Jump telegraph processes and a volatility smile

Nikita Ratanov[1]

Facultad de Economía, Universidad del Rosario,
C.14, No.4-69, Bogotá, Colombia
nratanov@urosario.edu.co

Abstract. We continue to study financial market models based on generalized telegraph processes with alternating velocities. The model is supplied with jumps occurring at the times of velocity switchings. This model is arbitrage-free and complete if the directions of jumps in stock prices are in a certain correspondence with their velocity and with the behaviour of the interest rates. A risk-neutral measure and arbitrage-free formulae for a standard call option are constructed. A new version of convergence under suitable scaling to the Black-Scholes model is proved, and the explicit limit is obtained. Next, we examine numerically the explicit formulae for call prices to obtain the behaviour of implied volatilities. Moreover, this model has some features of models with memory. The historical volatility of jump telegraph model is similar to historical volatility of the moving average type model.

Keywords. telegraph process, option pricing, volatility smile.
M.S.C. classification. 91B28, 60J75.
J.E.L. classification. G13.

1 Introduction

The famous Black-Scholes formula has well known shortages and it is rarely used to price options. It is commonly accepted that Black-Scholes pricing formula underprices deep-in-the-money and out-of-the-money options and it overprices at-the-money options, see [15] and e.g. [16]. This observation provokes a growing interest in the construction of more and more complicated stochastic volatility models based on stochastic dynamics of the Black-Scholes implied volatility (see a review of these activities e.g. in [4]). The volatility implied by the Black-Scholes formula is used in these models as common language to explain how the option should be priced. Usually the implied volatility as a function of moneyness K/S_0 forms a so called "volatility smile". On the other hand this "smile-consistent" approach proposes the quantity sophistication instead of fundamental explanation of difficulties. Moreover, models of this type increase Markov dimension of the market.

To explain market's movements we propose a rather new model based on telegraph-like processes. This paper continues our previous research [20] of such a model. Suppose that the log-returns are driven by a telegraph process, i.e. they move with pair of constant velocities alternating one to another at Poisson times. To make the model more adequate and to avoid arbitrage opportunities the log-return movement should be supplied with jumps occurring at times of the tendency switchings.

As a basis for building the model in Section 2, we take a counting Poisson process $N = N(t), t \geq 0$ with alternating transition intensities $\lambda_{\pm} > 0$. The process $\sigma(t) = (-1)^{N(t)}$ (or $\sigma(t) = -(-1)^{N(t)}$) with values ± 1 displays a current market state. Using $\sigma(t),\ t \geq 0$, we define processes $c_{\sigma(t)} = c_{\pm}, h_{\sigma(t)} = h_{\pm},\ h_{\pm} > -1,\ r_{\sigma(t)} = r_{\pm},\ r_{\pm} \geq 0$. Processes T_s and J_s are defined as $T_s(t) = \int_0^t c_{\sigma(\tau)} \mathrm{d}\tau$ and $J_s(t) = \int_0^t h_{\sigma(\tau)} \mathrm{d}N(\tau)$. The evolution of the risky asset $S(t)$ is determined by a stochastic exponential of the sum $T_s + J_s$. The risk-free asset is given by the usual exponential of the process $\mathcal{T}_s = \mathcal{T}_s(t) = \int_0^t r_{\sigma(\tau)} \mathrm{d}\tau, t \geq 0$. Here and below the subscript s indicates the starting value $s = \sigma(0) = \pm 1$ of the market's state $\sigma(t)$.

In view of such trajectories, the market is set up as a continuous process that evolves with velocity c_+ or c_-, changes the direction of movement from $c_{\pm}$ to $c_{\mp}$ and exhibits jumps of size $h_{\pm}$ whenever velocity changes. The interest rate in the market is stochastic with values $r_{\pm}$.

The processes $T_{\pm}(t) + J_{\pm}(t), t \geq 0$ are given by the pair of states $(c_{\pm}, \lambda_{\pm}, h_{\pm})$. They are called *jump telegraph processes with states* $(c_{\pm}, \lambda_{\pm}, h_{\pm})$. This model is regarded as jump telegraph market model.

In Section 2 we describe the model in detail. This section contains also the explicit expressions for means and variances which are exploited to describe historical volatility in Section 4. For the beginning all parameters are supposed to be deterministic, which leads to completeness of the market. The case of random jump values and random velocities creates incomplete market model and it will be reported anywhere later.

Such a model looks attractive because of finite propagation velocity and the intuitively clear comportment. Under respective scaling it converges to Black-Scholes model. Section 3 is concerned with this convergence and the definition of volatility in jump telegraph model. It contains a new version of scaling theorem (cf. [20] and [21]), and a new fundamental and natural explanation of volatility. It permits us to define the volatility of the jump telegraph model depending on the velocities $c_{\pm}$, the jumps values $h_{\pm}$ and the switching intensities $\lambda_{\pm}$. Further, (Section 4), we consider a historical volatility as $\mathrm{HV}(t) = \sqrt{\mathrm{Var} S(t)/t}$, and then an implied volatility as $\mathrm{IV}(t) = \sqrt{V_{\pm}(\mu, t)/t}$, where $V_{\pm}$ are implied variances of jump telegraph model with respect to the Black-Scholes dynamics. The implied volatility $\mathrm{IV}(t)$ with various values of log-moneyness μ forms the so called volatility smile. Volatility smiles of various shapes are presented in Section 5.

Telegraph processes have been studied before in different probabilistic aspects (see, for instance, Goldstein [9] (1951), Kac [11], [12] (1974) and Zacks [22] (2004)). These processes have been exploited for stochastic volatility modelling (Di Masi *et al.* [7] (1994)) as well as for obtaining a "telegraph analog" of the

Black-Scholes model (Di Crescenzo and Pellerey [6] (2002)). In contrast with the paper by Di Crescenzo and Pellerey, we use more complicated and delicate construction of such a model to avoid arbitrage and to develop an adequate option pricing theory in this framework. Recently telegraph processes was applied to actuarial problems [14].

Parameters of telegraph market model was calibrated in the working paper of De Gregorio and Iacus [5]. This calculations are based on weekly closings of the Dow-Jones industrial average July 1971 - Aug 1974 and returns of IBM stock closings. In Section 5 we use these calibrated data to estimate the implied volatility (see Table 3 and Figure 4).

2 Jump telegraph processes with alternating intensities

Let $(\Omega,\ \mathcal{F},\ \mathbb{P})$ be a probability space, $\lambda_\pm$ be positive numbers. First, we consider two counting Poisson processes $N_+ = N_+(t),\ N_- = N_-(t),\ t \geq 0$ with values $\{0,\ 1,\ 2,\ \ldots\}$ and alternating intensities $\lambda_{\pm 1} := \lambda_\pm$, i.e. for $n = 0, 1, 2, \ldots$ as $\Delta t \to 0$

$$\mathbb{P}\{N_-(t+\Delta t) = 2n+2 \mid N_-(t) = 2n+1\} = \lambda_+ \Delta t + o(\Delta t),$$

$$\mathbb{P}\{N_+(t+\Delta t) = 2n+2 \mid N_+(t) = 2n+1\} = \lambda_- \Delta t + o(\Delta t),$$

$$\mathbb{P}\{N_-(t+\Delta t) = 2n+1 \mid N_-(t) = 2n\} = \lambda_- \Delta t + o(\Delta t),$$

$$\mathbb{P}\{N_+(t+\Delta t) = 2n+1 \mid N_+(t) = 2n\} = \lambda_+ \Delta t + o(\Delta t).$$

Further we will consider all stochastic processes subscribed by $+$ or $-$ to be adopted to filtration $\mathbf{F} = (\mathbf{F}_t^+)_{t \geq 0}$ or $\mathbf{F} = (\mathbf{F}_t^-)_{t \geq 0}$ generated by $N_+ = N_+(t)$ and $N_- = N_-(t)$ respectively.

