Parameter estimation for differential equations using fractal-based methods and applications to economics and finance

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Abstract. Many problems from the area of economics and finance can be described using dynamical models. If time is the only independent variable and for which we work in a continuous framework, these models take the form of differential equations (DEs). These models can be studied through the direct problem and the inverse problem. The inverse problem consists of estimating the unknown parameters of the model starting from a set of observational data. We use fractal-based methods to get them. The method will be illustrated through several numerical examples and applications to economical and financial situations.

Keywords. Differential equations, collage methods, inverse problems, parameter estimation, stochastic differential equations, technological change, boat-fishery model.

M.S.C. classification. 65R32, 45Q05, 45R05.
J.E.L. classification. C02, C13, C63.

1 Introduction

Many problems from the area of economics and finance can be described using dynamical models. For them, in which time is the only independent variable and for which we work in a continuous framework, these models take the form of deterministic differential equations (DEs). We may study these models in two fundamental ways: the direct problem and the inverse problem. The *direct problem* is stated as follows: given all of the parameters in a system of DEs, find a solution or determine its properties either analytically or numerically. The *inverse problem* reads: given a system of DEs with unknown parameters and some observational data, determine the values of the parameters such that the system admits the data as an approximate solution. The inverse problem is crucial for the calibration of the model; starting from a series of data we wish to describe them using differential equations in which the parameters have to be estimated from data samples. The solutions to the inverse problems are the estimations of the unknown parameters and we use fractal-based methods to get them.

The paper is organized as follows: in sections 1 and 2 we present the basic results concerning the solution of inverse problems for fixed point equations through the so called "collage theorem." We then present some numerical examples: in section 3 and 4 we analyze inverse problems for two economic models arising in the contexts of technological change and resource management, and in section 5 we show how one can use the "collage method" for solving inverse problems for a class of stochastic differential equations.

2 Fixed point equations and inverse problems through the "collage theorem"

For the benefit of the reader, we now mention some important mathematical results which provide the basis for fractal-based approximation methods. Let us consider the fixed point equation $x = Tx$, where (X, d) is a complete metric space and T a *contractive operator* on X. The *direct problem* for a fixed point equation can be solved through the classical Banach theorem.

Theorem 1. *(Banach) Let (X, d) be a complete metric space. Also let $T : X \to X$ be a contraction mapping with contraction factor $c \in [0, 1)$, i.e., for all $x, y \in X$, $d(Tx, Ty) \leq cd(x, y)$. Then there exists a unique $\bar{x} \in X$ such that $\bar{x} = T\bar{x}$. Moreover, for any $x \in X$, $d(T^n x, \bar{x}) \to 0$ as $n \to \infty$.*

A simple triangle inequality along with Banach's theorem yields the following fundamental result.

Theorem 2. *("Collage Theorem" [2,1]) Let (X, d) be a complete metric space and $T : X \to X$ a contraction mapping with contraction factor $c \in [0, 1)$. Then for any $x \in X$,*

$$d(x, \bar{x}) \leq \frac{1}{1-c} d(x, Tx), \tag{1}$$

where $\bar{x}$ is the fixed point of T.

The *inverse problem* is: given a target element y, can we find an operator T with fixed point $\bar{x}$ so that $d(y, \bar{x})$ is sufficiently small. Thanks to the "Collage Theorem", most practical methods of solving the inverse problem for fixed point equations seek to find an operator T for which the *collage distance* $d(y, Ty)$ is as small as possible.

We now consider the case of random fixed point equations. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A mapping $T : \Omega \times X \to X$ is called a *random operator* if for any $x \in X$ the function $T(\cdot, x)$ is measurable. The random operator T is said to be continuous/Lipschitz/contractive if, for a.e. $\omega \in \Omega$, we have that $T(\omega, \cdot)$ is continuous/Lipschitz/contractive ([11]. A measurable mapping $x : \Omega \to X$ is called a *random fixed point* of the random operator T if x is a solution of the equation

$$T(\omega, x(\omega)) = x(\omega), \quad a.e. \omega \in \Omega. \tag{2}$$

We are concerned about the existence of solutions to such equations. Consider the space Y of all measurable functions $x : \Omega \to X$. If we define the operator $\tilde{T} : Y \to Y$ as $(\tilde{T}y)(\omega) = T(\omega, x(\omega))$ the solutions of this fixed point equation on Y are the solutions of the random fixed point equation $T(\omega, x(\omega)) = x(\omega)$. If the metric d is bounded then the space (Y, d_Y) is a complete metric space (see [4]) where

$$d_Y(x_1, x_2) = \int_\Omega d_X(x_1(\omega), x_2(\omega)) dP(\omega). \tag{3}$$

The following result follows from the completeness of (Y, d_Y) and Banach's fixed point theorem. It states sufficient conditions for the existence of solutions.

