Supermodular and ultramodular aggregation evaluators

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Abstract. This paper presents the construction of a new kind of evaluators, which are studied on a vector lattice. In particular, by using an important identity on a vector lattice, we prove a characterization of supermodular property and we construct supermodular evaluators, briefly named SM-evaluators. Then in a particular lattice SM-evaluators become *aggregation functions*. Similarly we construct ultramodular evaluators, briefly named UM-evaluators.

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1 Introduction

Normalized scalar evaluators were characterized in [9] as real functions which obey only two properties: boundary requirements and monotonicity. Evaluators on a complete lattice have been studied in [3] with an investigation about T_L and S_L evaluators. Our purpose in this work is to study supermodular property for normalized scalar evaluators on a complete vector lattice, which is not necessarily of finite dimension. In fact on vector lattices we have some simple but important results and in particular we study the complete vector lattice $[0,1]^n$ for studying aggregation of supermodular evaluators. First of all we recall some concepts and results largely discussed in [22]. Then we study normalized scalar evaluators from complete vector lattices and we recall some definitions given in [4].
A particular complete vector lattice is $[0,1]^n$ and it is interesting for our analysis to study aggregation operators, their compositions and relationships with evaluators and the particular case of supermodular aggregation evaluators.

2 Lattices and Vector Lattices

A partially ordered set (*poset*) is a set X on which there is a binary relation $\preceq$ that is reflexive, antisymmetric and transitive.
The set $\mathbb{R}^n = \{x = (x_1, \ldots, x_n) : x_i \in \mathbb{R}^1$ for $i = 1, \ldots, n\}$ with the ordering relation $\leq$ where $\mathbf{x}' \leq \mathbf{x}''$ in $\mathbb{R}^n$ if $x_i' \leq x_i''$ in $\mathbb{R}^1$ for $i = 1, \ldots, n$ is a partially ordered set. If two elements, x' and x'', of a partially ordered set X have a supremum (infimum) in X, it is their *join* (*meet*) and is denoted $\mathbf{x}' \vee \mathbf{x}''(\mathbf{x}' \wedge \mathbf{x}'')$. A partially ordered set that contains the join and the meet of each pair of its elements is a *lattice*.
For any positive integer n, $\mathbb{R}^n$ is a lattice with $\mathbf{x}' \vee \mathbf{x}'' = (x_1' \vee x_1'', \ldots, x_n' \vee x_n'')$ and $\mathbf{x}' \wedge \mathbf{x}'' = (x_1' \wedge x_1'', \ldots, x_n' \wedge x_n'')$ for $\mathbf{x}'$ and $\mathbf{x}''$ in $\mathbb{R}^n$.
If X' is a subset of a lattice X and X' contains the join and meet (with respect to X) of each pair of elements of X', then X' is a *sublattice* of X. For example the closed intervals of a lattice are sublattices. Moreover, a lattice in which each nonempty subset has a supremum and an infimum is *complete*. If X' is a sublattice of a lattice X and if, for each nonempty subset X'' of X', $\sup_X(X'')$ and $\inf_X(X'')$ exist and are in X', then X' is a *subcomplete* sublattice of X. A sublattice of $\mathbb{R}^n$ is subcomplete if and only if it is compact (Theorem 2.3.1 in [22]).
Moreover a *vector lattice* is an ordered vector space whose underlying poset is a lattice. We use the term *Riesz space* to mean a real vector lattice, i.e., a vector lattice over $\mathbb{R}$.
It is well known that spaces $\mathbb{R}^n$ of all vectors $[x_1, \ldots, x_n]$ with n real components, $\mathbb{R}^d$ of all infinite sequences $x_1, x_2, x_3, \ldots$ of real numbers, and $\mathbb{R}^c$ of all real functions $f(x)$ defined on the interval $0 \leq x \leq 1$ are complete vector lattices.
It is also clear that any subspace of a vector lattice, which is a sublattice as well, is a vector lattice. That is, if a subset contains with any f and g also $f+g$, $f \wedge g$, $f \vee g$ and every λf, then it is a vector lattice relative to the same operations.
We recall also that in a general vector lattice L_V we have the following identity (see page 207 in [19]), $\forall \mathbf{x}, \mathbf{y} \in L_V$

$$\mathbf{x} + \mathbf{y} = \mathbf{x} \vee \mathbf{y} + \mathbf{x} \wedge \mathbf{y}. \tag{1}$$

3 Supermodularity and ultramodularity on vector lattices

Now we consider a vector lattice L_V and a vector sublattice $X' \subset L_V$. We recall also the following definition of supermodularity.

