A note on the pricing of perpetual continuous-installment options

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Abstract. A perpetual continuous-installment option is an infinite maturity option in which the premium is paid continuously instead of up-front. The holder has the right to terminate payments at any time by either exercising the option or dropping the option contract. Within the standard Black-Scholes framework, the perpetual continuous-installment option pricing problem is discussed and solved as a free boundary problem for a parabolic inhomogeneous ordinary differential equation. The closed-form solution obtained for the special case of a non-dividend paying asset gives the possibility to observe some analytical properties of the initial premium and the optimal boundaries for the perpetual continuous-installment call option.

Keywords. Perpetual continuous-installment option, Black-Scholes equation, free boundary problem.

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1 Introduction

In this paper we consider a particular type of perpetual option in which the buyer pays a small up-front premium and then a constant stream of installment premiums to acquire and keep the right, but not the obligation, to exercise the option at any time during the infinite life of the option. However, the holder can choose at any time to stop making installment payments by either exercising the option or dropping the option contract. Crucially, though, there must be a critical value of the underlying asset at which it is optimal to exercise, as well as a critical value at which it is advantageous to drop the option contract.

Literature on installment options is quite recent. [7] and [8] derive no-arbitrage bounds for the initial premium of an installment option and study static versus dynamic hedging strategies within a Black-Scholes framework with stochastic volatility. Their analysis is restricted to European discrete-installment options, which allows for an analogy with compound options, previously considered by [12] and [20]. [9] values venture capital using an analogy with installment option. [3] develops a dynamic-programming procedure to price American discrete-installment options and derives some theoretical properties of the installment option contract within the geometric Brownian motion framework. This approach is applied to installment warrants, which are actively traded on the Australian Stock Exchange. [6] proposes three alternative approaches for valuing American continuous-installment options written on assets without dividends or with constant continuous dividend yield. This analysis can be applied to value installment derivatives on both non-dividend paying stocks and foreign currencies. [1] and [2] use a partial Laplace transform to derive an integral equation for the location of the free stopping boundary for a European continuous-installment option and study its asymptotic behavior close to expiry. Using the concept of compound options, [13] derives a closed-form solution to the initial premium of a European discrete-installment option in terms of multi-dimensional cumulative normal distribution functions and examines the limiting case of an installment option with a continuous payment plan. Finally, [14] and [15] apply the Laplace transform approach to solve the valuation problems of American and European continuous-installment options.

Installment options can be found embedded in other contracts, including life insurance contracts, and are also frequently used in financing capital investment projects with some examples given in [11]. In the field of real options a meaningful model is that due to [17], in which a firm invests in a project continuously and receives no payoff until the project is complete. Although the model of [17] bears many resemblances to a European continuous-installment option, it also has some differences, notably that the project can be resumed at a later time without loss of earlier capital outlays, whereas an installment option lapses if the holder halts installment payments.

The aim of this paper is to price perpetual continuous-installment options written on assets without dividends or with constant continuous dividend yield by using the standard Black-Scholes framework and extending the analysis developed by [6]. This option pricing problem presents some significant analogies with the investment decision problem under uncertainty analyzed by [10], in order to determine an optimal entry/exit strategy for a firm facing the decision whether or not to engage in an investment project, which is costly to both activate and suspend. Furthermore, it might be of interest to point out similarity between the analysis presented here and that concerned with the choice of optimal capital structure and risk management as studied in [4] and [16].

The ability to halt installment payments during an infinite time horizon by either exercising the option or dropping the option contract leads to two free boundaries separating the region where it is advantageous to hold from those where, respectively, exercise and stopping are optimal. In theory, exercise and stopping strategies should take place only on these free boundaries, which themselves are unknown and must be determined along with the option's initial premium. However, this sort of free boundary problem is substantially easier to handle than that arising for finite-lived American continuous-installment options, since the initial premium is not governed by an inhomogeneous partial differential equation and the free boundaries do not vary with respect to the time variable. In particular, it is possible to derive a closed-form solution to the free boundary problem for perpetual continuous-installment options written on a non-dividend paying asset, and provide some analytical properties of the initial premium and the optimal boundaries of the call option.

The layout of the remainder of this paper is as follows. In Section 2, we discuss the distinctive features of the pricing problem for infinite-lived continuous-installment options within the standard Black-Scholes framework and extending the analysis developed by [6]. The pricing problems of perpetual continuous-installment calls and puts are analyzed in Sections 3 and 4, respectively. We conclude in Section 5.

2 The reduction to the Black-Scholes ODE for perpetual continuous-installment options

We assume the standard model for perfect capital markets, continuous trading, no-arbitrage opportunities, a constant risk-free interest rate $r \geq 0$, and an asset without dividends or with constant continuous dividend yield $\delta \geq 0$ with price process $S = (S_t)_{t\geq 0}$ governed by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1}$$

where $\mu = (r - \delta)$ and $W = (W_t)_{t\geq 0}$ is a standard Wiener process under the risk-neutral probability measure. If the underlying asset is a foreign currency, δ is replaced by the foreign risk-free interest rate r_f.