Theorem 3. *Suppose that*

(i) for all $x \in Y$ the function $\xi(\omega) := T(\omega, x(\omega))$ belongs to Y,
(ii) $d_Y(\tilde{T}x_1, \tilde{T}x_2) \le c d_Y(x_1, x_2)$ with $c < 1$.

Then there exists a unique solution of $\tilde{T}\bar{x} = \bar{x}$, that is, $T(\omega, \bar{x}(\omega)) = \bar{x}(\omega)$ for a.e. $\omega \in \Omega$.

The inverse problem can be formulated as: given a function $\bar{x} : \Omega \to X$ and a family of operators $\tilde{T}_{\mathbf{a}} : Y \to Y$ find $\mathbf{a}$ such that $\bar{x}$ is the solution of random fixed point equation

$$\tilde{T}_{\mathbf{a}} \bar{x} = \bar{x}, \tag{4}$$

that is,

$$T_{\mathbf{a}}(\omega, \bar{x}(\omega)) = \bar{x}(\omega). \tag{5}$$

The collage theorem can also be reformulated for this setting, using the same hypotheses as in Theorem 3. In both theorems, hypothesis (i) can be avoided if X is a Polish space.

3 An inverse problem for a technological competition model

A classical technological competition model can be formulated ([10]) as

$$\frac{dx_1}{dt}(t) = f_1(x_1, x_2) = \frac{a_1}{K_1} x_1 (K_1 - x_1 - \alpha_2 x_2)$$
$$\frac{dx_2}{dt}(t) = f_2(x_1, x_2) = \frac{a_2}{K_2} x_2 (K_2 - x_2 - \alpha_1 x_1),$$

where all of the parameters are positive and a_1, a_2, α_1 and α_2 are less than one. Observe that the nonnegative quadrant is invariant. This means that if we start with $(x_1(0), x_2(0)) \geq (0,0)$ then we have $(x_1(t), x_2(t)) \geq (0,0)$ for all time. That is, in the applied meaningful cases, x_1 and x_2, as determined by our model, are always nonnegative. The linearization of the vector field (f_1, f_2) is

$$Df(x_1,x_2) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} a_1 - \frac{a_1}{K_1}(2x_1 + \alpha_2 x_2) & -\frac{a_1 \alpha_2 x_1}{K_1} \\ -\frac{a_2 \alpha_1 x_2}{K_2} & a_2 - \frac{a_2}{K_2}(2x_2 + \alpha_1 x_1)\,. \end{pmatrix}$$

Solving for equilibria, we obtain

$$(0,0), (0,K_2), (K_1,0), \text{ and } (x_1^\star, x_2^\star) = \left(\frac{K_1 - \alpha_2 K_2}{1 - \alpha_1 \alpha_2}, \frac{K_2 - \alpha_1 K_1}{1 - \alpha_1 \alpha_2} \right).$$

And so, evaluating the linearization at each of the equilibrium points, we calculate that

$$Df(0,0) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Since $Df(0,0)$ is positive definite, we know that $(0,0)$ is an unstable equilibrium (a source). On the other hand, at the next two points we obtain

$$Df(0,K_2) = \begin{pmatrix} \frac{a_1}{K_1}(K_1 - \alpha_2 K_2) & 0 \\ -a_2 \alpha_1 & -a_2 \end{pmatrix}$$

and

$$Df(K_1,0) = \begin{pmatrix} -a_1 & -a_1 \alpha_2 \\ 0 & \frac{a_2}{K_2}(K_2 - \alpha_1 K_1) \end{pmatrix}.$$

Each matrix has one negative eigenvalue, with the sign of other one determined by a relationship between K_1, K_2, and one of the α_is. All three of these equilibrium points have at least one component equal to zero, corresponding to one of the competing technologies begin eliminated from the market. The origin is a special equilibrium point, in that we only arrive at it if we start at it: if neither of the two technologies is present at the start, both of them will never appear. The equilibrium $(0, K_2)$ corresponds to technology x_2 triumphing over technology x_1. The single negative eigenvalue corresponds to the case that the market only has technology x_2 at the start. x_1 never appears, so we arrive at an equilibrium state where technology x_2 is the only one in the market. In the case that $K_1 - \alpha_2 K_2 < 0$, it is possible for a market with both technologies present to approach a state where x_2 has eliminated x_1. If $K_1 - \alpha_2 K_2 > 0$, then we can never reach $(0, K_2)$ if we start with both technologies present. Similar remarks can be made about $(K_1, 0)$. However, notice that if we try to make both of these boundary equilibria stable, we require both $K_1 - \alpha_2 K_2 < 0$ and $K_2 - \alpha_1 K_1 < 0$, which means that

$$K_1 < \alpha_2 K_2 < \alpha_2(\alpha_1 K_1) \Rightarrow 1 < \alpha_1 \alpha_2.$$