Processes $\sigma_+(t) = (-1)^{N_+(t)}$ and $\sigma_-(t) = -(-1)^{N_-(t)}$ indicate a current state: if $0 < \tau_1 < \tau_2 < \tau_3 < \ldots$ is a Poisson flow, then interarrival times $\tau_{n+1} - \tau_n,\ n = 0,\ 1,\ 2,\ \ldots$ are independent and exponentially distributed with parameter $\lambda_{\sigma_\pm(\tau_n -)}$. Subscripts $\pm$ here respect to the initial state of the medium.

Let c_-, c_+ and h_-, h_+ be real numbers, $h_\pm > -1,\ c_- < c_+$. Telegraph processes $T_- = T_-(t),\ T_+ = T_+(t),\ t \geq 0$ are defined as follows:

$$T_\pm(t) = \int_0^t c_{\sigma_\pm(t')} \mathrm{d}t'.$$

We define also right continuous pure jump processes $J_- = J_-(t),\ J_+ = J_+(t)$, $t \geq 0$, which are driven by the same Poisson processes:

$$J_\pm(t) = \int_0^t h_{\sigma_\pm(t')} \mathrm{d}N_\pm(t') = \sum_{j=1}^{N_\pm(t)} h_{\sigma_\pm(\tau_j -)}.$$

Processes $X_- = T_- + J_-$ and $X_+ = T_+ + J_+$ are referred to as jump telegraph processes with parameters $< c_\pm,\ \lambda_\pm,\ h_\pm >$.

The following theorem could be interpreted as a version of the Doob-Meyer decomposition for telegraph processes.

Theorem 1. *Jump telegraph processes $X_- = X_-(t)$ and $X_+ = X_+(t),\ t \geq 0$ with parameters $< c_\pm,\ h_\pm,\ \lambda_\pm >$ are martingales if and only if $c_+ = -\lambda_+ h_+$ and $c_- = -\lambda_- h_-$.*

Proof. See [20], Theorem 1.

Next, we study the properties of telegraph processes under a change of measure. Let T_s^* be the telegraph process with the states $(c_\pm^*,\ \lambda_\pm)$, and $J_s^* = -\sum_{j=1}^{N_s(t)} c^*_{\sigma(\tau_j-)}/\lambda_{\sigma(\tau_j-)}$ be the jump process with jump values $h_\pm^* = -c_\pm^*/\lambda_\pm > -1$. Consider a probability measure $\mathbb{P}_s^*$ with the following local density (with respect to $\mathbb{P}_s$):

$$Z_s(t) = \frac{\mathrm{d}\mathbb{P}_s^*}{\mathrm{d}\mathbb{P}_s}\,|_t = \mathcal{E}_t(T_s^* + J_s^*), \qquad 0 \leq t \leq T,\ s = \pm. \tag{1}$$

Here $\mathcal{E}_t(\cdot)$ denotes stochastic exponential.

Using properties of stochastic exponentials, we obtain

$$Z_s(t) = \mathrm{e}^{T_s^*(t)}\kappa_s^*(t), \tag{2}$$

where $\kappa_s^*(t) = \prod_{\tau \leq t}(1 + \varDelta J_s^*(\tau))$ with $\varDelta J_s^*(\tau) = J_s^*(\tau) - J_s^*(\tau-)$.

The process $\kappa_s^* = \kappa_s^*(t), t \geq 0$ can be represented as $\kappa_s^*(t) = \kappa^*_{N_s(t),s}$. Here the sequence $\kappa^*_{n,s}$ is defined as follows:

$$\kappa^*_{n,s} = \kappa^*_{n-1,-s}(1 + h_s^*),\ n \geq 1, \qquad \kappa^*_{0,s} \equiv 1. \tag{3}$$

It means that if $n = 2k$,

$$\kappa^*_{n,s} = (1 + h_s^*)^k(1 + h_{-s}^*)^k,$$

and if $n = 2k + 1$,

$$\kappa^*_{n,s} = (1 + h_s^*)^{k+1}(1 + h_{-s}^*)^k.$$

Theorem 2 (Girsanov theorem). *Under the probability measure $\mathbb{P}_s^*$,*

- *process $N_s = N_s(t),\ t \geq 0$ is a Poisson process with intensities $\lambda_-^* = \lambda_- - c_-^* = \lambda_-(1 + h_-^*)$ and $\lambda_+^* = \lambda_+ - c_+^* = \lambda_+(1 + h_+^*)$.*
- *process $T_s = T_s(t),\ t \geq 0$ is a telegraph process with states (c_-, λ_-^*) and (c_+, λ_+^*).*

Probability measure $\mathbb{P}_s^*$ becomes the martingale measure for jump telegraph process $T_s + J_s$, if it is constructed using parameters $c_-^* = \lambda_- + \frac{c_-}{h_-}$, $c_+^* = \lambda_+ + \frac{c_+}{h_+}$, $h_-^* = -1 - \frac{c_-}{\lambda_- h_-}$ and $h_+^* = -1 - \frac{c_+}{\lambda_+ h_+}$.

Proof. See [20], Theorem 2.

We obtain distributions of jump telegraph processes $X_\pm(t)$ in terms of generalized probability densities $p_\pm^{(n)} = p_\pm^{(n)}(x,\ t)$ and $p_\pm = p_\pm(x,\ t)$, which are defined as

$$\mathbb{P}(X_s(t) \in \varDelta, N_s(t)) = \int\limits_{\varDelta} p_s^{(n)}(x,t)\mathrm{d}x, \quad \mathbb{P}(X_s(t) \in \varDelta) = \int\limits_{\varDelta} p_s(x,t)\mathrm{d}x, \qquad (4)$$

$$s = \pm$$

for any borelian set $\varDelta$; $p_\pm(x,t) = \sum\limits_{n=0}^{\infty} p_\pm^{(n)}(x,t),\ x \in (-\infty,\infty),\ t \geq 0.$

Theorem 3. *Let $X_\pm(t),\ t \geq 0$ be a pair of jump telegraph processes with parameters $< c_\pm,\ h_\pm,\ \lambda_\pm >$. Their probability densities $p_\pm^{(n)}$ solve the system*

$$\begin{cases} \frac{\partial p_+^{(n)}}{\partial t} + c_+ \frac{\partial p_+^{(n)}}{\partial x} = -\lambda_+[p_+^{(n)}(x,t) - p_-^{(n-1)}(x-h_+,t)], \\ \frac{\partial p_-^{(n)}}{\partial t} + c_- \frac{\partial p_-^{(n)}}{\partial x} = -\lambda_-[p_-^{(n)}(x,t) - p_+^{(n-1)}(x-h_-,\ t)] \end{cases} \qquad (5)$$

with zero initial conditions $p_\pm^{(n)}\mid_{t=0} = 0,\ n \geq 1$ and $p_\pm^{(0)}(x,t) = \mathrm{e}^{-\lambda_\pm t}\delta(x - c_\pm t)$.

Proof. See [19], equation (2.12).

System (5) has the following solution (see e.g. [21]), $p_\pm^{(n)}(x,t) = g_\pm^{(n)}(x - j_\pm^{(n)}, t)$, where

$$j_\pm^{(n)} = \begin{cases} k(h_+ + h_-), & n = 2k, \\ k(h_+ + h_-) + h_\pm, & n = 2k+1 \end{cases}, \quad k = 0, 1, 2, \ldots$$

and

$$g_+^{(n)}(x,t) = \mathrm{e}^{-\mu t - \nu x} \frac{\lambda_+^{n-[n/2]}\lambda_-^{[n/2]}}{(c_+ - c_-)^n} \cdot \frac{(c_+ t - x)^{n-[n/2]-1}(x - c_- t)^{[n/2]}}{(n - [n/2] - 1)!\,[n/2]!} \mathbf{1}_{\{c_- t < x < c_+ t\}},$$

$$g_-^{(n)}(x,t) = \mathrm{e}^{-\mu t - \nu x} \frac{\lambda_+^{[n/2]}\lambda_-^{n-[n/2]}}{(c_+ - c_-)^n} \cdot \frac{(c_+ t - x)^{[n/2]}(x - c_- t)^{n-[n/2]-1}}{(n - [n/2] - 1)!\,[n/2]!} \mathbf{1}_{\{c_- t < x < c_+ t\}},$$

