Wavelet-analysis of time series with gap data. A preliminary study*

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Abstract. The paper is devoted to the the construction of spaces of scaling functions $V_j[a, b]$ in order to obtain multiresolution analysis on a finite set of intervals and determine the corresponding spaces of wavelets. Our approach is based on the method by Anderson, Hall, Jawerth and Peters (1994). For simplicity the only intervals with the rational end-points are on the consideration. We describe also an example of such construction for two disjoint intervals. This case is relevant to the application to the study of financial time-series with gaps.

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1 Introduction

The wavelet-analysis is nowadays one of the most powerful tools in the study of time series. This method allows to investigate both short-term and intermediate term series, as well as to make simultaneous localization as in spectral (frequency) area, as in scale (time) area. By using wavelet approach one can successfully determine the trend and eliminate a "white noise" from considered time series. The basic idea of wavelet-analysis is in a representation of a "signal" (e.g., time series) in the form of series with respect to specially constructed basis, made from a highly localized functions (wavelets) by means of scale changes and shifts along the time-axis.

At the study of time series, especially in practical applications, it is frequently necessary to handle the series with the data gaps. Due to incompleteness of

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the data set the prediction of the behavior of time series becomes in this case inconvenient or even practically impossible. The length of the gaps plays an important role in such an investigation. Moreover, if we use the incomplete data at the analysis of the time series, then we can miss the significant information presented in the series. All above said have to be taken into account at the developing of the method of an investigation and forecasting.

Various approaches to the study of time series with data gaps have been proposed (see, e.g., [11]). Traditional methods, which propose to fill in the gaps by using the first order interpolation, are not so effective for nonlinear and non-stationary time series. If we deal with financial time series it is necessary to note, that presence of data gaps for Saturday and Sunday is the usual situation for such series. On Monday, when stock exchange offices reopen, the prices for shares (indexes) are taken from Friday (so-called Monday's price) and only on Tuesday we have new so-called Tuesday's price. Such predicted gaps, as well as (what is more important) unexpected random gaps make significant influence on behavior of the series.

When a great number of the data is missed, and their distribution is casual, the evaluation of the "internal" behavior of the series and prediction on its base becomes a difficult problem. Thus it is very important to propose new methods of such kind evaluation for the time-series with gaps.

This paper is devoted to the application of the wavelet-analysis to the study of the time series containing data gaps. Our approach is based on the method by Anderson, Hall, Jawerth and Peters (1994). Their main idea, which is proposed for the construction of an orthonormal wavelet-bases for the unit interval (see [1]), is in splitting of the initial basis into three parts (analyzing the behaviour of a "signal" near end-points of interval and in its middle). Here we extend this approach to the construction of wavelet-bases for interval with arbitrary rational endpoints. This case is relevant to the above described application to the study of financial time-series with gaps.

We begin with the construction of wavelets on a bounded interval. There are several approaches to such construction. To illustrate these approaches it is better to use the functional (continuous) language, i.e. to extrapolate the discrete time series by the continuous function.

In this article we deal with the following questions: to study the main concepts of wavelet-transformations; to present a multiresolution analysis on the interval with rational endpoints. Such approach is close to that presented, e.g., in [1], [3], [5]. Our method differs from the above said by more explicit determination of the index sets corresponding to the wavelets. We start with the construction of spaces of scaling functions $V_j[a,b]$ forming the multiresolution analysis on the corresponding interval. Then we determine the spaces of wavelets $W_j[a,b]$ which form the wavelet bases for the case of arbitrary finite interval with rational endpoints. The main difficulty to be overcame is to determine the bases which are in a sense minimal and to evaluate more precisely the behavior of the data near the end-points. Then we generalize our construction for the situation with analyzing function given on two disjoint interval in order to prepare the study of financial time series with data gaps (cf., e.g. [2], [4]). The complete algorithm of the construction of sampling, as well as an application of this construction to the analysis of time series with data gaps is rather cumbersome and couldn't be include it into this article. It will be present in a forthcoming paper.

2 Multiresolution.

To give the definition of the multiresolution analysis [5], [12] we consider a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces $V_j \subset L_2(\mathbb{R})$ with the following properties:

1)
$$...V_{-1} \subset V_0 \subset V_1 \subset V_2...;$$

2)
$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_2(\mathbb{R})$$

$$(\text{or } \lim_{j\to+\infty} V_j = L_2(\mathbb{R}));$$

3)
$$\bigcap_{j\in\mathbb{Z}} V_j = 0$$

$$(\text{or } \lim_{j\to-\infty} V_j = \{0\}).$$

For all $f(x) \in L^2(\mathbb{R})$ we assume that the following conditions hold:

4)
$$f(x) \in V_j \Leftrightarrow f(2^j x) \in V_{j+1};$$

5) invariancy concerning shift: $f(x) \in V_0 \Leftrightarrow f(x-n) \in V_0$ for all $n \in \mathbb{N}$;

6) we also suppose that there exists a function $\phi(x) \in L^2(\mathbb{R})$ (called a *scaling function*) that the collection $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for V_0.

