967,678	550,194	184,773	202,953	532,014	406,901	68,463	870,452	12,838,130
2025	12,838,130	555,862	186,599	212,552	529,909	436,496	71,886	894,519	13,732,649
2026	13,732,649	556,398	186,627	231,957	511,068	466,910	75,480	902,497	14,635,146
2027	14,635,146	553,372	185,563	260,129	478,807	497,595	79,255	897,147	15,532,293
2028	15,532,293	540,360	180,946	295,673	425,633	528,098	83,217	870,514	16,402,807
2029	16,402,807	528,597	177,013	338,932	366,678	557,695	87,378	836,995	17,239,802
2030	17,239,802	514,022	172,243	386,708	299,557	586,153	91,747	793,963	18,033,766
2031	18,033,766	500,528	167,916	429,731	238,712	613,148	96,334	755,526	18,789,292
2032	18,789,292	488,378	164,259	477,128	175,508	638,836	101,151	713,193	19,502,485
2033	19,502,485	507,559	170,759	533,357	144,961	663,084	106,209	701,837	20,204,322
2034	20,204,322	482,169	162,691	587,370	57,490	686,947	111,519	632,918	20,837,240
2035	20,837,240	485,158	163,733	642,333	6,557	708,466	117,095	597,928	21,435,168
2036	21,435,168	443,015	147,875	720,059	-129,169	728,796	122,950	476,677	21,911,845
2037	21,911,845	418,296	139,557	799,093	-241,240	745,003	129,097	374,666	22,286,511
2038	22,286,511	391,102	130,357	885,043	-363,585	757,741	135,552	258,605	22,545,115
2039	22,545,115	364,139	121,175	952,225	-466,911	766,534	142,330	157,293	22,702,409
2040	22,702,409	348,623	115,898	999,893	-535,372	771,882	149,446	87,064	22,789,472
2041	22,789,472	337,174	111,961	1,030,520	-581,385	774,842	156,919	36,539	22,826,011
2042	22,826,011	333,977	110,795	1,041,047	-596,274	776,084	164,764	15,046	22,841,057
2043	22,841,057	336,503	111,606	1,029,905	-581,796	776,596	173,003	21,797	22,862,854
2044	22,862,854	339,549	112,603	1,000,520	-548,368	777,337	181,653	47,316	22,910,170
2045	22,910,170	343,496	113,914	977,384	-519,974	778,946	190,735	68,236	22,978,407
2046	22,978,407	347,699	115,319	960,813	-497,795	781,266	200,272	83,199	23,061,606

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