$n \geq 1$. Here

$$\nu = \frac{\lambda_+ - \lambda_-}{c_+ - c_-}, \qquad \mu = \lambda_\pm - \nu c_\pm = \frac{c_+\lambda_- - c_-\lambda_+}{c_+ - c_-}.$$

Next, we have

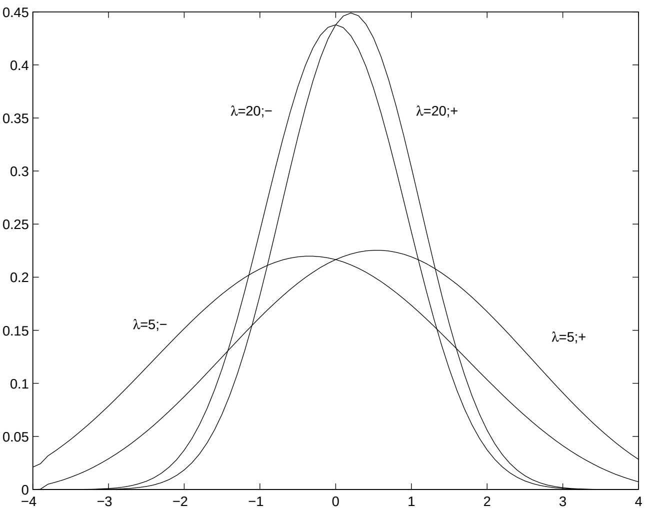

Fig. 1. Probability densities of telegraph process $T_\pm(t)$ (absolutely continuous part) with values $t = 1, c_\pm = \pm 4, h_\pm = \mp 0.2$ and with $\lambda_\pm = 5$ or $\lambda_\pm = 20$

$$p_\pm(x,t) = \sum_{n=0}^{\infty} p_\pm^{(n)}(x,t) = \sum_{n=0}^{\infty} g_\pm^{(n)}(x - j_\pm^{(n)}, t), \tag{6}$$

and functions $p_\pm$ satisfy the following system (see equation (2.9) in [21]):

$$\begin{cases} \frac{\partial p_+}{\partial t} + c_+ \frac{\partial p_+}{\partial x} = -\lambda_+ [p_+(x,t) - p_-(x - h_+, t)], \\ \\ \frac{\partial p_-}{\partial t} + c_- \frac{\partial p_-}{\partial x} = -\lambda_- [p_-(x,t) - p_+(x - h_-,\ t)] \end{cases} \tag{7}$$

with the initial condition $p_\pm(x, 0) = \delta(x)$.

The densities $p_\pm$ with certain $c_\pm$, $h_\pm$ and different $\lambda_\pm$ are presented in Fig.1.

Representation (6) of the telegraph process densities is adapted to the following rule of measure change. If the intensities $\lambda_\pm$ of the driving Poisson process are changed to $\bar{\lambda}_\pm$, then the densities of telegraph process will take the form

$$\bar{p}_\pm(x,t) = \mathrm{e}^{-(\bar{\mu}-\mu)t-(\bar{\nu}-\nu)x} \sum_{n=0}^{\infty} p_\pm^{(n)} \times \kappa_\pm^{(n)}, \tag{8}$$

where $\kappa_+^{(n)} = \left(\bar{\lambda}_+/\lambda_+\right)^{n-[n/2]} \left(\bar{\lambda}_-/\lambda_-\right)^{[n/2]}$, $\kappa_-^{(n)} = \left(\bar{\lambda}_+/\lambda_+\right)^{[n/2]} \left(\bar{\lambda}_-/\lambda_-\right)^{n-[n/2]}$, $\bar{\nu} = \frac{\bar{\lambda}_+ - \bar{\lambda}_-}{c_+ - c_-}$ and $\bar{\mu} = \frac{c_+\bar{\lambda}_- - c_-\bar{\lambda}_+}{c_+ - c_-}$.

Applying (7) one can easy obtain the following system for expectations (see [21], Corollary 2.6).

Lemma 1. *Let* $f = f(x)$ *and* $\mu_\pm = \mu_\pm(t)$, $t \geq 0$ *be smooth deterministic functions,* $X_\pm$ *be jump telegraph processes with parameters* $< c_\pm,\ h_\pm,\ \lambda_\pm >$. *Then functions*

$$u_\pm = u_\pm(x,\ t) = \mathbb{E}f(x - \mu_\pm(t) + X_\pm(t))$$

form a solution of the system

$$\begin{cases} \frac{\partial u_+}{\partial t} - (c_+ - \dot{\mu}_+)\frac{\partial u_+}{\partial x} = -\lambda_+\left[u_+(x,\ t) - u_-(x + \beta_+(t),\ t)\right] \\ \frac{\partial u_-}{\partial t} - (c_- - \dot{\mu}_-)\frac{\partial u_-}{\partial x} = -\lambda_-\left[u_-(x,\ t) - u_+(x + \beta_-(t),\ t)\right] \end{cases} \tag{9}$$

with $\beta_+(t) = h_+ - (\mu_+(t) - \mu_-(t))\,,\ \beta_-(t) = h_- - (\mu_-(t) - \mu_+(t))$.

Here $\dot{\mu}_\pm = \frac{\mathrm{d}\mu_\pm}{\mathrm{d}t}$.

From (9) we deduce formulae for mean value and variance of a jump telegraph process

$$m_\pm(t) = \mathbb{E}(X_\pm(t)), \qquad s_\pm(t) = \mathrm{Var}(X_\pm(t)).$$

Indeed, with the choices $f(x) = x, \mu_\pm = 0$ and $f(x) = x^2, \mu_\pm(t) = m_\pm(t)$ we get respectively

$$\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}t} = \Lambda\mathbf{m} + \mathbf{v_1} \tag{10}$$

and

$$\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} = \Lambda\mathbf{s} + \mathbf{v_2}. \tag{11}$$

Here

$$\Lambda = \begin{bmatrix} -\lambda_+ & \lambda_+ \\ \lambda_- & -\lambda_- \end{bmatrix}, \quad \mathbf{m} = \begin{bmatrix} m_+(t) \\ m_-(t) \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} s_+(t) \\ s_-(t) \end{bmatrix},$$

$\mathbf{v_1} = \begin{bmatrix} c_+ + \lambda_+h_+ \\ c_- + \lambda_-h_- \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} \lambda_+(h_+ + m_- - m_+)^2 \\ \lambda_-(h_- + m_+ - m_-)^2 \end{bmatrix}$.

System (10)-(11) can be explicitly resolved:

$$\mathbf{m}(t) = \frac{t}{2\lambda}\left(C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\Phi_\lambda(t)\begin{bmatrix} \lambda_+ \\ -\lambda_- \end{bmatrix}\right), \quad \Phi_\lambda(t) = \frac{1 - \mathrm{e}^{-2\lambda t}}{2\lambda t}, \tag{12}$$

where $C_1 = \lambda_-(c_+ + \lambda_+h_+) + \lambda_+(c_- + \lambda_-h_-),\ C_2 = c_+ - c_- + \lambda_+h_+ - \lambda_-h_-$. Then $m_+ - m_- = \frac{C_2}{2\lambda}(1 - \mathrm{e}^{-2\lambda t}) = C_2t\Phi_\lambda(t)$ and

$$\mathbf{s}(t) = \int_0^t \mathrm{e}^{(t-\tau)\Lambda}\mathbf{v_2}(\tau)\mathrm{d}\tau \tag{13}$$

with $\mathbf{v_2}(\tau) = \begin{bmatrix} \lambda_+(h_+ - C_2\tau\Phi_\lambda(\tau))^2 \\ \lambda_-(h_- + C_2\tau\Phi_\lambda(\tau))^2 \end{bmatrix}$.