The collection of the subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L_2(\mathbb{R})$ forms the multiresolution analysis genrated by the scaling function ϕ.

If we know additionally that the following representation of ϕ holds:

$$\phi(x) = \sum_{k=0}^{2N-1} h_k \sqrt{2}\phi(2x-k)$$

for certain $N \geq 1$ with coefficients h_k, which satisfy $\sum_{k=0}^{2N-1}(-1)^k h_k k^\alpha = 0$ for all $0 \leq \alpha \leq N-1$ then this function $\phi(x) \in L^2(\mathbb{R})$ has a compact support *supp* $\phi(x) = [0, 2N-1]$. It is called a *compactly supported scaling function*, which generates a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ of $L_2(\mathbb{R})$.

We can define a wavelet associated to the above introduced multiresolution analysis as follows. Let

$$\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k), j, k \in \mathbb{Z}.$$

Here the set $\{\phi_{jk}(x)|k \in \mathbb{Z}\}$ forms an orthonormal basis V_j in $L_2(\mathbb{R})$. The function $\psi(x)$ defined by

$$\psi(x) = \sum_k g_k \sqrt{2}\phi(2x-k),$$

where $g_k = (-1)^k h_{1-k}, j, k \in \mathbb{Z}$, is called a *wavelet* (corresponding the multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$).

For every $j \in \mathbb{Z}$ define W_j as orthogonal complement V_j in V_{j+1} as

$$V_{j+1} = V_j \bigoplus W_j.$$

By the construction we can write

$$L_2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j.$$

As all spaces W_j are mutually orthogonal then by combining all orthonormal bases in $L_2(\mathbb{R})$ we obtain our orthonormal wavelet basis in $L_2(\mathbb{R})$, namely $\{\psi_{jk}(x) := 2^{j/2}\psi(2^j x - k)|j, k \in \mathbb{Z}\}$. Note, that for the applications it is convenient to replace $\bigoplus_{j=-\infty}^{j=-1} W_j$ by V_0:

$$V_o \bigoplus \left\{ \bigoplus_{j=0}^{\infty} W_j \right\} = L_2(\mathbb{R}).$$

3 Multiresolution analysis on an interval

There are some alternative constructions of multiresolution approximation on $L_2([a,b])$ proposed by different researchers (see, e.g. [3], [10]). We follow here the Meyer's construction ([8], see also [6]).

For this purpose we need to determine functions ϕ_{jk} whose support has non empty intersection with interval $[a,b]$. So the following set of indexes is introduced

$$S_j = \{k : supp\ \phi_{jk} \bigcap (a,b) \neq 0\} = \{k : 2^j a - (2N-1) < k < 2^j b\}.$$

Here δ_L and δ_R are represent two fixed nonnegative integers as it was proposed in [1]. Let us define three subsets of the set S_j with the indexes associated with the left, right endpoints and with the interior of the interval respectively:

$$S_{j,L} = \left\{k : 2^j a - (2N-1) < k < 2^j a + \delta_L\right\},$$

$$S_{j,R} = \{k : 2^j b - (2N-1) - \delta_R < k < 2^j b\}$$

and

$$S_{j,I} = \{k : supp\ \phi_{j,k-\delta_L}\ \text{and}\ supp\ \phi_{j,k+\delta_R} \subset (a,b)\} =$$

$$= \left\{k : 2^j a + \delta_L \leq k \leq 2^j b - (2N-1) - \delta_R\right\}.$$

Each set $S_{j,L}$, $S_{j,R}$ independently of j contains $2N-2$ items, so for index j being large enough these sets $S_{j,L}$ and $S_{j,R}$ are disjoint and $S_j = S_{j,L} \bigcup S_{j,I} \bigcup S_{j,R}$.