But this is a contradiction if both α_is are less than one. As a result, we can make at most one of the nontrivial boundary equilibria stable. The final and most interesting equilibrium point can correspond to coexistence of the two technologies in the case that both $x_1^\star$ and $x_2^\star$ are positive. This situation occurs when

$$K_1 - \alpha_2 K_2 > 0 \text{ and } K_2 - \alpha_1 K_1 > 0. \tag{6}$$

These conditions are familiar, corresponding to the case when both boundary equilibria cannot be reached by interior solutions. In this case, we calculate that

$$Df(x_1^\star, x_2^\star) = \begin{pmatrix} -\frac{a_1}{K_1}\frac{K_1-\alpha_2 K_2}{1-\alpha_1\alpha_2} & -\frac{a_1\alpha_2}{K_1}\frac{K_1-\alpha_2 K_2}{1-\alpha_1\alpha_2} \\ -\frac{a_2\alpha_2}{K_2}\frac{K_2-\alpha_1 K_1}{1-\alpha_1\alpha_2} & -\frac{a_2}{K_2}\frac{K_2-\alpha_1 K_1}{1-\alpha_1\alpha_2} \end{pmatrix},$$

with determinant

$$a_1 a_2 \frac{(K_1 - \alpha_2 K_2)(K_2 - \alpha_1 K_1)}{K_2 K_1 (1 - \alpha_1 \alpha_2)} > 0.$$

Since the determinant is positive and $(Df(x_1^\star, x_2^\star))_{11} < 0$, we conclude that if we are in the case where our system exhibits a positive equilibrium then it is asymptotically stable—in fact with basis of attraction the positive quadrant! Notice that if either inequality in (6) is replaced by the corresponding equation then our equilibrium point coalesces with one of the boundary equilibria. If either inequality is in fact negative, then the equilibrium point we are discussing is not physically realizable.

Figure 1 presents a solution trajectory in the case that $K_1 = 320$, $K_2 = 100$, $\alpha_1 = 0.2$, $\alpha_2 = 0.5$, $a_1 = 0.3$, and $a_2 = 0.6$. Note that the inequalities in (6) are satisfied.

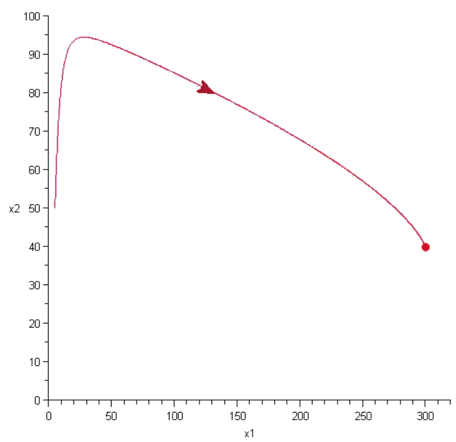

Fig. 1. All solution trajectories starting in the positive quadrant approach the positive equilibrium because (6) holds

Figure 2 presents a solution trajectory in the case that $K_1 = 125$, $K_2 = 320$, $\alpha_1 = 0.2$, $\alpha_2 = 0.5$, $a_1 = 0.3$, and $a_2 = 0.6$. In this case, we have no positive equilibrium, but the equilibrium point $(0, 320)$ is asymptotically stable.

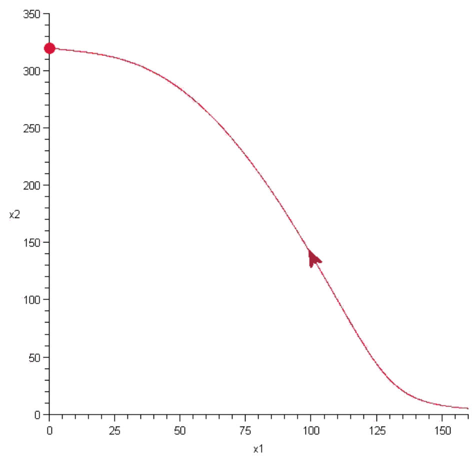

Fig. 2. All solution trajectories starting in the positive quadrant approach $(0, K_2)$ in this case

The inverse problem of interest to us is: given observed values $x_1(t_i)$ and $x_2(t_i)$ for $1 \le i \le N$, say, approximate the values of the parameters a_1, a_2, α_1, α_2, K_1, and K_2.