With this in hand we can easy find the limits of $s_\pm(t)/t$ as $t \to 0$ and as $t \to \infty$:

$$\lim_{t\to 0} s_\pm(t)/t = \lambda_\pm h_\pm^2,$$

$$\lim_{t\to\infty} \frac{s_\pm(t)}{t} = \frac{\lambda_+\lambda_-}{(\lambda_+ + \lambda_-)^3}\left[(\lambda_-(h_+ + h_-) + c_- - c_+)^2 + (\lambda_+(h_+ + h_-) + c_+ - c_-)^2\right]. \tag{14}$$

We shall use these limits in Section 4 to evaluate the comportment of historical volatilities.

In symmetric case $\lambda_+ = \lambda_- := \lambda$ the formulae for solutions of (10)-(11) can be simplified as follows. Setting $A = (c_+ + c_-)/2$, $a = (c_+ - c_-)/2$, $B = (h_+ + h_-)/2$, $b = (h_+ - h_-)/2$, $\gamma_+ = -2a(a/\lambda + h_+)$, $\gamma_- = -2a(a/\lambda - h_-)$ we have

$$m_\pm(t) = [A + \lambda B \pm (a + \lambda b)\Phi_\lambda(t)]\, t, \tag{15}$$

$$s_\pm(t) = \left[a^2/\lambda + \lambda B^2 + (a + \lambda b)^2 \Phi_{2\lambda}(t)/\lambda + \gamma_\pm \Phi_\lambda(t) \pm 2B(a + \lambda b)\mathrm{e}^{-2\lambda t}\right] t. \tag{16}$$

These formulae are presented in [21] (see Theorem 2.7, formulae (2.25)-(2.26)).

3 Jump telegraph market model and diffusion rescaling

We consider a market with one stock and a bond.

The stock price $S(t) = S_\pm(t),\ t \geq 0$ follows the equation

$$\mathrm{d}S(t) = S(t-)\mathrm{d}X(t), \quad S(0) = S^0,\ \sigma(0) = \pm 1, \tag{17}$$

where $X(t) = X_\pm(t) = T_\pm(t) + J_\pm(t),\ t \geq 0$ is the jump telegraph process with parameters $< c_\pm,\ h_\pm,\ \lambda_\pm >$, $\sigma(0) = \pm 1$ indicates initial market trend. Integrating we have

$$S(t) = S^0 \mathcal{E}_t(X) = S^0 \exp(T(t))\kappa(t), \qquad \kappa(t) = \prod_{n=0}^{N(t)} (1 + h_{\sigma(\tau_j -)}). \tag{18}$$

The bond price is

$$B(t) = \exp(\mathcal{T}(t)), \tag{19}$$

where $\mathcal{T} = \mathcal{T}_\pm(t),\ t \geq 0$ be the telegraph process with velocities $r_\pm \geq 0$, which is driven by the same inhomogeneous Poisson process:

$$\mathcal{T}(t) = \int_0^t r_{\sigma_\pm(t')}\mathrm{d}t'.$$

Model (18)-(19) is named jump telegraph market model.

The probability measure $\mathbb{P}^*$ is the martingale measure for pricing process $\tilde{S}(t) \equiv B(t)^{-1}S(t)$. Process $\tilde{S}(t) = S_0 \exp(T(t) - \mathcal{T}(t))\kappa(t),\ t \geq 0$ is again the stochastic exponent of jump telegraph process with parameters $< c_\pm - r_\pm,\ h_\pm,\ \lambda_\pm >$. So with no loss of generality we assume $r_\pm = 0$. Thus, the stock price process $S(t)$ is a nonnegative $\mathbb{P}^*$-martingale. By Theorem 1 under measure $\mathbb{P}^*$ the driving Poisson process N has intensities $\lambda_-^* = -c_-/h_-$ and $\lambda_+^* = -c_+/h_+$, and by Theorem 2 change of measure is defined by $c_\pm^* = \lambda_\pm - \lambda_\pm^*$.

It is well known that under suitable scaling the telegraph process $T = T(t), t \geq 0$ converges to a Brownian motion: if $c_+ \to +\infty$, $c_- \to -\infty$, $\lambda_\pm \to \infty$ such that $c_+/\sqrt{\lambda_+} \to \sigma,\ c_-/\sqrt{\lambda_-} \to -\sigma$, then the telegraph process $T(t), t \geq 0$ converges in distribution to $\sigma w(t), t \geq 0$, where w denotes the standard Brownian motion. This convergence was first proved in [11]; see more details and some extensions in [18].

Thus, it is reasonable to obtain a similar rescaling result for jump telegraph model (18). To separate the drift from the diffusion component we consider the telegraph processes $\tilde{T}_\pm(t), t \geq 0$, driven by the same Poisson process as $T_\pm$ and with velocities a_+ and $-a_-$, where $a_+ = \frac{c_+ - c_-}{\sqrt{\lambda_+}+\sqrt{\lambda_-}}\sqrt{\lambda_+}$ and $a_- = \frac{c_+ - c_-}{\sqrt{\lambda_+}+\sqrt{\lambda_-}}\sqrt{\lambda_-}$. Notice that $a_+ + a_- = c_+ - c_-$ and $\frac{a_\pm}{a_+ + a_-} = \frac{\sqrt{\lambda_\pm}}{\sqrt{\lambda_+}+\sqrt{\lambda_-}}$.

It is easy to see that

$$T_+(t) - \tilde{T}_+(t) = T_-(t) - \tilde{T}_-(t) = At, \tag{20}$$

where $A = \frac{c_+\sqrt{\lambda_-} + c_-\sqrt{\lambda_+}}{\sqrt{\lambda_+}+\sqrt{\lambda_-}} = \frac{c_+ a_- + c_- a_+}{c_+ - c_-}$.

Further we assume $\lambda_\pm \to +\infty$, $c_+ - c_- \to +\infty$ and

$$\frac{c_+ - c_-}{\sqrt{\lambda_+}+\sqrt{\lambda_-}} \to \sigma, \qquad \sqrt{\frac{\lambda_+}{\lambda_-}} \to \gamma \tag{21}$$

for some $\sigma, \gamma \geq 0$.

To control jump and drift components we suppose $h_\pm \to 0$ such that for some $\alpha_\pm, \delta \in (-\infty, \infty)$

$$\sqrt{\lambda_\pm} h_\pm \to \alpha_\pm \tag{22}$$

and

$$\varDelta := A + \frac{\sqrt{\lambda_+\lambda_-}}{\sqrt{\lambda_+}+\sqrt{\lambda_-}}(\sqrt{\lambda_+}h_+ + \sqrt{\lambda_-}h_-) \to \delta. \tag{23}$$

Notice that $\varDelta = \frac{\sqrt{\lambda_+}}{\sqrt{\lambda_+}+\sqrt{\lambda_-}}(c_- + \lambda_- h_-) + \frac{\sqrt{\lambda_-}}{\sqrt{\lambda_+}+\sqrt{\lambda_-}}(c_+ + \lambda_+ h_+)$ and set $\beta^2 = \frac{\alpha_+^2 + \gamma\alpha_-^2}{1+\gamma}$.

The following theorem generalizes previous author's results (see Theorem 3.3 [21] and Theorem 4 [20]). The scaling property (24) can be applied to interpret a volatility in jump telegraph model (18).

Theorem 4. *Under scaling (21)-(23) jump telegraph model (18) converges in distribution to the Black-Scholes model:*

$$S(t) \xrightarrow{D} S_0 \exp\{vw(t) + (\delta - \beta^2/2)t\}, \tag{24}$$

where $v = \sqrt{\left(\sigma + (\gamma\alpha_+ - \alpha_-)/(1+\gamma)\right)^2 + \beta^2}$.

Remark 1. Under the martingale measure $\mathbb{P}^*$ transition intensities take a form $-c_\pm/h_\pm$. Thus the drift vanishes, $\Delta = \frac{\sqrt{\lambda_+}}{\sqrt{\lambda_+}+\sqrt{\lambda_-}}(c_- + \lambda_- h_-) + \frac{\sqrt{\lambda_-}}{\sqrt{\lambda_+}+\sqrt{\lambda_-}}(c_+ + \lambda_+ h_+) = 0$. Moreover, in this case $\sigma = \lim \frac{c_+ - c_-}{\sqrt{\lambda_+}+\sqrt{\lambda_-}} = -\lim \frac{\lambda_+ h_+ - \lambda_- h_-}{\sqrt{\lambda_+}+\sqrt{\lambda_-}} = -\frac{\gamma\alpha_+ - \alpha_-}{1+\gamma}$. The limiting volatility v in this case coincides with β: $v = \beta = \sqrt{\frac{\alpha_+^2 + \gamma\alpha_-^2}{1+\gamma}}$.