We can construct the spaces $V_j[a,b]$ by

$$V_j[a,b] = \overline{\{(X-a)^{\alpha}_{j,L}\}}_{0\le\alpha\le N-1} \bigcup \overline{\{\phi_{jk}\}}_{k\in S_{j,I}} \bigcup \overline{\{(X-b)^{\alpha}_{j,R}\}}_{0\le\alpha\le N-1}.$$

Here

$$(X-a)^{\alpha}_{j,L}(t) = \sum_{k\in S_{j,L}} (t-a)^{\alpha}_{j}(k)\phi_{jk}(t)|_{[a,b]},$$

$$(X-b)^{\alpha}_{j,R}(t) = \sum_{k\in S_{j,R}} (t-b)^{\alpha}_{j}(k)\phi_{jk}(t)|_{[a,b]}.$$

Let $\{(t-a)^{\alpha}_{j,L}(k)\}_{0\le\alpha\le M-1}$ and $\{(t-b)^{\alpha}_{j,R}(k)\}_{0\le\alpha\le M-1}$ be components of $(X-a)^{\alpha}_{j,L}(t)$ and $(X-b)^{\alpha}_{j,R}(t)$, respectively:

$$\overline{\{(t-a)^{\alpha}_{j,L}(k)\}}_{0\le\alpha\le M-1} = \overline{\{\{(k-2^j a)^{\alpha}\}_{k\in S_{j,L}}\}}_{0\le\alpha\le M-1},$$

$$\overline{\{(t-b)^{\alpha}_{j,R}(k)\}}_{0\le\alpha\le M-1} = \overline{\{\{(k-2^j b)^{\alpha}\}_{k\in S_{j,R}}\}}_{0\le\alpha\le M-1}, M \le N.$$

In the following we use a notation

$$t^{\alpha}_{j}(k) =$$

$$= \langle 2^{j/2}(2^j t)^{\alpha}, \phi_{jk}\rangle = \langle 2^{j/2}(2^j t)^{\alpha}, 2^{j/2}\phi(2^j t-k)\rangle = \int 2^{j(1+\alpha)}t^{\alpha}\phi(2^j t-k)dt =$$

$$= [2^j t - k = y, dt = dy/2^j] = \int (y+k)^{\alpha}\phi(y)dy = \sum_{i=0}^{\alpha} C^i_{\alpha} k^i M_{\alpha-i},$$

where $M_i = \int y^i\phi(y)dy$ is the i-th moment of ϕ. We can notice that $t^{\alpha}_{j}(k)$ are independent of j.

To provide the process of orthonormalization we can follow [8] and take the square root inverse of the mass matrices of $(X-a)^{\alpha}_{j,L}$ and $(X-b)^{\alpha}_{j,R}$. Let $\mathbf{X}^L = (X^L_{mn})$ and $\mathbf{X}^R = (X^R_{mn})$ for $0 \le m,n \le N-1$ where

$$X^L_{mn} = < (X-a)^m_{j,L}, (X-a)^n_{j,L} >$$

$$X^R_{mn} = < (X-b)^m_{j,R}, (X-b)^n_{j,R} > .$$

Both matrices $\mathbf{X}^L$ and $\mathbf{X}^R$ are symmetric and positively defined. There exist $\mathbf{S} = \left(\mathbf{X}^L\right)^{-1/2}$ and $\mathbf{T} = \left(\mathbf{X}^R\right)^{-1/2}$. Let

$$\begin{pmatrix} \varphi_{j,2^j a+\delta_L-N+1}(x) \\ \cdots \\ \cdots \\ \cdots \\ \varphi_{j,2^j a+\delta_L}(x) \end{pmatrix} = \mathbf{S} \begin{pmatrix} (X-a)^0_{j,L}(x) \\ \cdots \\ \cdots \\ \cdots \\ (X-a)^{N-1}_{j,L}(x) \end{pmatrix}$$

$$\begin{pmatrix} \varphi_{j,2^jb-(2N-1)-\delta_R}(x) \\ \dots \\ \dots \\ \dots \\ \varphi_{j,2^jb-N-\delta_R}(x) \end{pmatrix} = \mathbf{T} \begin{pmatrix} (X-b)^0_{j,R}(x) \\ \dots \\ \dots \\ \dots \\ (X-b)^{N-1}_{j,R}(x) \end{pmatrix}.$$

Hence we can build for $k \in S_{j,I}$ the set of functions which form an orthonormal basis of $V_j[a,b]$:

$$\{\varphi_{j,2^ja+\delta_L-N+1}, \dots, \varphi_{j,2^ja+\delta_L}, \Phi_{jk}, \varphi_{j,2^jb-(2N-1)-\delta_R}, \dots, \varphi_{j,2^jb-N-\delta_R}\}.$$

The basic elements of $V_j[a,b]$ are defined by the following notation:

$$\varphi_{jk} = \begin{cases} \varphi^\alpha_{j,L} & \text{if } k = 2^ja + \delta_L - \alpha \text{ for } \alpha = 0, \dots, N-1; \\ \Phi_{jk} & \text{if } k \in S_{j,I}; \\ \varphi^\alpha_{j,R} & \text{if } k = 2^jb - (2N-1) - \delta_R + \alpha \text{ for } \alpha = 0, \dots, N-1. \end{cases}$$