In [9], [3], [7], [8], [6], the collage theorem presented in Section 2 is used to solve such an inverse problem. Starting from the differential equation,

$$\dot{x} = f(t, x), \quad x(0) = x_0, \tag{7}$$

we consider the Picard integral operator associated with it,

$$(Tx)(t) = x_0 + \int_0^t f(s, x(s))\, ds. \tag{8}$$

If f is Lipschitz in the variable x, that is, $|f(s, x_1) - f(s, x_2)| \le K|x_1 - x_2|$, then T is Lipschitz on the space $C([-\delta, \delta] \times [-M, M])$ with Lipschitz constant $c = \delta K$ [9]. Thus, for δ sufficiently small, T is contractive with respect to the L^2 metric. Now let $\delta' > 0$ be such that $\delta' K < 1$. Let $\{\phi_i\}$ be a basis of functions in $L^2([-\delta', \delta'] \times [-M, M])$, then

$$f_{\mathbf{a}}(s, x) = \sum_{i=1}^{+\infty} a_i \phi_i(s, x). \tag{9}$$

Each sequence of coefficients $\mathbf{a} = \{a_i\}_{i=1}^{+\infty}$, then defines a Picard operator $T_{\mathbf{a}}$. Suppose further that each function $\phi_i(s, x)$ is Lipschitz in x with constants K_i.

Theorem 4. *[9] Let* $\|\mathbf{K}\|_2 = \left(\sum_{i=1}^{+\infty} K_i^2\right)^{\frac{1}{2}}$ *and* $\|\mathbf{a}\|_2 = \left(\sum_{i=1}^{+\infty} a_i^2\right)^{\frac{1}{2}}$. *Then*

$$|f_{\mathbf{a}}(s, x_1) - f_{\mathbf{a}}(s, x_2)| \le \|\mathbf{a}\|_2 \|\mathbf{K}\|_2 |x_1 - x_2| \tag{10}$$

for all $s \in [-\delta', \delta']$ *and* $x_1, x_2 \in [-M, M]$.

Given a target solution $x(t)$, we now seek to minimize the collage distance $\| x - T_{\mathbf{a}}x \|_2$. The square of the collage distance becomes

$$\varDelta(\mathbf{a})^2 = \|x - T_{\mathbf{a}}x\|_2^2 = \int_{-\delta}^{\delta} \left| x(t) - \int_0^t \sum_{i=1}^{+\infty} a_i \phi_i(s, x(s)) ds \right|^2 dt \tag{11}$$

and the inverse problem can be formulated as

$$\min_{\mathbf{a} \in \varLambda} \varDelta(\mathbf{a}), \tag{12}$$

where $\varLambda = \{\mathbf{a} \in \mathbb{R}^{+\infty} : \|\mathbf{K}\|_2 \|\mathbf{a}\|_2 < 1\}$. The minimization may be performed by means of classical minimization methods on a subspace of finite dimension. Of course, the approximation error goes to zero when the dimension goes to infinity.

We apply this approach to our technological competition model inverse problem. We use collage coding, finding the system of the form

$$\frac{dx_1}{dt}(t) = b_1 x_1 + c_1 x_1^2 + d_1 x_1 x_2 \tag{13}$$

$$\frac{dx_2}{dt}(t) = b_2 x_1 + c_2 x_2^2 + d_2 x_1 x_2 \tag{14}$$

for which the corresponding L^2 collage distance is minimized. Having found the coefficients b_i, c_i, and d_i, $i = 1, 2$, we obtain the approximation of the physical parameters via

$$a_i = b_i,\ K_i = -\frac{b_i}{c_i},\ \text{and } \alpha_2 = \frac{d_1}{c_1},\ \alpha_1 = \frac{d_2}{c_2}.$$

Example 1. We set $K_1 = 125$, $K_2 = 320$, $\alpha_1 = 0.2$, $\alpha_2 = 0.5$, $a_1 = 0.3$, and $a_2 = 0.6$, and solve numerically the system of differential equations. We gather observed data by adding low amplitude Gaussian noise to sampled values of the numerical solution. For $x_1(t)$, we gather 100 sample values at the times $t = \frac{i}{100}$, $0 \le i \le 99$; we add normally distributed noise with distribution noise$_1$. We fit a piecewise tenth-degree polynomial to each consecutive set of ten data points to produce our target function for $x_1(t)$. We follow the same procedure to produce a target function for $x_2(t)$, this time with noise distribution noise$_2$. Finally, we minimize the collage distance to recover values of b_i, c_i, d_i, and x_{i0}, from which we recover the approximations of a_i, K_i, and α_i, $i = 1, 2$. The results (to five decimal places) obtained for different noise distributions are presented in Table 1. The values in the table are quite close to the true values, with the accuracy decreasing as the noise is increased.