Definition 1. *A function $f : X' \to \mathbb{R}$ is said supermodular if for every $\mathbf{x}$, $\mathbf{y} \in X'$*

$$f(\mathbf{x} \wedge \mathbf{y}) + f(\mathbf{x} \vee \mathbf{y}) \geq f(\mathbf{x}) + f(\mathbf{y}).$$

The concept of supermodularity is strictly connected to the concept of increasing differences, so that we give the following definition and result.

Definition 2. *A function $f : X' \to \mathbb{R}$ is said to have increasing differences on X' iff $f(\mathbf{x}+\mathbf{h}+\mathbf{h}') - f(\mathbf{x}+\mathbf{h}) \geq f(\mathbf{x}+\mathbf{h}') - f(\mathbf{x})$, for all $\mathbf{x} \in X'$ and all $\mathbf{h}, \mathbf{h}' \in X'_+$ with $\mathbf{h} \perp \mathbf{h}'$, where $X'_+ = \{\mathbf{x} \in X' : \mathbf{x} \geq 0\}$ is the positive cone of X'.*

Theorem 1. *Let $X' \subset L_V$ and $f : X' \to \mathbb{R}$. Then f is supermodular on X' if and only if f has increasing differences on X'.*

Proof. ($\Leftarrow$) Pick $\mathbf{x}$ and $\mathbf{y} \in X'$ and set $\mathbf{h} = \mathbf{x} - \mathbf{x} \wedge \mathbf{y}$, $\mathbf{h}' = \mathbf{y} - \mathbf{x} \wedge \mathbf{y}$. Then $\mathbf{h} \wedge \mathbf{h}' = (\mathbf{x} - \mathbf{x} \wedge \mathbf{y}) \wedge (\mathbf{y} - \mathbf{x} \wedge \mathbf{y}) = (\mathbf{x} \wedge \mathbf{y}) - (\mathbf{x} \wedge \mathbf{y}) = 0$, i.e. $\mathbf{h} \perp \mathbf{h}'$. Since $\mathbf{x} + \mathbf{y} = \mathbf{x} \vee \mathbf{y} + \mathbf{x} \wedge \mathbf{y}$, for all $\mathbf{x}$, $\mathbf{y} \in L_V$, replacing $\mathbf{x}$ by $\mathbf{h} + \mathbf{x} \wedge \mathbf{y}$ and $\mathbf{y}$ by $\mathbf{h}' + \mathbf{x} \wedge \mathbf{y}$ we get $\mathbf{x} \vee \mathbf{y} = \mathbf{x} \wedge \mathbf{y} + \mathbf{h} + \mathbf{h}'$. Since $f(\mathbf{a}+\mathbf{h}+\mathbf{h}') - f(\mathbf{a}+\mathbf{h}) \geq f(\mathbf{a}+\mathbf{h}') - f(\mathbf{a})$ for $\mathbf{h}$ and $\mathbf{h}' \in X'_+$ as chosen and every $\mathbf{a} \in L_V$, taking $\mathbf{a} = \mathbf{x} \wedge \mathbf{y}$ we get supermodularity. ($\Rightarrow$) Pick $\mathbf{a} \in L_V$ and $\mathbf{h}, \mathbf{h}' \in X'_+$ such that $\mathbf{h} \perp \mathbf{h}'$. Let $\mathbf{x} = \mathbf{a} + \mathbf{h}$ and $\mathbf{y} = \mathbf{a} + \mathbf{h}'$. Clearly, $\mathbf{x} \wedge \mathbf{y} = \mathbf{a} + \mathbf{h} \wedge \mathbf{h}' = \mathbf{a}$. Hence, $\mathbf{x} \vee \mathbf{y} = \mathbf{a} + \mathbf{h} \vee \mathbf{h}' = \mathbf{a} + \mathbf{h} + \mathbf{h}' - \mathbf{h} \wedge \mathbf{h}' = \mathbf{a} + \mathbf{h} + \mathbf{h}'$. Using the definition of supermodularity we get the property of increasing differences.

Now we consider the ultramodular (or directionally convex) functions, a class of functions that generalizes scalar convexity and that naturally arises in some economic and statistical applications (see [13] and [20]).

Definition 3. *A function $f : X' \to \mathbb{R}$ is said to be ultramodular iff*

$$f(\mathbf{x}+\mathbf{h}+\mathbf{k}) - f(\mathbf{x}+\mathbf{k}) \geq f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})$$

for all $\mathbf{x} \in X'$ with $\mathbf{h}, \mathbf{k} \in X'_+$.