Consider a perpetual continuous-installment option written on an asset whose price process S follows (1) and with constant installment per unit time $q \geq 0$ and plain vanilla payoff

$$H(S_t) = \begin{cases} (S_t - K)^+, & \text{for a call option} \\ (K - S_t)^+, & \text{for a put option} \end{cases} \qquad \forall\, t \geq 0, \tag{2}$$

where $(x)^+ = \max(x, 0)$ and $K \geq 0$ is the exercise (or strike) price of the option. In order to price such an infinite maturity option contract, the standard Black-Scholes framework is adopted and the analysis developed by [6] to the valuation of American continuous-installment options is extended.

From [6], the initial premium $V_t = V(S_t, t; q)$ of a finite-lived continuous-installment option is governed by the inhomogeneous Black-Scholes partial differential equation (PDE)

$$\frac{\partial V_t}{\partial t} + \mu S_t \frac{\partial V_t}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2} - rV_t = q\,. \tag{3}$$

The term $q \geq 0$ represents the continual input of cash via the installment premium: in a time period dt a constant amount $q\,dt$ must be paid to keep the option alive. If $q = 0$ we have the usual Black-Scholes PDE for equity options.

Independently from the type of exercise, equation (3) is valid only on the continuation region, that is, on the region where it is advantageous to continue paying installment premiums since the option is worth more alive than dead. For American continuous-installment options, specifically, equation (3) must be solved together with the appropriate boundary conditions at the free stopping and exercise boundaries, which themselves are unknown and must be solved for. In the free boundary problems studied in [6], the conditions at the two free boundaries are that the initial premium function and its Delta ($\partial V_t/\partial S$) must be continuous across each free boundary.

However, since the dividend yield, the installment premium and the boundary conditions are time-independent, then the initial premium of the continuous-installment option with infinite maturity will not depend upon time either and so PDE (3) reduces to an ordinary differential equation. This is a property of perpetual options when the contract details are time-homogeneous, provided that there is a finite solution. If the contract has no finite solution, then it has no financial meaning. Furthermore, the free stopping and exercise boundaries, which are time paths of critical asset prices when the life of the option is finite, become time-invariant constants for the theoretical case of infinite maturity or perpetual options.

Let us denote the Black-Scholes initial premium of a perpetual continuous-installment option at time $t \geq 0$ by $V_t = V(S_t; q)$, defined on domain $\mathcal{D} \equiv \big\{S_t \in [0, \infty)\big\}$. Over the range of underlying asset values at which it is optimal to keep the option alive paying continuously the installment premium q, the initial premium V_t of this option must satisfy the following equation

$$\frac{1}{2}\,\sigma^2 S_t^2 \frac{d^2 V_t}{dS^2} + \mu S_t \frac{dV_t}{dS} - rV_t = q. \tag{4}$$

Equation (4) is the inhomogeneous Black-Scholes ordinary differential equation (ODE) we get when the initial premium of the option is a function of S_t only.

Proposition 1. *The solution $V(S_t; q)$ to the Black-Scholes ODE (4) is given by*

$$V(S_t; q) = \alpha S_t^{\gamma_1} + \beta S_t^{\gamma_2} - \frac{q}{r}, \tag{5}$$

with

$$\gamma_{1,2} = \frac{\left(\frac{1}{2}\,\sigma^2 - \mu\right) \pm \sqrt{\left(\frac{1}{2}\,\sigma^2 - \mu\right)^2 + 2r\sigma^2}}{\sigma^2}, \tag{6}$$

and where α and β are constants to be determined.

Proof. The Black-Scholes ODE (4) is a second-order, linear and inhomogeneous ordinary differential equation. To find the complementary function for the homogeneous part of (4), we put $V(S_t;q) = S_t^\gamma$. Substitution yields

$$f(\gamma) \equiv \frac{1}{2}\sigma^2\gamma^2 - \Big(\frac{1}{2}\sigma^2 - \mu\Big)\gamma - r = 0,$$

and

$$\gamma_{1,2} = \frac{\Big(\frac{1}{2}\sigma^2 - \mu\Big) \pm \sqrt{\Big(\frac{1}{2}\sigma^2 - \mu\Big)^2 + 2r\sigma^2}}{\sigma^2}.$$

We note that the root γ_1 is positive and the other γ_2 is negative. In fact, if we rearrange this to

$$\gamma_{1,2} = \frac{\Big(\frac{1}{2}\sigma^2 - \mu\Big) \pm \sqrt{\Big(\frac{1}{2}\sigma^2 + \mu\Big)^2 + 2(r-\mu)\sigma^2}}{\sigma^2},$$

we see immediately that $\gamma_1 \geq 1$, since $(r - \mu) = \delta \geq 0$.