Proof. Let $f_\pm(z,\ t) = \mathbb{E}\mathcal{E}_t(zX_\pm) = \mathbb{E}\mathrm{e}^{zT_\pm(t)}\kappa_\pm(t)^z$ be the moment-generating function. We prove here the convergence

$$f_\pm(z,\ t) \to \exp\{(\delta - \beta^2/2)zt + v^2 z^2 t/2\}, \tag{25}$$

which is sufficient for the convergence of pointwise distributions in (24).

Using (20) and the rule (8) we note that

$$f_\pm(z,t) = \mathrm{e}^{Azt}\mathbb{E}\mathrm{e}^{z\tilde{T}_\pm(t)}\kappa_\pm(t)^z = \mathrm{e}^{Azt + (\bar{\mu}-\mu)t}\int\limits_{-\infty}^{\infty} \mathrm{e}^{(z+\bar{\nu}-\nu)x}\bar{p}_\pm(x,t)\mathrm{d}x.$$

Here $\bar{p}_\pm$ are (generalized) probability densities of the telegraph processes $\bar{T}_\pm$ with velocities a_+ and $-a_-$, which are controlled by the Poisson process with alternating intensities $\bar{\lambda}_\pm = \lambda_\pm(1+h_\pm)^z$, furthermore $\bar{\mu} = (a_-\bar{\lambda}_+ + a_+\bar{\lambda}_-)/(a_+ + a_-)$, $\mu = (a_-\lambda_+ + a_+\lambda_-)/(a_+ + a_-)$ and $\nu = (\bar{\lambda}_+ - \bar{\lambda}_-)/(a_+ + a_-)$, $\nu = (\lambda_+ - \lambda_-)/(a_+ + a_-)$.

Since under the scaling (21) $a_+/\sqrt{\lambda_+},\ a_-/\sqrt{\lambda_-} \to \sigma$ and thus the processes $\bar{T}_+(t), t \ge 0$ and $\bar{T}_-(t), t \ge 0$ converge to $\sigma w(t), t \ge 0$, then

$$\bar{p}_\pm(x,\ t) \to \frac{1}{\sigma\sqrt{2\pi t}}\mathrm{e}^{-x^2/(2t\sigma^2)}.$$

Further notice that

$$\bar{\nu} - \nu = \frac{\lambda_+[(1+h_+)^z - 1] - \lambda_-[(1+h_-)^z - 1]}{a_+ + a_-} \sim z\cdot\frac{\lambda_+ h_+ - \lambda_- h_-}{a_+ + a_-} \to z\cdot\frac{\gamma\alpha_+ - \alpha_-}{\sigma(1+\gamma)}. \tag{26}$$

Moreover

$$\bar{\mu} - \mu = \frac{a_+(\bar{\lambda}_- - \lambda_-) + a_-(\bar{\lambda}_+ - \lambda_+)}{a_+ + a_-}$$

$$= \frac{a_+\lambda_-}{a_+ + a_-}\left[(1+h_-)^z - 1\right] + \frac{a_-\lambda_+}{a_+ + a_-}\left[(1+h_+)^z - 1\right]$$

$$= \frac{z\sqrt{\lambda_+\lambda_-}}{\sqrt{\lambda_+} + \sqrt{\lambda_-}}\left[\sqrt{\lambda_+}h_+ + \sqrt{\lambda_-}h_-\right] \tag{27}$$

$$+\frac{z^2 - z}{2}\left[\frac{\sqrt{\lambda_-}}{\sqrt{\lambda_+} + \sqrt{\lambda_-}}\lambda_+ h_+^2 + \frac{\sqrt{\lambda_+}}{\sqrt{\lambda_+} + \sqrt{\lambda_-}}\lambda_- h_-^2\right].$$

Applying (21)-(23) and summarizing the above statements, we obtain the convergence (25). □

Remark 2. Condition (23) in this theorem means that the total drift $\Delta \equiv A + \frac{\sqrt{\lambda_+\lambda_-}}{\sqrt{\lambda_+}+\sqrt{\lambda_-}}(\sqrt{\lambda_+}h_+ + \sqrt{\lambda_-}h_-)$ is asymptotically finite. Here $A = \frac{a_-c_+ + a_+c_-}{c_+ - c_-}$ is generated by the velocities of the telegraph process, and the summand $\frac{\sqrt{\lambda_+\lambda_-}}{\sqrt{\lambda_+}+\sqrt{\lambda_-}}(\sqrt{\lambda_+}h_+ + \sqrt{\lambda_-}h_-)$ represents the drift component (possibly with infinite asymptotics) that is motivated only by jumps. If the limits of $\lambda_\pm h_\pm$ are finite, then $A \to const$, and $\alpha_+ = \alpha_- = 0$. In this case the volatility of limit is $v = \sigma = \lim a_\pm/\sqrt{\lambda_\pm}$.

Hence in model (18)-(19) value $a_+/\sqrt{\lambda_+} = a_-/\sqrt{\lambda_-} = (c_+ - c_-)/(\sqrt{\lambda_+} + \sqrt{\lambda_-})$ can be interpreted as "telegraph" component of volatility, and $\sqrt{\lambda_\pm}h_\pm$ are volatility components engendered by jumps.

In general, the limiting volatility $v = \sqrt{(\sigma + (\gamma\alpha_+ - \alpha_-)/(1+\gamma))^2 + \beta^2}$ depends both on "telegraph" and jump components. So it is natural to define volatility in jump telegraph model as (see (26)-(27))

$$vol = \sqrt{\left(\frac{c_+ - c_-}{\sqrt{\lambda_+} + \sqrt{\lambda_-}}\right)^2\left(1 + \frac{\lambda_+h_+ - \lambda_-h_-}{c_+ - c_-}\right)^2 + \frac{\sqrt{\lambda_-}\lambda_+h_+^2 + \sqrt{\lambda_+}\lambda_-h_-^2}{\sqrt{\lambda_+} + \sqrt{\lambda_-}}}. \tag{28}$$

4 Historical and implied volatilities in the jump telegraph model

4.1 Historical volatility

Historical volatility is defined as

$$\mathrm{HV}(t) = \sqrt{\frac{\mathrm{Var}\{\log S(t+\tau)/S(\tau)\}}{t}}. \tag{29}$$

For classical Black-Scholes model $\log S(t+\tau)/S(\tau) \stackrel{D}{=} at + \sigma w(t)$ (where $w = w(t),\ t \geq 0$ is a standard Brownian motion), the historical volatility is constant: $\mathrm{HV}_{\mathrm{BS}}(t) \equiv \sigma$.

In a moving-average type model, which is described by (see [2])

$$\log S(t)/S(0) = at + \sigma w(t) - \sigma \int_0^t \mathrm{d}\tau \int_{-\infty}^{\tau} p\mathrm{e}^{-(q+p)(\tau-u)}\mathrm{d}w(u),$$

$(\sigma,\ q,\ q+p > 0)$ the historical volatility has a more tricky structure

$$\mathrm{HV} = \frac{\sigma}{2\lambda}\sqrt{q^2 + p(2q+p)\Phi_\lambda(t)} \tag{30}$$

with $2\lambda = q + p$ and $\Phi_\lambda(t) = \frac{1-\mathrm{e}^{-2\lambda t}}{2\lambda t}$. Recently this type of models have been applied to capture memory effects of the market [8], [10].