Proposition 1. *A function $f\colon X' \to \mathbb{R}$ is ultramodular if and only if*

$$f(\mathbf{x}_3) - f(\mathbf{x}_1) \leq f(\mathbf{x}_4) - f(\mathbf{x}_2)$$

for all collections $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ of vectors in X' such that $\mathbf{x}_1 \leq \mathbf{x}_2 \leq \mathbf{x}_4$ and $\mathbf{x}_1 + \mathbf{x}_4 = \mathbf{x}_2 + \mathbf{x}_3$.

Proof. By definition and replacing $\mathbf{x}_4$ by $\mathbf{x}+\mathbf{h}+\mathbf{k}$, $\mathbf{x}_3$ or $\mathbf{x}_2$ by $\mathbf{x}+\mathbf{h}$ or $\mathbf{x}+\mathbf{k}$, and $\mathbf{x}_1$ by $\mathbf{x}$, where $\mathbf{h}$, $\mathbf{k} \in X'_+$, we immediately get the result.

4 Supermodular and ultramodular evaluators

As in [4] we consider a complete sublattice $(X', \leq, \perp, \top)$, which is in our case a vector sublattice too with the least and the greatest elements $\perp$ and $\top$, respectively and we will focus on the evaluation of elements from L_V by real numbers in the unit interval.

Definition 4. *A function $\phi : X' \to [0,1]$ is said to be an evaluator on X' iff it satisfies the following properties:*

– $\phi(\perp) = 0$, $\phi(\top) = 1$,
– $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$ *for all* $\mathbf{x}, \mathbf{y} \in X'$ *such that* $\mathbf{x} \leq \mathbf{y}$.

An evaluator ϕ is called *existential* if for arbitrary $\mathbf{x} \in X'$,

$$\phi(\mathbf{x}) = 0 \Rightarrow \mathbf{x} = \perp.$$

An evaluator ϕ is called *universal* if for arbitrary $\mathbf{x} \in X'$,

$$\phi(\mathbf{x}) = 1 \Rightarrow \mathbf{x} = \top.$$

The standard comparison of real numbers allows us to compare evaluators on the same system of objects. In this case we utilize the usual pointwise ordering of functions. This means that if ϕ_1 and ϕ_2 are two evaluators on X', we say that ϕ_1 is greater than ϕ_2, with notation $\phi_2 \leq \phi_1$ if for all $\mathbf{x} \in X'$, $\phi_2(\mathbf{x}) \leq \phi_1(\mathbf{x})$. The greatest evaluator is the existential evaluator ϕ_E defined for all $\mathbf{x} \in X'$ by

$$\phi_E(\mathbf{x}) = \begin{cases} 0 & \text{if} \quad \mathbf{x} = \perp, \\ 1 & \text{otherwise.} \end{cases}$$

The smallest evaluator is the universal evaluator ϕ_U defined for all $\mathbf{x} \in X'$ by

$$\phi_U(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad \mathbf{x} = \top, \\ 0 & \text{otherwise.} \end{cases}$$

Now we introduce the modular, supermodular and ultramodular evaluators on the general complete vector sublattice $X' \subset L_V$.

Definition 5. *An operation $SM : X' \to [0, 1]$ is said to be a supermodular evaluator iff it is an evaluator and it satisfies the following property:*

$$SM(\mathbf{x} \wedge \mathbf{y}) + SM(\mathbf{x} \vee \mathbf{y}) \geq SM(\mathbf{x}) + SM(\mathbf{y}).$$

In the case of equality in the above equation, we have the *modular* evaluator.

Definition 6. *An operation $U : X' \to [0, 1]$ is said to be a ultramodular evaluator (UM evaluator for short) iff U is an evaluator satisfying the property:*

$$U(\mathbf{x}_1) + U(\mathbf{x}_4) \geq U(\mathbf{x}_2) + U(\mathbf{x}_3)$$

for all collections $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ of vectors in X' such that $\mathbf{x}_1 \leq \mathbf{x}_2 \leq \mathbf{x}_4$ and $\mathbf{x}_1 + \mathbf{x}_4 = \mathbf{x}_2 + \mathbf{x}_3$.