In the particular case where $\delta = 0$ (i.e., the underlying asset pays no dividends), we get

$$\gamma_1 = 1, \qquad \text{and} \qquad \gamma_2 = -\frac{2r}{\sigma^2}.$$

Hence, the general solution of the homogeneous part is given by

$$V(S_t;q) = \alpha S_t^{\gamma_1} + \beta S_t^{\gamma_2},$$

where α and β are constants to be determined. Trying a linear form $V(S_t;q) = m + nS$ for the inhomogeneous part gives

$$n\mu S - r(m + nS) = q.$$

In order to find the coefficients m and n, we impose the following conditions

$$\begin{cases} -rm = q \\ n(\mu - r)S = 0 \end{cases} \Rightarrow \quad \begin{cases} m = -q/r \\ n = 0 \end{cases}.$$

Finally, we can write the general solution of the Black-Scholes ODE (4) as

$$V(S_t;q) = \alpha S_t^{\gamma_1} + \beta S_t^{\gamma_2} - \frac{q}{r},$$

which is equation (5). □

The last term in the above expression for $V(S_t;q)$ is the present value of a constant stream of installment premiums to be paid to hold the option for an infinite time horizon. Therefore, the remaining part of the solution must be the value of the option to stop installment payments optimally.

3 Pricing of a perpetual continuous-installment call option

Consider a perpetual continuous-installment call written on an asset whose price process S follows (1) and with constant installment per unit time $q \geq 0$ and plain vanilla payoff $H(S_t)$. Let $C(S_t;q)$ be the initial premium function of this option, defined on the domain $\mathcal{D}$.

Independently from time t of entering this option contract, there exists a sufficiently low underlying price, $\bar{S}_l^c$, for which it will be advantageous to terminate payments by dropping the option contract, as well as a sufficiently high underlying price, $\bar{S}_u^c$, for which it will be advantageous to terminate payments by exercising the option. The stopping and exercise boundaries $\bar{S}_l^c$ and $\bar{S}_u^c$ are time-invariant constants and divide the domain $\mathcal{D}$ into a *stopping region* $\mathcal{D}_1 = \{S_t \in [0, \bar{S}_l^c]\}$, a *continuation region* $\mathcal{D}_2 = \{S_t \in (\bar{S}_l^c, \bar{S}_u^c)\}$ and an *exercise region* $\mathcal{D}_3 = \{S_t \in [\bar{S}_u^c, \infty)\}$.

The initial premium function $C(S_t;q)$ satisfies the inhomogeneous Black-Scholes ODE (4) in $\mathcal{D}_2$, that is,

$$\frac{1}{2}\sigma^2 S_t^2 \frac{d^2 C(S_t;q)}{dS^2} + \mu S_t \frac{dC(S_t;q)}{dS} - rC(S_t;q) = q, \quad \text{on } \mathcal{D}_2. \tag{7}$$

Following the analysis of [18], [19] and [6], we determine that $C(S_t;q)$ and the stopping and exercise boundaries $\bar{S}_l^c$ and $\bar{S}_u^c$ jointly solve a free boundary problem consisting of (7), subject to the following boundary conditions

$$C(S_t;q) = 0, \qquad \text{on } \mathcal{D}_1, \tag{8}$$

$$\frac{dC(S_t;q)}{dS} = 0, \qquad \text{on } \mathcal{D}_1, \tag{9}$$

$$C(S_t;q) = S_t - K, \qquad \text{on } \mathcal{D}_3, \tag{10}$$

$$\frac{dC(S_t;q)}{dS} = 1, \qquad \text{on } \mathcal{D}_3. \tag{11}$$

Substituting the expression of $C(S_t;q)$ from (5) into the left-hand side of equations (8–11) and calculating each pair of them at the free stopping and exercise boundaries respectively, yields

$$\alpha_c(\bar{S}_l^c)^{\gamma_1} + \beta_c(\bar{S}_l^c)^{\gamma_2} - \frac{q}{r} = 0, \tag{12}$$

$$\gamma_1\alpha_c(\bar{S}_l^c)^{(\gamma_1-1)} + \gamma_2\beta_c(\bar{S}_l^c)^{(\gamma_2-1)} = 0, \tag{13}$$

$$\alpha_c(\bar{S}_u^c)^{\gamma_1} + \beta_c(\bar{S}_u^c)^{\gamma_2} - \frac{q}{r} = \bar{S}_u^c - K, \tag{14}$$

$$\gamma_1\alpha_c(\bar{S}_u^c)^{(\gamma_1-1)} + \gamma_2\beta_c(\bar{S}_u^c)^{(\gamma_2-1)} = 1. \tag{15}$$

Equations (12–15) allow us to determine the unknown constants α_c and β_c of the initial premium function $C(S_t;q)$, and the unknown optimal stopping and exercise boundaries $\bar{S}_l^c$ and $\bar{S}_u^c$. Since, in general, no closed-form solution for the system of nonlinear equations (12–15) can be found, we have to resort to a suitable numerical method. In order to solve numerically the system of equations (12–15), arising from the boundary conditions of the free boundary problem, we use the Newton-Raphson method.

As an example, we will consider $K = 100$, $r = 0.07$, $\delta = 0.05$, $\sigma = 0.25$ and $q = 1$, for which we obtain the following stopping and exercise boundaries: $\bar{S}_l^c = 35.965$ and $\bar{S}_u^c = 213.692$. Figure 1 shows the initial premium function $C(S_t;q)$ of the perpetual continuous-installment call for our standard example. For comparison, the value (lump-sum premium) of the perpetual standard call is also shown.