The historical volatility of jump telegraph model (18) takes the form

$$\mathbf{HV}(t) = \sqrt{\frac{1}{t}\int_0^t \mathrm{e}^{(t-\tau)\Lambda}\mathbf{v}(\tau)\mathrm{d}\tau}, \tag{31}$$

where $\mathbf{v} = \begin{bmatrix} v_+(\tau) \\ v_-(\tau) \end{bmatrix}$ is defined as in (13), but with $\ln(1+h_\pm)$ instead of $h_\pm$:

$$v_+(\tau) = \lambda_+\left[\ln(1+h_+) - C\tau\Phi_\lambda(\tau)\right]^2, \quad v_-(\tau) = \lambda_-\left[\ln(1+h_-) + C\tau\Phi_\lambda(\tau)\right]^2.$$

Here as usual, subscripts $\pm$ denote the initial state of the market, $C = c_+ - c_- + \lambda_+\ln(1+h_+) - \lambda_-\ln(1+h_-)$ and $\Phi_\lambda(\tau) = \frac{1-\mathrm{e}^{-(\lambda_++\lambda_-)\tau}}{(\lambda_++\lambda_-)\tau}$.

Historical volatility in jump telegraph model has the following very natural limiting behaviour (see (14)):

$$\lim_{t\to 0}\mathrm{HV}_\pm(t) = \sqrt{\lambda_\pm}\ln(1+h_\pm),$$

$$\lim_{t\to\infty}\mathrm{HV}_\pm(t) = \sqrt{\frac{\lambda_+\lambda_-}{2\lambda^3}[(\lambda_- B - a)^2 + (\lambda_+ B + a)^2]}$$

($B = \frac{1}{2}\ln(1+h_+)(1+h_-)$, $a = (a_+ + a_-)/2$; see (14)). These limits look reasonable: the limit at 0 is engendered by jumps only, the limit at ∞ contains both "velocity" component and a long term influence of jumps.

Using (16) and (29), in the symmetric case $\lambda_+ = \lambda_- = \lambda$ formula (31) takes the form similar to (30)

$$\mathrm{HV}_\pm(t) = \sqrt{a^2/\lambda + \lambda B^2 + (a+\lambda b)^2\Phi_{2\lambda}(t)/\lambda + \gamma_\pm\Phi_\lambda(t) \pm 2B(a+\lambda b)\mathrm{e}^{-2\lambda t}}.$$

The limits of historical volatility under a standard diffusion scaling (see Theorem 4) are more complicated. Nevertheless, in the symmetric case $\lambda_+ =$

$\lambda_- = \lambda$, we have under the scaling conditions $\lambda,\ a \to \infty,\ h_\pm \to 0,\ a^2/\lambda \to \sigma^2,\ \sqrt{\lambda}h_\pm \to \alpha_\pm$ that the historical volatility $\mathbb{HV}_\pm(t)$ defined by (31) converges to $\sqrt{\sigma^2 + (\alpha_+ + \alpha_-)^2/4}$.

Notice, that under the martingale measure $\mathbb{P}^*$, we have $\lambda = -c_\pm/h_\pm$, $\sigma = (-\alpha_+ + \alpha_-)/2$, and the diffusion limit of historical volatility equals to $v = \sqrt{(\alpha_+^2 + \alpha_-^2)/2}$, which coincides with the volatility expression in Remark 1.

4.2 Implied volatility

Define the Black-Scholes call price function $f(\mu, v)$, $\mu = \log K$ by

$$f(\mu, v) = \begin{cases} F\left(\frac{-\mu}{\sqrt{v}} + \frac{\sqrt{v}}{2}\right) - \mathrm{e}^\mu F\left(\frac{-\mu}{\sqrt{v}} - \frac{\sqrt{v}}{2}\right), & \text{if } v > 0, \\ \\ (1 - \mathrm{e}^\mu)^+, \qquad \text{if } v = 0. \end{cases}$$

The processes $V_\pm(\mu, t),\ t \geq 0,\ \mu \in \mathbb{R}$ defined by the equation

$$\mathbb{E}\left[(S(t+\tau)/S(\tau) - \mathrm{e}^\mu)^+ | \mathcal{F}_\tau\right] = f(\mu,\ V_{\sigma(\tau)}(\mu, t)) \tag{32}$$

are referred to as implied variance processes.

The implied volatilities $\mathbb{IV}_\pm(\mu, t)$ are

$$\mathbb{IV}_\pm(\mu, t) = \sqrt{\frac{V_\pm(\mu, t)}{t}}. \tag{33}$$

The LHS of (32) is defined exactly by following. In the framework of this model option pricing formulae and hedging strategies are completely constructed (see [20]).

$$\mathbb{E}\left[(S(t+\tau)/S(\tau) - \mathrm{e}^\mu)^+ | \mathcal{F}_\tau\right] = u_s(\mu, t;\ \bar{\lambda}_\pm) - \mathrm{e}^\mu u_s(\mu, t;\ \lambda^*_\pm), \qquad s = \sigma(\tau), \tag{34}$$

where $\bar{\lambda}_\pm = \lambda^*_\pm(1 + h_\pm)$, $\lambda^*_\pm = -c_\pm/h_\pm > 0$. Functions $u_\pm$ can be expressed as

$$u_s(\mu, t;\ \lambda_\pm) = \sum_{n=0}^{\infty} u_s^{(n)}(\mu - b_s^{(n)}, t), \qquad s = \pm,$$

where $b_\pm^{(n)} = \ln \kappa_n = \sum_{j=0}^{n} \ln(1 + h_{\sigma_\pm(\tau_j -)})$ are drift parameters engendered by jumps. Summands $u_\pm^{(n)}$ of this sum has the following structure: for $n \geq 1$

$$u_\pm^{(n)}(y,\ t) = \begin{cases} 0, & y > c_+ t \\ w_\pm^{(n)}(p,\ q), & c_- t \leq y \leq c_+ t, \quad p = \dfrac{c_+ t - y}{c_+ - c_-}, q = \dfrac{y - c_- t}{c_+ - c_-}, \\ \rho_\pm^{(n)}(t), & y < c_- t \end{cases} \tag{35}$$

and $u_+^{(0)}(y,t) = \begin{cases} 0, & \text{if } p < 0 \\ \mathrm{e}^{-\lambda_+ t}, & \text{if } p \geq 0 \end{cases}$, $u_-^{(0)}(y,t) = \begin{cases} \mathrm{e}^{-\lambda_- t}, & \text{if } q < 0 \\ 0, & \text{if } q \geq 0 \end{cases}$.

Functions $\rho_\pm^{(n)}$ in (35) have a form

$$\rho_\pm^{(n)}(t) = \mathrm{e}^{-\lambda_- t} \Lambda_\pm^{(n)} P_\pm^{(n)}(t). \tag{36}$$

Here $\Lambda_+^{(n)} = (\lambda_+)^{[(n+1)/2]}(\lambda_-)^{[n/2]}$, $\Lambda_-^{(n)} = (\lambda_-)^{[(n+1)/2]}(\lambda_+)^{[n/2]}$ and

$$P_\pm^{(n)}(t) = \frac{t^n}{n!} \cdot {}_1F_1(m_n^{(\pm)}+1;\ n+1;\ -\delta t),\ m_n^{(+)} = [n/2]\,,\ m_n^{(-)} = [(n-1)/2]\,,$$

$\delta = \lambda_+ - \lambda_-$. Here we exploit a hypergeometric function ${}_1F_1(\alpha;\ \beta;\ z)$ which is defined as

$${}_1F_1(\alpha;\ \beta;\ z) = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+k-1)}{k!\beta(\beta+1)\dots(\beta+k-1)} z^k = 1 + \sum_{k=1}^{\infty} \frac{(\alpha)_k}{k!(\beta)_k} z^k.$$

(see Abramowitz and Stegun [1]). Notice that $P_+^{(2n+1)} \equiv P_-^{(2n+1)} := P^{(2n+1)}$, $n = 0, 1, 2, \dots$