For a detailed study of the properties of supermodular and ultramodular functions we refer to [13], [14], [20], [21] and [22] as well as the references therein.
One of the most important consequence is that the t-norms are SM evaluators, while the t-conorms are not.
It is known that aggregation of evaluators yields an evaluator (see Proposition 1 in [4]). Now we continue the discussion about aggregation of several kinds of evaluators and we will focus on aggregation of supermodular evaluators. We

would like to know which aggregation of supermodular evaluators yields a supermodular evaluator.

Let $K_1, \dots, K_m : X' \to [0,1]$, $i = 1, 2, \dots, m$ be SM evaluators. A vector function $\mathbf{K} : X' \to [0,1]^m$ given by $\mathbf{K}(\mathbf{x}) = (K_1(\mathbf{x}), \dots, K_m(\mathbf{x}))$ is said to be an SM evaluator.

Proposition 2. *If* $\psi \colon [0,1]^m \to [0,1]$ *is an increasing UM evaluator and* $\mathbf{K} \colon X' \to [0,1]^m$ *is an increasing SM evaluator, then the function*

$$H \colon X' \to [0,1] \textit{given by} H(x_1, \dots, x_n) = \psi(\mathbf{K})(\mathbf{x}) = \psi(K_1(\mathbf{x}), \dots, K_m(\mathbf{x}))$$

is an SM evaluator.

Proof. We consider 3 vectors $\mathbf{x}, \mathbf{h}, \mathbf{k}$ such that $\mathbf{h}, \mathbf{k} \geq 0$ and $\mathbf{h} \perp \mathbf{k}$. For all $i = 1, \dots, m$, $K_i(\mathbf{x}+\mathbf{h}+\mathbf{k}) - K_i(\mathbf{x}+\mathbf{k}) \geq K_i(\mathbf{x}+\mathbf{h}) - K_i(\mathbf{x})$ and then there exist s_i, t_i with $s_i \geq t_i \geq 0$ such that

$$K_i(\mathbf{x}+\mathbf{h}+\mathbf{k}) = K_i(\mathbf{x}+\mathbf{k}) + s \qquad K_i(\mathbf{x}+\mathbf{h}) = K_i(\mathbf{x}) + t_i.$$

So there exist $\mathbf{s}, \mathbf{t}$ vectors in $\mathbb{R}^m$ such that $\mathbf{s} \geq \mathbf{t} \geq \mathbf{0}$ and

$$\mathbf{K}(\mathbf{x}+\mathbf{h}+\mathbf{k}) = \mathbf{K}(\mathbf{x}+\mathbf{k}) + \mathbf{s} \qquad \mathbf{K}(\mathbf{x}+\mathbf{h}) = \mathbf{K}(\mathbf{x}) + \mathbf{t}.$$

Since ψ is an increasing UM evaluator and $\mathbf{K}$ is increasing in each variable one has:

$$\psi(\mathbf{K})(\mathbf{x}+\mathbf{h}+\mathbf{k}) - \psi(\mathbf{K})(\mathbf{x}+\mathbf{k}) = \psi(\mathbf{K}(\mathbf{x}+\mathbf{k})) + \mathbf{s}) -$$

$$-\psi(\mathbf{K}(\mathbf{x}+\mathbf{k})) \geq \psi(\mathbf{K}(\mathbf{x}+\mathbf{k}) + \mathbf{t}) - \psi(\mathbf{K}(\mathbf{x}+\mathbf{k})) \geq$$

$$\geq \psi(\mathbf{K}(\mathbf{x}) + \mathbf{t})) - \psi(\mathbf{K}(\mathbf{x})) =$$

$$\psi(\mathbf{K}(\mathbf{x}+\mathbf{h})) - \psi(\mathbf{K}(\mathbf{x})) = \psi(\mathbf{K})(\mathbf{x}+\mathbf{h}) - \psi(\mathbf{K})(\mathbf{x}).$$

Corollary 1. *Let A be an SM evaluator. If $\varphi : [0,1] \to [0,1]$ is a continuous increasing and convex function with $\varphi(0) = 0$ and $\varphi(1) = 1$ then the function*

$$A_\varphi(\mathbf{x}) := \varphi\left(A(x_1, \dots, x_n)\right)$$

is an SM evaluator.

Proof. It is obvious that $A_\varphi(\perp) = 0$ and $A_\varphi(\top) = 1$. Then, it suffices to apply the above theorem to the function $H(x_1, \dots, x_n) = \psi(K_1(\mathbf{x}))$, with $\psi = \varphi$ and $K_1 = A$. In fact scalar convex functions are ultramodular and so ψ is increasing and ultramodular.