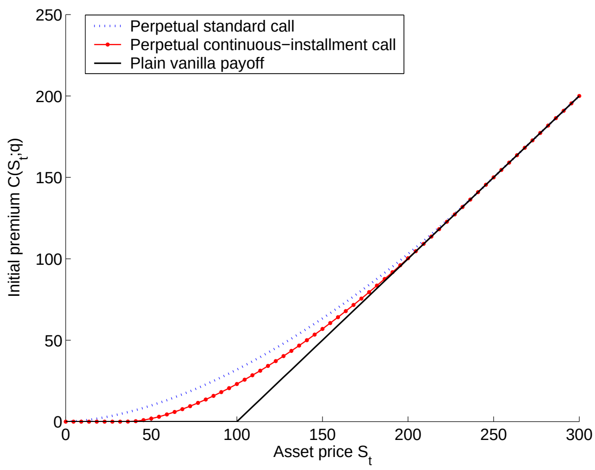

Fig. 1. Initial premium of perpetual continuous-installment call and value of perpetual standard call when underlying asset pays dividends. Parameters used: $K = 100$, $r = 0.07$, $\delta = 0.05$, $\sigma = 0.25$ and $q = 1$.

3.1 Closed-form formulas for the case of a non-dividend paying asset

An interesting special case occurs when the underlying asset does not pay dividends, i.e., the continuous dividend yield δ equals zero. In the literature, it is well known that in this situation it is never optimal to exercise the perpetual standard call, independently of the strike price[3]. The value of the option simply coincides with the value of the underlying asset.

For the perpetual continuous-installment call we cannot obtain the same result, since in the absence of dividends early exercise may be optimal. Firstly, for the case of a non-dividend paying asset we are able to find a closed-form solution to both the initial premium $C(S_t;q)$ and the optimal boundaries $\bar{S}_l^c$ and $\bar{S}_u^c$, because it is possible to perform an algebraic simplification of the nonlinear equations (12–15) and thus solve them analytically.

Proposition 2. *If the underlying asset does not pay dividends, i.e., $\delta = 0$, the initial premium function of a perpetual continuous-installment call is defined by*

$$C(S_t;q) = \alpha_c S_t + \beta_c S_t^{\gamma_2} - \frac{q}{r}, \tag{16}$$

where the constants α_c and β_c are given by

$$\alpha_c = \frac{1}{1 - \left(1 - \frac{rK}{q}\right)^{1-\frac{1}{\gamma_2}}}, \tag{17}$$

$$\beta_c = -\frac{1}{\gamma_2}\left[\frac{q}{\left(r + \frac{\sigma^2}{2}\right)}\right]^{1-\gamma_2}\left[\frac{1}{1 - \left(1 - \frac{rK}{q}\right)^{1-\frac{1}{\gamma_2}}}\right]^{\gamma_2}, \tag{18}$$

with $\gamma_2 = -\frac{2r}{\sigma^2}$. Furthermore, the optimal stopping and exercise boundaries are defined respectively by

$$\bar{S}_l^c = \frac{q}{\left(r + \frac{\sigma^2}{2}\right)}\left[1 - \left(1 - \frac{rK}{q}\right)^{1-\frac{1}{\gamma_2}}\right], \tag{19}$$

$$\bar{S}_u^c = \frac{q}{\left(r + \frac{\sigma^2}{2}\right)}\left[\left(1 - \frac{rK}{q}\right)^{\frac{1}{\gamma_2}} - \left(1 - \frac{rK}{q}\right)\right]. \tag{20}$$

[3] By not exercising, the owner loses nothing, because cash does not leak out via the dividend. Thus the owner can wait until the present value of paying the strike price is arbitrary small. Essentially, the option holder gets the underlying asset for nothing in terms of the present value of the strike price. Hence, the option value must be the same as the underlying value in order to avoid an arbitrage opportunity.

Proof. From Proposition 1 we know that $\gamma_1 = 1$ when the dividend yield δ equals zero. Then, it follows that the general solution $C(S_t; q)$ to the inhomogeneous Black-Scholes ODE (7) reduces to (16), and the system of nonlinear equations (12–15) can be easily simplified to the form

$$\alpha_c \bar{S}_l^c + \beta_c (\bar{S}_l^c)^{\gamma_2} - \frac{q}{r} = 0, \tag{12$'$}$$

$$\alpha_c + \gamma_2 \beta_c (\bar{S}_l^c)^{(\gamma_2 - 1)} = 0, \tag{13$'$}$$

$$\alpha_c \bar{S}_u^c + \beta_c (\bar{S}_u^c)^{\gamma_2} - \frac{q}{r} = \bar{S}_u^c - K, \tag{14$'$}$$

$$\alpha_c + \gamma_2 \beta_c (\bar{S}_u^c)^{(\gamma_2 - 1)} = 1. \tag{15$'$}$$

Finding an expression for the term $(\bar{S}_l^c)^{\gamma_2}$ from (13$'$) and substituting it into (12$'$), we have

$$\bar{S}_l^c = \frac{q}{\left(r + \frac{\sigma^2}{2}\right)} \left(\alpha_c\right)^{-1}.$$