Next, we build a projection $V_{j+1}[a,b]$ onto $V_j[a,b]$. Consider a low pass filter H which we can define as $\Phi_j = H\Phi_{j+1}$, where

$$\Phi_j = \begin{pmatrix} \varphi_{j,2^ja+\delta_L-N+1} \\ \dots \\ \dots \\ \dots \\ \varphi_{j,2^ja+\delta_L} \\ \dots \\ \dots \\ \dots \\ \varphi_{j,2^jb-(2N-1)-\delta_R} \\ \dots \\ \dots \\ \dots \\ \varphi_{j,2^jb-N-\delta_R} \end{pmatrix}, \Phi_{j+1} = \begin{pmatrix} \varphi_{j+1,2^{j+1}a+\delta_L-N+1} \\ \dots \\ \dots \\ \dots \\ \varphi_{j+1,2^{j+1}a+\delta_L} \\ \dots \\ \dots \\ \dots \\ \varphi_{j+1,2^{j+1}b-(2N-1)-\delta_R} \\ \dots \\ \dots \\ \dots \\ \varphi_{j+1,2^{j+1}b-N-\delta_R} \end{pmatrix}.$$

We can construct the block-matrixmatrix $H = H_{mn} =< \varphi_{jm}, \varphi_{(j+1)n} >$ of the general form as is

$$H = \begin{pmatrix} H_{LL} & H_{LI} & H_{LR} \\ H_{IL} & H_{II} & H_{IR} \\ H_{RL} & H_{RI} & H_{RR} \end{pmatrix}.$$

Let us describe in details the components of this matrix. By the orthogonality conditions the matrices $H_{LR}, H_{IL}, H_{IR}, H_{RL}$ will be zero matrices. Now we build the remaining matrices. $N \times N$ matrices H_{LL} and H_{RR} are defined as follows:

$$H_{LL} = \begin{pmatrix} < \varphi_{j,2^ja-N+1+\delta_L}, \varphi_{j+1,2^{j+1}a-N+1+\delta_L} > & \dots & < \varphi_{j,2^ja-N+1+\delta_L}, \varphi_{j+1,2^{j+1}a+\delta_L} > \\ \dots & \dots & \dots \\ < \varphi_{j,2^ja+\delta_L}, \varphi_{j+1,2^{j+1}a-N+1+\delta_L} > & \dots & < \varphi_{j,2^ja+\delta_L}, \varphi_{j+1,2^{j+1}a+\delta_L} > \end{pmatrix} =$$

$$= \mathbf{S} \begin{pmatrix} < (X-a)^0_{j,L}, (X-a)^0_{j+1,L} > & \dots & < (X-a)^0_{j,L}, (X-a)^{N-1}_{j+1,L} > \\ \dots & \dots & \dots \\ < (X-a)^{N-1}_{j,L}, (X-a)^0_{j+1,L} > & \dots & < (X-a)^{N-1}_{j,L}, (X-a)^{N-1}_{j+1,L} > \end{pmatrix} \mathbf{S}^*.$$

$$H_{RR} = \begin{pmatrix} < \varphi_{j,2^j b-2N+1-\delta_R}, \varphi_{j+1,2^{j+1} b-2N+1-\delta_R} > & \dots & < \varphi_{j,2^j b-2N+1-\delta_R}, \varphi_{j+1,2^{j+1} b-N-\delta_R} > \\ \dots & \dots & \dots \\ < \varphi_{j,2^j b-N-\delta_R}, \varphi_{j+1,2^{j+1} b-2N+1-\delta_R} > & \dots & < \varphi_{j,2^j b-N-\delta_R}, \varphi_{j+1,2^{j+1} b-N-\delta_R} > \end{pmatrix} =$$

$$= \mathbf{T} \begin{pmatrix} < (X-b)^0_{j,R}, (X-b)^0_{j+1,R} > & \dots & < (X-b)^0_{j,R}, (X-b)^{N-1}_{j+1,R} > \\ \dots & \dots & \dots \\ < (X-b)^{N-1}_{j,R}, (X-b)^0_{j+1,R} > & \dots & < (X-b)^{N-1}_{j,R}, (X-b)^{N-1}_{j+1,R} > \end{pmatrix} \mathbf{T}^*.$$