Moreover $w_\pm^{(n)} = \mathrm{e}^{-\lambda_+ q - \lambda_- p} \Lambda_\pm^{(n)} v_\pm^{(n)}$, $p, q > 0$, where $v_\pm^{(1)} = P^{(1)}(p) = (1-\mathrm{e}^{-\delta p})/\delta$, and

$$\begin{aligned} v_+^{(2n)} &= v_+^{(2n)}(p,\ q) = P_+^{(2n)}(p) + qP^{(2n-1)}(p) + \sum_{k=2}^{n} \tfrac{q^k}{k!} \sum_{j=0}^{k-2} \delta^{k-j-2} \beta_{k-1,j} P_-^{(2n-j-2)}(p), \\ v_-^{(2n)} &= v_-^{(2n)}(p,\ q) = P_-^{(2n)}(p) + \sum_{k=1}^{n-1} \tfrac{q^k}{k!} \sum_{j=0}^{k} \delta^{k-j} \beta_{k+1,j} P_-^{(2n-j)}(p), \\ v_\pm^{(2n+1)} &= v_\pm^{(2n+1)}(p,\ q) = P^{(2n+1)}(p) + \sum_{k=1}^{n} \tfrac{q^k}{k!} \sum_{j=0}^{k-1} \delta^{k-j-1} \beta_{k,j} P_-^{(2n-j)}(p), \end{aligned} \tag{37}$$

Here the coefficients $\beta_{k,j},\ j < k$ are defined as follows: $\beta_{k,0} = \beta_{k,1} = \beta_{k,k-2} = \beta_{k,k-1} = 1$,

$$\beta_{k,j} = \frac{(k-j)_{[j/2]}}{[j/2]!}.$$

Remark 3. In the symmetric case $\lambda_+ = \lambda_- = \lambda$ we have $P_\pm^{(n)}(t) = \frac{t^n}{n!}$ and functions $u_\pm^{(n)}$ can be simplified as follows

$$u_\pm^{(n)}(y,t) = \mathrm{e}^{-\lambda t} \frac{\lambda^n}{n!} \begin{cases} 0, & \text{if } p < 0 \\ \sum_{k=0}^{m_n^{(\pm)}} \begin{pmatrix} n \\ k \end{pmatrix} p^{n-k} q^k & \text{if } p, q > 0 \\ t^n, & \text{if } q < 0 \end{cases}$$

The detailed proof of (34)-(37) see in [20].

5 Numerical results

We performed the numerical valuation of the jump telegraph volatility (28) and the historical volatility (31), which are compared with the implied volatilities (33) with respect to different moneyness and to the initial market states. The implied volatilities are calculated by the explicit formulae (33)-(37). First, we consider the symmetric case: $\lambda_\pm = 10, c_\pm = \pm 1$ and $h_\pm = \mp 0.1$. In Figure 2 we plot implied volatilities of this simple case. Table 1 lists call prices and implied volatilities of this volatility smile numerically. Notice that these frowned smiles of implied volatilities $\mathbb{IV}_-$ and $\mathbb{IV}_+$ intersect at $K/S_0 \approx 1.17$.

Table 1. Symmetric smile, $t = 1,\ S_0 = 100,\ \lambda_\pm = 10,\ h_\pm = \mp 0.1,\ c_\pm = \pm 1$

K	40	70	100	117	130	160	190	220	250	280
c_-	60.0013	31.6774	12.7370	6.9036	4.1565	1.1433	0.2632	0.0478	0.0058	0.0002
c_+	60.0026	31.7257	12.7680	6.9039	4.1382	1.1128	0.2430	0.0390	0.0032	0.0
$\mathbb{IV}_-$	0.2670	0.3147	0.3206	0.3200	0.3186	0.3128	0.3045	0.2935	0.2787	0.2545
$\mathbb{IV}_+$	0.2811	0.3175	0.3214	0.3200	0.3180	0.3109	0.3010	0.2875	0.2671	0.0

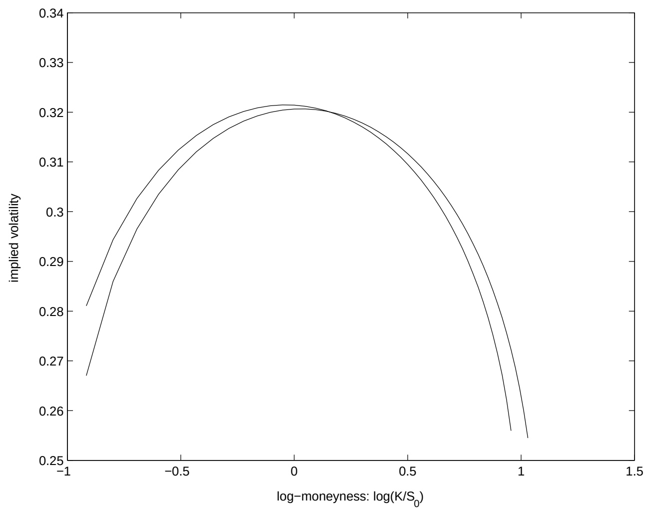

Fig. 2. Symmetric smile, $t = 1, S_0 = 100, \lambda_\pm = 10, h_\pm = \mp 0.1, c_\pm = \pm 1$, $\mathbb{HV}_\pm = 0.3162$, jump telegraph volatility=0.3162

Table 2 and Figure 3 show the implied volatility picture for skewed movement, when the market prices have a drift: both velocities are positive, and to avoid an arbitrage we suppose jump values to be negative. This figure has unstable oscillations for deep-out-of-the-money options. Moreover, only in this case historical and jump telegraph volatilities are less then implied volatilities values for at-the-money options.

Table 2. Skewed smile, $t = 1,\ S_0 = 100,\ \lambda_\pm = 10,\ h_- = -0.03,\ h_+ = -0.19,\ c_- = 0.3,\ c_+ = 1.9$

K	50	100	150	200	250	300	350	400	450	500
c_-	50.8133	17.6956	5.5624	1.8243	0.6350	0.2325	0.0882	0.0347	0.0127	0.0053
c_+	50.9762	18.5944	6.3367	2.2640	0.8586	0.3454	0.1413	0.0621	0.0279	0.0099
IV_-	0.4475	0.4473	0.4539	0.4590	0.4620	0.4632	0.4630	0.4624	0.4577	0.4565
IV_+	0.4662	0.4704	0.4776	0.4827	0.4856	0.4875	0.4868	0.4873	0.4867	0.4766

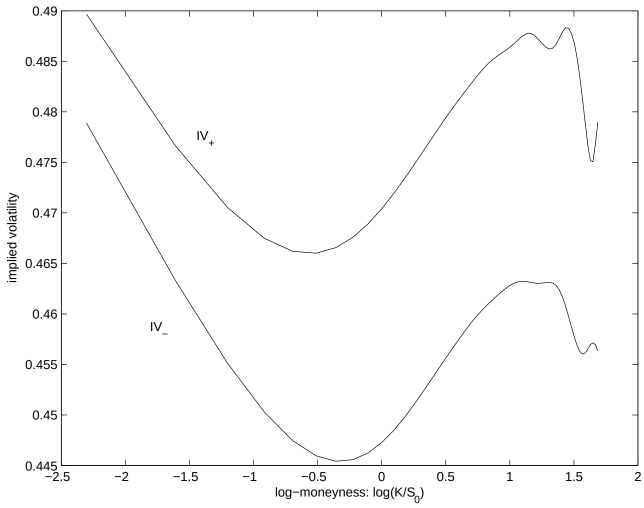

Fig. 3. Skewed smile, $t = 1, S_0 = 100, \lambda_\pm = 10, h_- = -0.03, h_+ = -0.19, c_- = 0.3, c_+ = 1.9$, $HV_- = 0.4198, HV_+ = 0.4402$; jump telegraph volatility=0.4301

Finally, we calculate exactly the case which was considered in the work of A. De Gregorio and S.M. Iacus [5]. In this paper values of the parameters was statistically estimated. The numerical work are based on weekly closings of the Dow-Jones industrial average July 1971 - Aug 1974. We admit the values of alternating intensities $\lambda_\pm$ and alternating market trends $c_\pm$, proposed by [5]. Assuming these parameters have respect to martingale measure we calibrate jump values as $h_\pm = -c_\pm/\lambda_\pm$.