Substituting for $\bar{S}_l^c$ into (13$'$), yields

$$\beta_c = -\frac{1}{\gamma_2} \left[\frac{q}{\left(r + \frac{\sigma^2}{2}\right)} \right]^{1-\gamma_2} \left(\alpha_c\right)^{\gamma_2}.$$

Substituting for β_c into (15$'$), gives

$$\bar{S}_u^c = \frac{q}{\left(r + \frac{\sigma^2}{2}\right)} \left[\left(\alpha_c - 1\right) \left(\alpha_c\right)^{-\gamma_2} \right]^{\frac{1}{\gamma_2 - 1}}.$$

Finally, substituting for $\bar{S}_u^c$ and β_c into (14$'$) we obtain expression (17), and combining this with the above results we can found expressions (18–20). □

Since the initial premium $C(S_t; q)$ and the optimal boundaries $\bar{S}_l^c$ and $\bar{S}_u^c$ must be positive real numbers, we impose the constraint

$$\left(1 - \frac{rK}{q}\right) > 0. \tag{21}$$

The financial interpretation of (21) is given in Proposition 3. While for perpetual standard calls the early exercise is never optimal when the underlying asset does not pay dividends, a perpetual continuous-installment call will be exercised early also when the dividend yield equals zero.

Proposition 3. *Let the payoff function be given by (2), with the strike price $K > 0$. If the underlying asset does not pay dividends, i.e., $\delta = 0$, the risk-free interest rate r is a positive constant and*

$$q > rK, \tag{22}$$

then it is optimal to exercise early an American continuous-installment call with both finite and infinite maturity.

Proof. To prove condition (22), we demonstrate using no-arbitrage arguments that the early exercise is never optimal when $q \leq rK$. Let us consider the following two portfolios:

1. *Portfolio A* consisting of

- an American continuous-installment call written on a non-dividend paying asset whose price process S follows (1) and with maturity time $T \geq 0$, plain vanilla payoff $H(S_t)$ and a constant installment per unit time $q \geq 0$;
- K units of pure discount bonds at a price of $b(t, T)$ each;
- a steady stream of q units of pure discount bonds at prices $b(t, s)$, for $t \leq s \leq T$;

2. *Portfolio B* consisting of one unit of the asset underlying the option contract.

Both portfolios are performed at current time $t \leq T$ and $b(t, s) \equiv e^{-r(s-t)}$, for $s \in [t, T]$, is the price at time t of a default-free pure discount bond paying one monetary unit at maturity time s.

At any time $t' \in [t, T]$, the value of portfolio A, $\Pi_A(t')$, is given by

$$\Pi_A(t') = (S_{t'} - K)^+ \, + \, Ke^{-r(T-t')} \, + \, q \int_{t'}^{T} e^{-r(s-t')} ds,$$

while the value of portfolio B is $\Pi_B(t') = S_{t'}$.

Suppose that at time t', the spot price of the underlying asset, $S_{t'}$, is at some level greater than K such that the call could be exercised. Then, we have

$$\begin{aligned} \Pi_A^e(t') &= (S_{t'} - K) \, + \, Ke^{-r(T-t')} \, + \, q \int_{t'}^{T} e^{-r(s-t')} ds \\ &= S_{t'} \, - \, K\Big(1 - e^{-r(T-t')}\Big) \, + \, \frac{q}{r}\Big(1 - e^{-r(T-t')}\Big) \\ &= S_{t'} \, - \, \Big(1 - e^{-r(T-t')}\Big)\Big(K - \frac{q}{r}\Big), \end{aligned}$$

which is obviously not greater than $S_{t'}$ when the quantity $(K - \frac{q}{r})$ is not negative, since K, $r > 0$ and $q \geq 0$. Hence, when $q \leq rK$, portfolio A is worth less than or equal to portfolio B if the call is exercised prior to maturity.

If the call is held to expiration, the value of portfolio A at maturity time T is equal to

$$\begin{aligned} \Pi_A(T) &= \lim_{t' \to T} \Pi_A(t') \\ &= \max(S_T,\, K), \end{aligned}$$

while the value of portfolio B is $\Pi_B(T) = S_T$. Hence, at expiration, portfolio A is greater than or equal in value to portfolio B.

Therefore, when $\delta = 0$, K, $r > 0$ and $q \leq rK$, the value of portfolio A is not greater than that of portfolio B if the call is exercised early, while it is at least as much as the value of portfolio B if the call is only exercised at maturity.