Table 1. Collage Coding Results for the Technological Competition Model

noise_1	noise_2	a_1	K_1	α_1	a_2	K_2	α_2
$\mathcal{N}(0, 0.02)$	$\mathcal{N}(0, 0.04)$	0.28235	123.72647	0.17625	0.58718	319.86466	0.50565
$\mathcal{N}(0, 0.10)$	$\mathcal{N}(0, 0.15)$	0.26212	122.09029	0.15334	0.57498	319.74032	0.51273
$\mathcal{N}(0, 0.30)$	$\mathcal{N}(0, 0.20)$	0.23672	119.72296	0.14602	0.57147	319.69409	0.52283
$\mathcal{N}(0, 0.50)$	$\mathcal{N}(0, 0.45)$	0.22304	118.25901	0.11801	0.55675	319.55805	0.52880
$\mathcal{N}(0, 0.80)$	$\mathcal{N}(0, 0.75)$	0.20793	116.46787	0.09335	0.54418	319.44395	0.53596
$\mathcal{N}(0, 1.00)$	$\mathcal{N}(0, 1.00)$	0.20028	115.47621	0.07632	0.53568	319.36985	0.53980
$\mathcal{N}(0, 2.00)$	$\mathcal{N}(0, 2.00)$	0.17142	111.14858	0.02406	0.51072	319.15910	0.55635

4 An inverse problem for an economic resource model

We consider a common access fishery model,

$$\begin{aligned} \dot{a}(t) &= \gamma(\bar{p}Hb(t) - \bar{c})a(t) \\ \dot{b}(t) &= B(\bar{b} - b(t))b(t) - Ha(t)b(t), \end{aligned}$$

where the first equation models fishing effort by the fishermen, quantified by the number of boats on the water, and the second equation models the fish population. (With some tweaking, this model is the equivalent to the self-regulating predator-prey model found in biomathematics. Here, the fish are analogous to the prey and fishermen (or boats) to the predators.)

This model is referred to as a "common access" model because there are no barriers to entry. That is, fishermen are free to enter and exit the industry as they wish without penalty, cost, legal restriction, or any other stipulations which make entry difficult. In practice, we have entry so long as profits are positive. In the case of zero profits, we will neither have entry nor exit until other factors influence the dynamics of interaction between our players, the fish and fishermen. For instance, if there is suddenly a large number of fish, more fishermen will enter the industry in hopes of realizing potential gains from profit. The meaning of each term in our model is given in the following list:

$$\begin{aligned} a(t) &= \text{number of boats at time } t \\ b(t) &= \text{number of fish at time } t \\ \bar{b} &= \text{sustainable fish population, } \bar{b} > 0 \\ B &= \text{scaling term} = \frac{\text{growth rate of fish}}{\bar{b}},\ 0 < B < 1 \\ H &= \text{technological constant, converts effort into catch}, 0 < H < 1 \\ \bar{c} &= \text{marginal constant cost per boat}, \bar{c} > 0 \\ \bar{p} &= \text{market price per fish}, \bar{p} > 0 \\ R &= pHb(t)a(t) = \text{total industry revenue at time } t, R > 0 \\ E &= R - ca(t) = \text{industry profits at time } t, E > 0 \\ \gamma &= \text{scaling term}, \gamma > 0 \end{aligned}$$

$\gamma c =$ rate at which fishermen leave the water, $c > 0$.

In the first equation, $\bar{c}a(t)$ is the total cost of the boats per unit time, at time t. The product $\bar{p}Hb(t)a(t)$ is the total revenue per unit time. The difference $\bar{p}Hb(t)a(t) - \bar{c}a(t)$ is the profit at time t. The first equation says that the rate of change of the fishing effort is proportional to the profit. The first term of the left hand side of the second equation, $B(\bar{b} - b(t))b(t)$, is in the usual logistic form, describing the natural dynamics of the fish population. The bracketed term $(\bar{b} - b(t))$ is the element which makes this model self-regulating. The second term, $Ha(t)b(t)$, represents the total harvest.