The model was taken asymmetric with $\lambda_- = 48.53,\ \lambda_+ = 34.61,\ h_- = -0.0126,\ h_+ = -0.0358,\ c_- = 0.61,\ c_+ = 1.24$. It respects to simulations of a preferably bullish market with small jump corrections. The main feature of this market is in the redundancy of small jumps. The calibrated martingale distribution is strongly asymmetric.

The behaviour of implied volatility in the jump-telegraph model for these data surprisingly resembles the calibration results for stochastic volatility models of the Ornstein-Uhlenbeck type (see [17], fig. 5.1, where implied volatilities of OU-stochastic volatility model was depicted) and for jump-diffusion models (see Table 2 of [3] which contains the implied volatilities calibrated with respect to jump-diffusion model. All calculations there are prepared considering a data set of European call options on S&P 500 index).

Figure 5 depicts an implied volatility surface with respect to strike prices and maturity times.

Table 3. Dow-Jones smile, $t = 1, S_0 = 100, \lambda_- = 48.53, \lambda_+ = 34.61, h_- = -0.0126, h_+ = -0.0358, c_- = 0.61, c_+ = 1.24$

K	50	70	100	130	160	190	220	250
c_-	50.0002	30.1167	6.8313	0.4913	0.0146	0.0002	0.0000	0.0000
c_+	50.0002	30.1215	6.8838	0.5117	0.0162	0.0003	0.0000	0.0000
IV_-	0.1809	0.1762	0.1714	0.1684	0.1663	0.1645	0.1628	0.1608
IV_+	0.1819	0.1773	0.1728	0.1699	0.1679	0.1662	0.1646	0.1629

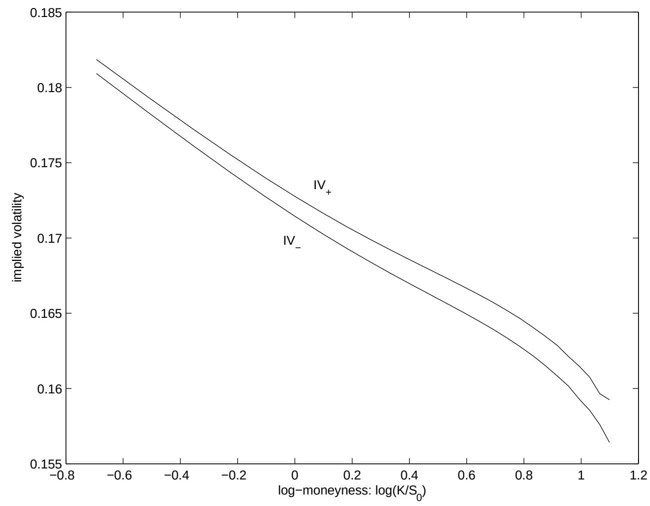

Fig. 4. Dow-Jones smile, $t = 1, S_0 = 100, \lambda_- = 48.53, \lambda_+ = 34.61, h_- = -0.0126, h_+ = -0.0358, c_- = 0.61, c_+ = 1.24$, $HV_- = 0.1630, HV_+ = 0.1642$; jump telegraph volatility=0.1661

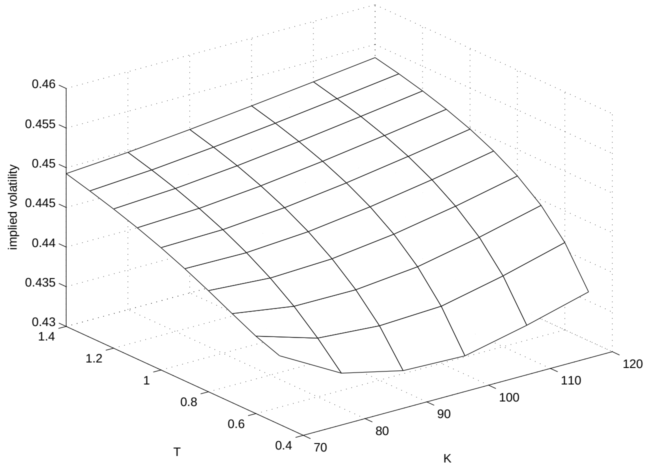

Fig. 5. Skewed smile, $S_0 = 100, \lambda_{\pm} = 10, h_- = -0.03, h_+ = -0.19, c_- = 0.3, c_+ = 1.9$

References

1. Abramowitz, M., and Stegun, I. A.: Handbook of mathematical functions. Dover, New York (1972)
2. Anh, V., and Inoue, A.: Financial markets with memory I: Dynamic models. Stochastic Analysis and Applications **23** (2005) 275-300
3. Andersen, L., and Andreasen, J.: Jump-diffusion processes: volatility smile fitting and numerical methods for option pricing. Rev. Deriv. Research **4** (2000) 231-262
4. Boyarchenko, S. I. and Levendorskiĭ, S. Z.: Non-Gaussian Merton-Black-Scholes theory. Vol.9 of Advanced Series on Statistical Science and Applied Probability. World Scientific Publishing Co. Inc., River Edge, NJ (2002)
5. De Gregorio, A., and Iacus, S. M.: Change point estimation for the telegraph process observed at discrete time. Working paper, Dipartamento di Scienze Economiche, Aziendali e Statistiche, University of Milan (2007) http://services.bepress.com/unimi/statistics/art14
6. Di Crescenzo, A., and Pellerey, F.: On prices' evolutions based on geometric telegrapher's process. Appl. Stoch. Models Business Industry **18** (2002) 171-184
7. Di Masi, G., Kabanov, Y. and Runggaldier, W.: Mean-variance hedging of options on stocks with Markov volatilities. Theory Probab. Appl. **39** (1994) 172-182
8. Elliott, R. J., and Osakwe, C.-J. U.: Option pricing for pure jump processes with Markov switching compensators. Finance & Stochastics **10** (2006) 250-275
9. Goldstein, S.: On diffusion by discontinuous movements and on the telegraph equation. Quart. J.Mech.Apl. Math. **4** (1951) 129-156
10. Inoue, A., Nakano, Y., and Anh, V.: Linear filtering of systems with memory and application to finance. Journal of Applied Mathematics and Stochastic Analysis Volume 2006 (2006), Article ID 53104, 26 pages, doi:10.1155/JAMSA/2006/53104 http://downloads.hindawi.com/journals/jamsa/2006/053104.pdf

11. Kac, M.: Some stochastic problems in physics and mathematics. Colloquium Lectures in the Pure and Applied Sciences, No. 2, hectographed, Field Research Laboratory, Soconi Mobil Oil Company, Inc., Dallas, TX (1956)
12. Kac, M.: A stochastic model related to the telegraph equation. Rocky Mountain J. Math. **4** (1974) 497-509
13. Karatzas, I, and Shreve, S. E.: Methods of mathematical finance. Spriger-Verlag, New York (1998)
14. Mazza, C., and Rullière, D.: A link between wave governed random motions and ruin processes. Insurance: Mathematics and Economics **35** (2004) 205-222
15. Merton, R.C.: Option pricing when the underlying stock returns are discontinuous. Journal of Financial Economics **3** (1976) 125-144
16. Navas, J.F.: On jump-diffusion processes for asset returns. IE working paper, WP 18/02 (2002)
17. Nicolato, E., and Venardos, E.: Option pricing in stochastic volatility models of the Ornstein-Uhlenbeck type. Math. Finance **13** (2003) 445-466
18. Ratanov, N.: Telegraph evolutions in inhomogeneous media. Markov Processes Relat. Rields **5** (1999) 53-68
19. Ratanov, N.: Telegraph models of financial markets. Rev. Colombiana de Matem. **41** (2007) 247-252
20. Ratanov, N.: A jump telegraph model for option pricing. Quant. Finance **7** (2007) 575-583
21. Ratanov, N.: Jump telegraph models and financial markets with memory. J. Appl. Math. Stoch. Anal. Volume 2007 (2007)
22. Zacks, S.: Generalized integrated telegraph processes and the distribution of related stopping times. J. Appl. Prob. **41** (2004) 497-507