Using the same no-arbitrage arguments, it is easy to show that this property also holds when $T \to \infty$, since the value of both portfolios at maturity depends only on the underlying asset price. Consequently, when the installment premium is less than or equal to interest earned on the strike price, i.e., $q \leq rK$, then it is never optimal to exercise early an American continuous-installment call with both finite and infinite maturity. □

From Proposition 3 follows that if the condition (22) is not satisfied, i.e., $q \leq rK$, then a finite-lived American continuous-installment call written on a non-dividend paying asset is equivalent to its European counterpart, since the additional feature of the early exercise privilege is worthless. Under these circumstances, American and European continuous-installment calls are priced the same, as well as it happens for American and European standard calls when the underlying asset does not pay dividends. Furthermore, if $\delta = 0$ and $q \leq rK$, then it has no financial meaning to writing a perpetual American continuous-installment call due to the fact that there is no finite time at which it is optimal to exercise early the option.

It is worth noting that in the limiting case as $q \to rK^+$, we obtain a similar result to that of the perpetual standard call with no dividends, namely, the optimal exercise boundary $\bar{S}_u^c$ reaches the positive infinity and no early exercise would occur. Indeed, substituting the expressions of α_c and β_c from (17–18) into (16), and taking the limit of both the resulting equation and equations (19–20) as $q \to rK^+$, yields

$$\lim_{q \to rK^+} C(S_t; q) = S_t - \frac{1}{\gamma_2} \left[\frac{rK}{\left(r + \frac{\sigma^2}{2} \right)} \right]^{1-\gamma_2} S_t^{\gamma_2} - K,$$

and

$$\begin{aligned} \lim_{q \to rK^+} S_l^c &= \frac{rK}{\left(r + \frac{\sigma^2}{2} \right)}, \\ \lim_{q \to rK^+} S_u^c &= +\infty. \end{aligned}$$

For our example with $K = 100$ and $r = 0.07$, the installment premium q must be greater than $rK = 7$. Figure 2 shows the initial premium $C(S_t; q)$ of a perpetual continuous-installment call with $q = 7.5$ and the value of the perpetual standard call when the underlying asset pays no dividends.

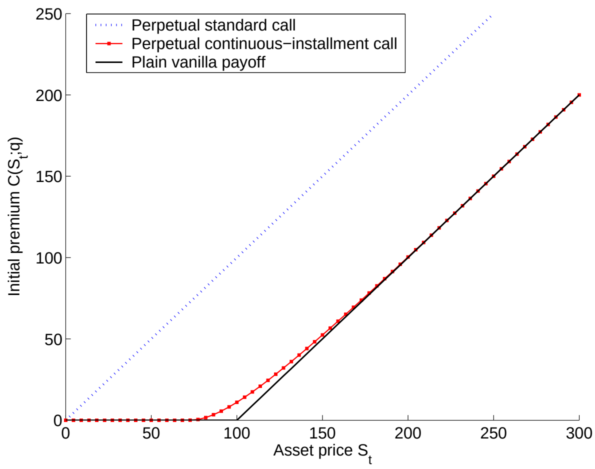

Fig. 2. Initial premium of perpetual continuous-installment call and value of perpetual standard call when underlying asset pays no dividends. Parameters used: $K = 100$, $r = 0.07$, $\delta = 0$, $\sigma = 0.25$ and $q = 7.5$.

4 Pricing of a perpetual continuous-installment put option

For the pricing of a perpetual continuous-installment put we proceed in the same way as for the call. Consider a perpetual continuous-installment put written on an asset whose price process S follows (1) and with constant installment per unit time $q \geq 0$ and plain vanilla payoff $H(S_t)$. Let us denote the initial premium function of this option by $P(S_t; q)$, defined on the domain $\mathcal{D}$.

The optimal exercise boundary, $\bar{S}_l^p$, is defined as the critical asset price below which it is optimal to terminate payments by exercising the option. Similarly, the optimal stopping boundary, $\bar{S}_u^p$, is defined as the critical asset price above which it is advantageous to terminate payments by dropping the option contract. The exercise and stopping boundaries, which are time-invariant constants, divide the domain $\mathcal{D}$ into an *exercise region* $\widetilde{\mathcal{D}}_1 = \{S_t \in [0, \bar{S}_l^p]\}$, a *continuation region* $\widetilde{\mathcal{D}}_2 = \{S_t \in (\bar{S}_l^p, \bar{S}_u^p)\}$ and a *stopping region* $\widetilde{\mathcal{D}}_3 = \{S_t \in [\bar{S}_u^p, \infty)\}$.

The initial premium function $P(S_t;q)$ satisfies the inhomogeneous Black-Scholes ODE (4) in the continuation region $\widetilde{\mathcal{D}}_2$, that is,

$$\frac{1}{2}\,\sigma^2 S_t^2 \frac{d^2P(S_t;q)}{dS^2} + \mu S_t \frac{dP(S_t;q)}{dS} - rP(S_t;q) = q, \quad \text{on } \widetilde{\mathcal{D}}_2, \tag{23}$$

subject to the following boundary conditions

$$P(S_t;q) = (K - S_t), \qquad \text{on } \widetilde{\mathcal{D}}_1, \tag{24}$$

$$\frac{dP(S_t;q)}{dS} = -1, \qquad \text{on } \widetilde{\mathcal{D}}_1, \tag{25}$$

$$P(S_t;q) = 0, \qquad \text{on } \widetilde{\mathcal{D}}_3, \tag{26}$$

$$\frac{dP(S_t;q)}{dS} = 0, \qquad \text{on } \widetilde{\mathcal{D}}_3. \tag{27}$$