We find that the model has three equilibria

$$(0,0),\ (0,\bar{b}),\ \text{and}\,(a^\star, b^\star) = \left(\frac{B}{H}\left[\bar{b} - \frac{\bar{c}}{H\bar{p}}\right], \frac{\bar{c}}{H\bar{p}}\right).$$

The final equilibrium corresponds to coexistence of the fish and fisherman populations in the case that $\bar{b} - \frac{\bar{c}}{H\bar{p}}$ is positive. The linearization of the vector field is

$$Df(a,b) = \begin{pmatrix} \gamma\bar{c} & \gamma\bar{p}Ha \\ -Hb & B\bar{b} - 2Bb - Ha \end{pmatrix}$$

Evaluating at the origin, we have

$$Df(0,0) = \begin{pmatrix} \gamma\bar{c} & 0 \\ 0 & B\bar{b} \end{pmatrix}.$$

We conclude that $(0,0)$ is unstable. At the equilibrium point $(0,\bar{b})$, we find

$$Df(0,\bar{b}) = \begin{pmatrix} \gamma\bar{c} & 0 \\ -H\bar{b} & -B\bar{b} \end{pmatrix},$$

with eigenvalues of each sign. The equilibrium point is an unstable saddle point. The stable ray of contraction on the b-axis corresponds to the fact that in the absence of fishermen the fish population approaches the sustainable fish population value, $\bar{b}$. Finally, in the case $\bar{b} - \frac{\bar{c}}{H\bar{p}} > 0$, at the positive equilibrium point $(a^\star, b^\star)$, we determine that

$$Df\left(a^\star, b^\star\right) = \begin{pmatrix} 0 & \gamma\bar{p}B\left(\bar{b} - \frac{\bar{c}}{H\bar{p}}\right) \\ -\frac{\bar{c}}{\bar{p}} & -\frac{B\bar{c}}{H\bar{p}} \end{pmatrix}.$$

We calculate that

$$\det\left(Df\left(a^\star, b^\star\right)\right) = \gamma\bar{c}B\left(\bar{b} - \frac{\bar{c}}{H\bar{p}}\right) > 0$$
$$\operatorname{trace}\left(Df\left(a^\star, b^\star\right)\right) = -\frac{B\bar{c}}{H\bar{p}} < 0.$$

We conclude that the coexistence equilibrium is a stable sink.

To illustrate the results, we set $\bar{b} = 1000000$, $B = \frac{7}{480000}$, $H = 0.5$, $\bar{c} = 100000$, $\bar{p} = 2$, $\gamma = \frac{1}{20000}$, and use the the initial values $b_0 = 100000$ and $a_0 = 22$ to generate the phase portrait in Figure 3.

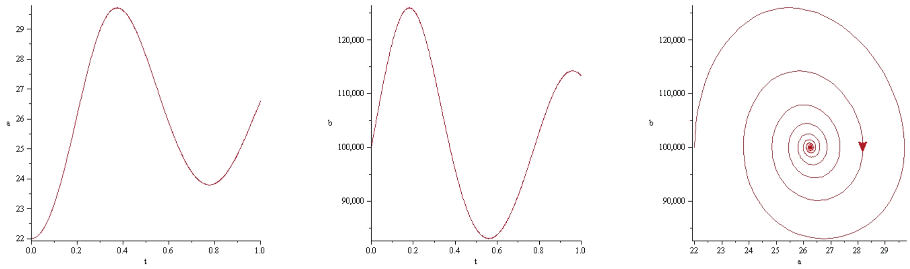

Fig. 3. Left to right: graphs of $a(t)$ versus t, $b(t)$ versus t, and the phase portrait $b(t)$ versus $a(t)$

We are now interested in solving the inverse problems: given data values $b(t_i)$, $i = 1, \ldots, M$ and $a(t_j)$, $j = 1, \ldots, N$, find values of the physical variables b, B, H, c, p, and γ so that the solution to the system agrees approximately with the data.

Example 2. To generate solution data, we set $\bar{b} = 1000000$, $B = \frac{7}{480000}$, $H = 0.5$, $\bar{c} = 100000$, $\bar{p} = 2$, and $\gamma = \frac{1}{20000}$, solve numerically for $b(t)$ and $a(t)$, and sample the solutions at uniformly-spaced times in $[0, 1]$, adding low-amplitude Gaussian noise with amplitude ε_b and ε_a, respectively. We fit piecewise polynomial target functions to these noisy data values and minimize the L^2 collage distance corresponding to the differential equations

$$\dot{b}(t) = c_1 b(t) + c_2 b^2(t) + c_3 a(t) b(t) \quad (15)$$

$$\dot{a}(t) = c_4 a(t) b(t) + c_5 a(t). \quad (16)$$

The results for different noise amplitudes are summarized in Table 2.

We observe that $c_1 = B\bar{b}$, $c_2 = -B$, $c_3 = -H$, $c_4 = \gamma \bar{p} H$, and $c_5 = -\gamma \bar{c}$. If we assume that $\bar{p} = 2$ is known, since it is the price determined by the market, we can calculate the remaining parameters from the minimal collage distance coefficient values. We obtain the results in Table 3. The values in the table lie quite close to the true values.