5 The $[0,1]^n$ case

A particular and important kind of evaluator is the aggregation function.
Let $n \in \mathbb{N}, n \geq 2$. An *$n$–ary aggregation function* is a mapping A from $\bigcup_{n\in\mathbb{N}} [0,1]^n$ into $[0,1]$ that satisfies the following properties:

(A1) $A(0,\ldots,0) = 0$ and $A(1,\ldots,1) = 1$;
(A2) A is increasing in each component.

Each aggregation function A can be canonically represented by a family $(A_{(n)})_{n\in\mathbb{N}}$ of n-ary operations, e.g., functions $A_{(n)} : [0,1]^n \to [0,1]$, given by

$$A_{(n)}(x_1,\ldots,x_n) = A(x_1,\ldots,x_n).$$

Each function $A_{(n)}$ is an evaluator on the complete vector lattice $([0,1]^n, \leq, \perp, \top)$. If $A(x_1,\ldots,x_n) = 0$ implies that $x_i = 0$ for $i = 1,\ldots,n$, we say that aggregation operator A does not have zero divisors. In this case, function $A_{(n)}$ is an existentional evaluator and A is an existentional aggregator. If $A(x_1,\ldots,x_n) = 1$ implies that $x_i = 1$ for $i = 1,\ldots,n$, function $A_{(n)}$ is a universal evaluator and A is a universal aggregator. Now we study particular families of SM evaluators on the complete vector lattice $([0,1]^n, \leq, \perp, \top)$ used for the aggregation of evaluated elements from $[0,1]^n$.

Proposition 3. *The quasi–arithmetic mean*

$$M_f(\mathbf{x}) := f^{-1}\left(\frac{f(x_1) + \cdots + f(x_n)}{n}\right),$$

where $f : [0,1] \to \mathbb{R}$ is a continuous strictly monotone function, is an SM evaluator if and only if f^{-1} is convex.

Proof. M_f is an SM aggregation evaluator if, and only if, for any couple of integers α, β, such that $1 \leq \alpha < \beta \leq n$,

$$\overline{M}_f(x_\alpha, x_\beta) = M_f(a_1,\ldots,a_{\alpha-1}, x_\alpha, a_{\alpha+1},\ldots,a_{\beta-1}, x_\beta, a_{\beta+1},\ldots,a_n)$$

is an SM evaluator, if and only if f^{-1} is convex.

Due to wellknown characterization of quasi-arithmetic means M_f bounded from above by the arithmetic mean M (see lemma 1 in [5]) we have the next result.

Corollary 2. *M_f is an SM aggregation evaluator if and only if $M_f \leq M$.*

A similar result holds for weighted quasi-arithmetic means, that is with n-dimensional weighting vector $\mathbf{w} = (w_1,\ldots,w_n) \in [0,1]^n$ such that $\sum_{i=1}^n w_i = 1$ and $W_f(\mathbf{x}) := f^{-1}\left(\frac{\sum_{i=1}^n w_i f(x_i)}{n}\right)$ we have:

Corollary 3. *W_f is an SM aggregation evaluator if and only if $W_f \leq W$.*

Proof. From convexity of f^{-1} and Jensen's inequality we have:

$$W_f(\mathbf{x}) = f^{-1}\left(\frac{\sum_{i=1}^n w_i f(x_i)}{n}\right) \leq \left(\frac{\sum_{i=1}^n w_i f^{-1}(f(x_i))}{n}\right) =$$
$$= \frac{\sum_{i=1}^n w_i x_i}{n} = W(\mathbf{x}).$$

Proposition 4. *Let $f_i : [0,1] \to [0,1]$ be increasing functions such that $f_i(0) = 0$ and $f_i(1) = 1$, $i = 1, 2 \ldots, n$. If A is an SM evaluator, then the function defined by*

$$A_{f_1,\ldots,f_n}(x_1,\ldots,x_n) := A(f_1(x_1),\ldots,f_n(x_n))$$

is an SM aggregation evaluator.

Proof. It is obvious that $A_{f_1,\ldots,f_n}(0,\ldots,0) = 0$, $A_{f_1,\ldots,f_n}(1,\ldots,1) = 1$ and $A_{f_1,\ldots,f_n}$ is increasing in each place. Moreover, given $x_1^j \leq x_2^j$, $\forall j = 1,\ldots,n$, one obtains

$$A_{f_1,\ldots,f_n}(x_1^1, x_1^2, \ldots, x_1^n) + A_{f_1,\ldots,f_n}(x_2^1, x_2^2, \ldots, x_2^n) \geq$$
$$A_{f_1,\ldots,f_n}(x_2^1, \ldots, x_2^h, x_1^{h+1}, \ldots, x_1^n) + A_{f_1,\ldots,f_n}(x_1^1, \ldots, x_1^h, x_2^{h+1}, \ldots, x_2^n),$$

because A is an SM evaluator and f_i are increasing.