Similarly, substituting the expression of $P(S_t;q)$ from (5) into the left-hand side of equations (24–27) and calculating each pair of them at the free exercise and stopping boundaries respectively, yields

$$\alpha_p(\bar{S}_l^p)^{\gamma_1} + \beta_p(\bar{S}_l^p)^{\gamma_2} - \frac{q}{r} = (K - \bar{S}_l^p), \tag{28}$$

$$\gamma_1\alpha_p(\bar{S}_l^p)^{(\gamma_1-1)} + \gamma_2\beta_p(\bar{S}_l^p)^{(\gamma_2-1)} = -1, \tag{29}$$

$$\alpha_p(\bar{S}_u^p)^{\gamma_1} + \beta_p(\bar{S}_u^p)^{\gamma_2} - \frac{q}{r} = 0, \tag{30}$$

$$\gamma_1\alpha_p(\bar{S}_u^p)^{(\gamma_1-1)} + \gamma_2\beta_p(\bar{S}_u^p)^{(\gamma_2-1)} = 0. \tag{31}$$

As for the call, the system of nonlinear equations (28–31) is solved numerically using the Newton-Raphson method, since in general no closed-form solution can be found.

For our standard example, we obtain the following exercise and stopping boundaries: $\bar{S}_l^p = 64.375$ and $\bar{S}_u^p = 253.368$. Figure 3 displays the initial premium function $P(S_t;q)$ of the perpetual continuous-installment put and the value (lump-sum premium) of the perpetual standard put when the underlying asset pays a continuous dividend yield.

For the case of a non-dividend paying asset, the system of nonlinear equations (28–31) simplifies to a more tractable form and then it can be solved analytically. Thus, the closed-form solution for the initial premium of the perpetual continuous-installment put is defined by

$$P(S_t;q) = \alpha_p S_t + \beta_p S_t^{\gamma_2} - \frac{q}{r},$$

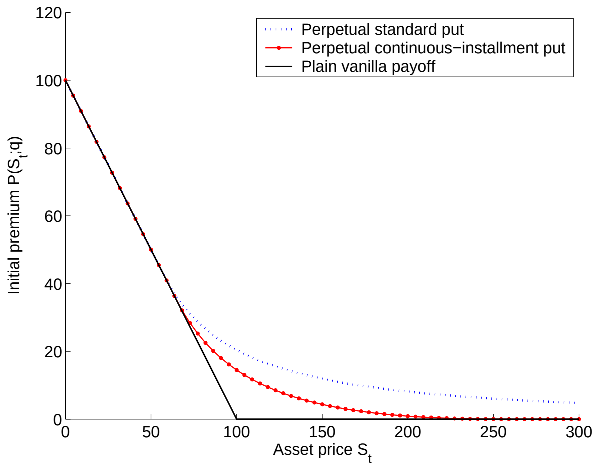

Fig. 3. Initial premium of perpetual continuous-installment put and value of perpetual standard put when underlying asset pays dividends. Parameters used: $K = 100$, $r = 0.07$, $\delta = 0.05$, $\sigma = 0.25$ and $q = 1$.

with $\gamma_2 = -\frac{2r}{\sigma^2}$, and where the constants α_p and β_p are given by

$$\alpha_p = \frac{1}{\left(1 + \frac{rK}{q}\right)^{1-\frac{1}{\gamma_2}} - 1}, \tag{32}$$

$$\beta_p = -\frac{1}{\gamma_2}\left[\frac{q}{\left(r + \frac{\sigma^2}{2}\right)}\right]^{1-\gamma_2}\left[\frac{1}{\left(1 + \frac{rK}{q}\right)^{1-\frac{1}{\gamma_2}} - 1}\right]^{\gamma_2}. \tag{33}$$

The optimal exercise and stopping boundaries are defined respectively by

$$\bar{S}_l^p = \frac{q}{\left(r + \frac{\sigma^2}{2}\right)}\left[\left(1 + \frac{rK}{q}\right) - \left(1 + \frac{rK}{q}\right)^{\frac{1}{\gamma_2}}\right], \tag{34}$$

$$\bar{S}_u^p = \frac{q}{\left(r + \frac{\sigma^2}{2}\right)}\left[\left(1 + \frac{rK}{q}\right)^{1-\frac{1}{\gamma_2}} - 1\right]. \tag{35}$$

For the perpetual continuous-installment put there is no condition to impose, since both the initial premium function and the two optimal boundaries are always positive for any given value of the model parameters. Figure 4 shows the initial premium $P(S_t; q)$ of the perpetual continuous-installment put and the value of the perpetual standard put when the underlying asset pays no dividends.