5 An inverse problem for a class of stochastic differential equations

Let us consider the following system of stochastic differential equations:

$$\begin{cases} \frac{d}{dt} X_t = A X_t dt + B_t, \\ x(0) = x_0. \end{cases} \quad (17)$$

Table 2. Minimal Collage Distance Coefficients for the Resource Model Inverse Problem

ε_b	ε_a	b_0	a_0	c_1	c_2	c_3	c_4	c_5
0	0	100000	22.0000	14.5833	-0.00001	-0.5000	0.00005	-5.00000
0.05	0.02	100680	21.9916	14.5735	-0.00001	-0.4998	0.00005	-4.9535
0.08	0.04	101134	22.0046	14.5343	-0.00001	-0.4998	0.00005	-4.9128
0.10	0.06	101476	22.0279	14.4795	-0.00001	-0.4998	0.00005	-4.8755
0.15	0.05	101964	21.9608	14.5714	-0.00001	-0.4990	0.00005	-4.8661
0.20	0.10	102815	22.0237	14.4100	-0.00001	-0.4983	0.00005	-4.7674

Table 3. Minimal Collage Distance Parameter Values for the Resource Model Inverse Problem

ε_b	ε_a	b_0	a_0	$\bar{b}$	B	H	$\bar{c}$	$\bar{p}$	γ
0	0	100000	22.00	1000000	0.0000100	0.500	100000	2	0.0000500
0.05	0.02	100680	21.99	995241	0.0000146	0.500	99878	2	0.0000496
0.08	0.04	101134	22.01	1014389	0.0000143	0.500	99848	2	0.0000492
0.10	0.06	101476	22.03	1046756	0.0000138	0.500	99852	2	0.0000488
0.15	0.05	101964	21.96	971935	0.0000150	0.499	99543	2	0.0000489
0.20	0.10	102815	22.02	1056825	0.0000136	0.498	99455	2	0.0000479

where $X : \mathbb{R} \times \Omega \to \mathbb{R}^n$, A is a (deterministic) matrix of coefficients and B_t is a classical vector Brownian motion. An inverse problem for this kind of equation can be formulated as: given an i.d. sample of observations of $X(t,\omega)$, say $(X(t,\omega_1),\ldots,X(t,\omega_n))$, get an estimation of the matrix A. For this purpose, let us take the integral over Ω of both sides of the previous equation and suppose that $X(t,\omega)$ is sufficiently regular; recalling that $B_t \sim \mathcal{N}(0,t)$, we have

$$\int_\Omega \frac{dx}{dt} dP(\omega) = \frac{d}{dt}\mathbb{E}(X(t,\cdot)) = A\mathbb{E}(X(t,\cdot)) \tag{18}$$

This is a deterministic differential equation in $\mathbb{E}(X(t,\cdot))$. From the sample of observations of $X(t,\omega)$ we can then get an estimation of $\mathbb{E}(X(t,\cdot))$ and then use of approach developed for deterministic differential equations to solve the inverse problem for A. The essential idea from [5] is that each realization $x(\omega_j,s)$, $j = 1,\ldots,N$, of the random variable $x(\omega,s)$ is the solution of a fixed point equation

$$\begin{aligned} x(\omega_j,s) &= \int_0^s \phi(\omega_j,t,x(\omega_j,t))dt + x_0(\omega_j) \\ &= \int_0^s \big(a_0(\omega_j) + a_1(\omega_j)t + a_2(\omega_j)x(\omega_j,t) \\ &\quad + a_3(\omega_j)t^2 + a_4(\omega_j)tx(\omega_j,t) + a_5(\omega_j)(x(\omega_j,t))^2 + \cdots\big)\, dt + x_0(\omega_j). \end{aligned}$$

Thus, for each target function $x(\omega_j,s)$, we can find the constant values $x_0(\omega_j)$ and $a_i(\omega_j)$ via collage coding. Upon treating each realization, we will have determined $x_0(\omega_j)$ and $a_i(\omega_j)$, $i = 1, \dots, M$, $j = 1, \dots, N$. We then construct the approximations

$$\mu \approx \mu_N = \frac{1}{N} \sum_{j=1}^{N} x_0(\omega_j) \text{ and } \nu_i \approx (\nu_i)_N = \frac{1}{N} \sum_{j=1}^{N} a_i(\omega_j), \tag{19}$$

where we note that results obtained from collage coding each realization are independent. Using our approximations of the means, we can also calculate that

$$\sigma^2 \approx \sigma_N^2 = \frac{1}{N-1} \sum_{j=1}^{N} (x_0(\omega_j) - \mu_N)^2, \ \sigma_i^2 \approx (\sigma_i)_N^2 = \frac{1}{N-1} \sum_{j=1}^{N} (a_i(\omega_j) - (\nu_i)_N)^2.$$

As a numerical example, we consider the first-order system

$$\begin{aligned} \frac{d}{dt} x_t &= a_1 x_t + a_2 y_t + b_t \\ \frac{d}{dt} y_t &= b_1 x_t + b_2 y_t + c_t \end{aligned}$$

Setting $a_1 = 0.5$, $a_2 = -0.4$, $b_1 = -0.3$, $b_2 = 1$, $x_0 = 0.9$, and $y_0 = 1$,we construct observational data values for x_t and y_t for $t_i = \frac{i}{N}$, $1 \leq i \leq N$, for various values of N. For each of M data sets, different pairs of Brownian motion are simulated for b_t and c_t. Figure 4 presents several plots of b_t and c_t for $N = 100$.