Corollary 4. *Let A be an SM aggregation evaluator and $\varphi : [0,1] \to [0,1]$ be a continuous strictly monotone function with $\varphi(0) = 0$ and $\varphi(1) = 1$. Then the following statements are equivalent:*

(a) *φ is concave;*
(b) *the function $A_\varphi(\mathbf{x}) := \varphi^{-1}(A(\varphi(x_1),\ldots,\varphi(x_n)))$ is an SM aggregation evaluator.*

Proof. $(a) \Rightarrow (b)$ If φ is concave and positive, then φ^{-1} is convex. By Corollary 1 and Proposition 4 we have the result.
$(b) \Rightarrow (a)$ If $A_\varphi(\mathbf{x})$ is an SM aggregation evaluator, we can consider

$$M_\varphi(\mathbf{x}) := \varphi^{-1}\left(\frac{\varphi(x_1) + \cdots + \varphi(x_n)}{n}\right).$$

By Proposition 3, $M_\varphi(\mathbf{x})$ an SM aggregation evaluator if and only if φ^{-1} is convex. So, φ is concave.

A special subclass of SM evaluators is that formed by modular aggregation functions, i.e. those A's for which

$$A(\mathbf{x} \wedge \mathbf{y}) + A(\mathbf{x} \vee \mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y}),$$

for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$. For these operators the following characterization holds.

Proposition 5. *For an aggregation operator A the following statements are equivalent:*

(a) *A is modular;*
(b) *there exist increasing functions $f_i : [0,1] \to [0,1]$ such that $f_i(0) = 0$, $i = 1,\dots,n$, $\sum_{i=1}^n f_i(1) = 1$, and*

$$A(x_1,\dots,x_n) = \sum_{i=1}^{n} f_i(x_i); \tag{2}$$

(c) *A is strongly additive, i.e., if $\mathbf{x} \wedge \mathbf{y} = \mathbf{0}$ and $\mathbf{x}+\mathbf{y} \in [0,1]^n$, then $A(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y})$.*

Proof. $(a) \Rightarrow (b)$ If A is modular, set $f_i(x_i) := A(0,\dots,x_i,\dots,0)$, $\forall i = 1,\dots,n$. From modularity of A we get

$$\begin{aligned} A(\mathbf{x}) + A(\mathbf{0}) &= A(x_1, 0\dots,0) + A(0,x_2,\dots,x_n) = \\ &= f_1(x_1) + A(0,x_2,\dots,x_n), \end{aligned}$$

and also,

$$\begin{aligned} A(0,x_2,\dots,x_n) + A(\mathbf{0}) &= A(0,x_2,0\dots,0) + A(0,0,x_3\dots,x_n) = \\ &= f_2(x_2) + A(0,0,x_3,\dots,x_n), \end{aligned}$$

which implies (b) recursively.
$(b) \Rightarrow (c)$ $A(\mathbf{x}+\mathbf{y}) = A(x_1+y_1,\dots,x_n+y_n) = \sum_{i=1}^n f_i(x_i+y_i)$.
Since $\mathbf{x} \wedge \mathbf{y} = \mathbf{0}$, then $\sum_{i=1}^n f_i(x_i+y_i) = \sum_{i=1}^n (f_i(x_i) + f_i(y_i))$, i.e. $A(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y})$.
$(c) \Rightarrow (a)$ By (1) $\mathbf{x}+\mathbf{y} = \mathbf{x}\wedge\mathbf{y} + \mathbf{x}\vee\mathbf{y}$. So,

$$A(\mathbf{x}\wedge\mathbf{y}) + A(\mathbf{x}\vee\mathbf{y}) = A(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y}).$$

6 Conclusion

Aggregation of evaluators by an aggregation operator yields an evaluator [4]. In this work we have shown that aggregation of supermodular evaluators yields a supermodular evaluator and in several cases a supermodular aggregation evaluator. In particular we have analyzed supermodularity property for quasi-arithmetic means, by using appropriate functions on $[0,1]$. So we have studied this kind of transformation in general to present some interesting applications in our future work.

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