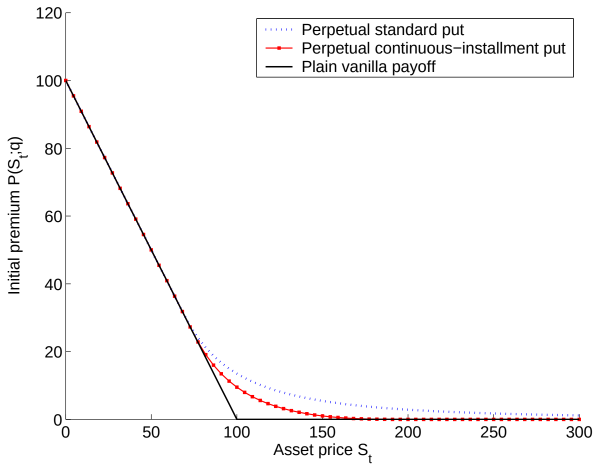

Fig. 4. Initial premium of perpetual continuous-installment put and value of perpetual standard put when underlying asset pays no dividends. Parameters used: $K = 100$, $r = 0.07$, $\delta = 0$, $\sigma = 0.25$ and $q = 1$.

5 Concluding remarks

In this paper we studied the pricing problem of perpetual continuous-installment options within the standard Black-Scholes framework and extending the analysis developed by [6].

Firstly, we discussed the distinctive features of the pricing problem by comparing it with respect to the free boundary problem for an American continuous-installment option with finite maturity. Since the contract details and the free boundary conditions are time-independent, the inhomogeneous Black-Scholes partial differential equation governing the initial premium function of a finite-lived continuous-installment option reduces to an ordinary differential equation when the life of the option becomes infinite. Moreover, the optimal stopping and exercise boundaries of a perpetual continuous-installment option do not vary with respect to the time variable and then their determination, along with that of the initial premium, is more easy to handle.

Using these results, the perpetual continuous-installment option pricing problem was formulated as a free boundary problem for the inhomogeneous Black-Scholes ordinary differential equation. Although the resulting system of nonlinear equations can be solved in general by using an appropriate numerical technique, e.g., Newton-Raphson method, closed-form solutions were found for both the initial premium and the optimal exit and exercise thresholds when the option is written on a non-dividend paying asset. Furthermore, it was shown that also in the absence of dividends it is optimal to exercise early a perpetual continuous-installment call if the installment premium is greater than the return on the riskless investment of the strike price.

References

1. Alobaidi, G., Mallier, R., and Deakin, A.S.: Laplace transforms and installment options. Mathematical Models and Methods in Applied Sciences **14** (2004) 1167-1189
2. Alobaidi, G., and Mallier, R.: Installment options close to expiry. Journal of Applied Mathematics and Stochastic Analysis (2006) 1-9
3. Ben-Ameur, H., Breton, M., and François, P.: A dynamic programming approach to price installment options. European Journal of Operational Research **169** (2006) 667-676
4. Black, F., and Cox, J.C.: Valuing corporate securities: some effects of bond indenture provisions. Journal of Finance **31** (1976) 351-367
5. Black, F., and Scholes, M.: The pricing of options and corporate liabilities. Journal of Political Economy **81** (1973) 637-659
6. Ciurlia, P., and Roko, I.: Valuation of American continuous-installment options. Computational Economics **25** (2005) 143-165
7. Davis, M., Schachermayer, W., and Tompkins, R.: Pricing, no-arbitrage bounds and robust hedging of instalment options. Quantitative Finance **1** (2001) 597-610
8. Davis, M., Schachermayer, W., and Tompkins, R.: Installment options and static hedging. Journal of Risk Finance **3** (2002) 46-52
9. Davis, M., Schachermayer, W., and Tompkins, R.: The evaluation of venture capital as an instalment option: valuing real options using real options. Zeitschrift für Betriebswirtschaft **3** (2004) 77-96
10. Dixit, A.R.: Entry and exit decisions under uncertainty. Journal of Political Economy **97** (1989) 620-638
11. Dixit, A.R., and Pindyck, R.S.: Investment under uncertainty. Princeton University Press, New Jersey (1994)
12. Geske, R.: The valuation of corporate liabilities as compound options. Journal of Financial and Quantitative Analysis **12** (1977) 541-552
13. Griebsch, S., Kühn, C. and Wystup, U.: Instalment options: A closed-form solution and the limiting case. Research Report No 5, Center for Practical Quantitative Finance, Frankfurt School of Finance & Management (2007)
14. Kimura, T.: Valuing continuous-installment options. Discussion Paper Series A, No 2007-184, Hokkaido University (2007)
15. Kimura, T.: Valuing American continuous-installment options. Discussion Paper Series A, No 2007-185, Hokkaido University (2007)
16. Leland, H.E.: Agency costs, risk management, and capital structure. Journal of Finance **53** (1998) 1213-1243
17. Majd, S., and Pindyck, R.S.: Time to build, option value, and investment decisions. Journal of Financial Economics **18** (1987) 7-27
18. McKean, H.P.: Appendix: A free boundary problem for the heat equation arising from a problem in mathematical economics. Industrial Management Review **6** (1965) 32-39
19. Merton, R.C.: Theory of rational option pricing. Bell Journal of Economics and Management Science **4** (1973) 141-183
20. Selby, M.J.P., and Hodges, S.D.: On the evaluation of compound options. Management Science **33** (1987) 347-355