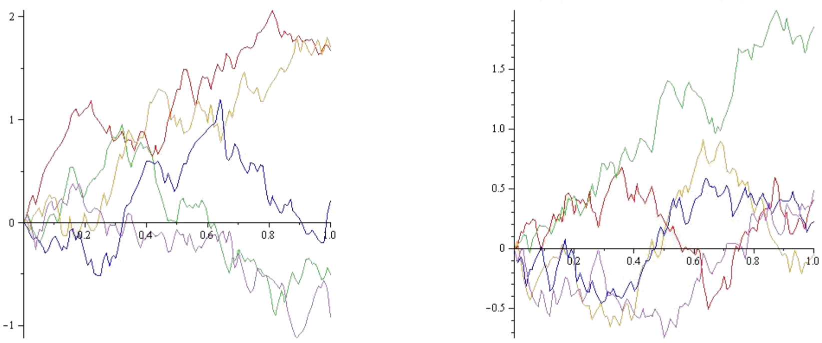

Fig. 4. Example plots of b_t and c_t for $N = 100$

In Figure 5, we present some plots of our generated x_t and y_t, as well as phase portraits for x_t versus y_t.

For each sample time, we construct the mean of the observed data values, $x^*_{t_i}$ and $x^*_{t_i}$, $1 \leq i \leq N$. We minimize the squared collage distances

$$\Delta_x^2 = \frac{1}{N} \sum_{i=1}^{N} \left(x^*_{t_i} - x_0 - \frac{1}{N} \sum_{j=1}^{i} \left(a_1 x^*_{t_j} + a_2 y^*_{t_j} \right) \right)^2$$

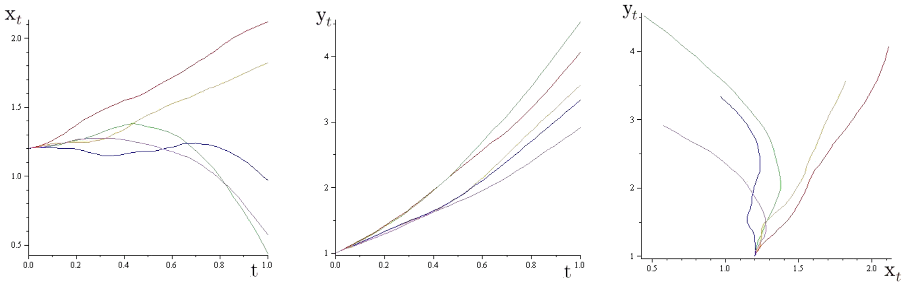

Fig. 5. Example plots of x_t, y_t, and x_t versus y_t for $N = 100$

and

$$\Delta_y^2 = \frac{1}{N} \sum_{i=1}^{N} \left(y_{t_i}^* - y_0 - \frac{1}{N} \sum_{j=1}^{i} \left(b_1 x_{t_j}^* + b_2 y_{t_j}^* \right) \right)^2$$

to determine the minimal collage parameters a_1, a_2, b_1, and b_2. The results of the process are summarized in Table 4.

Table 4. Minimal collage distance parameters for different N and M

N	M	a_1	a_2	b_1	b_2
100	100	0.2613	-0.2482	-0.2145	0.9490
100	200	0.3473	-0.3496	-0.2447	0.9709
100	300	0.3674	-0.3523	-0.2494	0.9462
200	100	0.3775	-0.3015	-0.1989	0.9252
200	200	0.3337	-0.3075	-0.2614	0.9791
200	300	0.4459	-0.3858	-0.2822	0.9718
300	100	0.4234	-0.3246	-0.2894	0.9838
300	200	0.3834	-0.3263	-0.3111	1.0099
300	300	0.5094	-0.4260	-0.3157	0.9965

6 Concluding Remarks

In this paper, we have considered three inverse problems drawn from applications in economics and finance. The fundamental approach for solving the problems is rooted in fractal-based analysis. The results in the paper demonstrate the usefulness of the collage method. It is worth mentioning that the method does not require significant computational power or time.

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