H_{LI} and H_{RI} are matrices most elements of which are zero:

$$H_{LI} = \begin{pmatrix} < \varphi_{j,2^j a+\delta_L-N+1}, \varphi_{j+1,2^{j+1} a+1+\delta_L} > & \dots & < \varphi_{j,2^j a+\delta_L-N+1}, \varphi_{j+1,2^{j+1} b-2N-\delta_R} > \\ \dots & \dots & \dots \\ < \varphi_{j,2^j a+\delta_L}, \varphi_{j+1,2^{j+1} a+1+\delta_L} > & \dots & < \varphi_{j,2^j a+\delta_L}, \varphi_{j+1,2^{j+1} b-2N-\delta_R} > \end{pmatrix}$$

$$H_{RI} = \begin{pmatrix} < \varphi_{j,2^j b-2N+1-\delta_R}, \varphi_{j+1,2^{j+1} a+1+\delta_L} > & \dots & < \varphi_{j,2^j b-2N+1-\delta_R}, \varphi_{j+1,2^{j+1} b-2N-\delta_R} > \\ \dots & \dots & \dots \\ < \varphi_{j,2^j b-N-\delta_R}, \varphi_{j+1,2^{j+1} a+1+\delta_L} > & \dots & < \varphi_{j,2^j b-N-\delta_R}, \varphi_{j+1,2^{j+1} b-2N-\delta_R} > \end{pmatrix}.$$

The main part of matrix H makes a band matrix H_{II}:

$$H_{II} = \begin{pmatrix} < \varphi_{j,2^j a+1+\delta_L}, \varphi_{j+1,2^{j+1} a+1+\delta_L} > & \dots & < \varphi_{j,2^j a+1+\delta_L}, \varphi_{j+1,2^{j+1} b-2N-\delta_R} > \\ \dots & \dots & \dots \\ < \varphi_{j,2^j b-2N-\delta_R}, \varphi_{j+1,2^{j+1} a+1+\delta_L} > & \dots & < \varphi_{j,2^j b-2N-\delta_R}, \varphi_{j+1,2^{j+1} b-2N-\delta_R} > \end{pmatrix}.$$

Even in the case of intervals with rational endpoints we can get the incommensurability of scales. It means that we can obtain the different spaces V_j of scaling functions and the different spaces W_j of wavelets whenever the scaling coefficients h_k and the sets of indexes are the same for two different intervals.

4 Wavelets on an interval

Consider the construction of the wavelet spaces $W_j[a,b]$. By definition $W_j[a,b]$ has to be an orthonormal complement of $V_j[a,b]$ in $V_{j+1}[a,b]$

$$\dim W_j[a,b] = \dim V_{j+1}[a,b] - \dim V_j[a,b] =$$

$$= 2^{j+1} - \delta_L - \delta_R + 1 - 2^j + \delta_L + \delta_R - 1 = 2^j.$$

Let us introduce the functions

$$\Psi_{jk}(x) = \sum_m g_{m-2k} \Phi_{j+1,m}(x), \; g_{m-2k} \neq 0. \tag{1}$$

The subspace $V_{j+1}[a,b]$ is splitted into three parts:

$$V_{j+1}[a,b] = \overline{\{(X-a)^\alpha_{j+1,L}\}} \bigcup \overline{\{\Phi_{j+1,k}\}} \bigcup \overline{\{(X-b)^\alpha_{j+1,R}\}}.$$

As left collection could be rewritten in a way

$$(X-a)^\alpha_{j,L}(t) = \sum_{k \in S_{j,L}} (t-a)^\alpha_j(k) \sum_{m \in S_{j+1}} g_{m-2k} \Phi_{j+1,m}(x) =$$

$$= \sum_{m\in S_{j+1}} \sum_{k\in S_{j,L}} (t-a)^{\alpha}_{j}(k) g_{m-2k} \Phi_{j+1,m}(x)$$

and based on the Proposition 4.1 from [1] not difficult to see that the "left" and "right" collections consist of combinations of functions from $V_j[a,b]$. Hence we need to provide more precise description of functions from $\overline{\{\Phi_{j+1,k}\}}$. For this we represent the function $\Psi_{jk}(x)$ in the form (1) and use the change of a variable $x = 2^j x - k$ in the corresponding terms in the right hand-side.

The set of all integers k such that $k \in S_{j+1,I}$ and $\Psi_{jk}(x) \in V_{j+1}[a,b]$ is follows: $\left\{k : 2^j a + \frac{\delta_L}{2} + N - 1 \le k \le 2^j b - N + \frac{1}{2} - \frac{\delta_R}{2}\right\}$ and as it is known the demension of the $W_j[a,b]$ we need for our construction approximately $2^j - (2^j b - N + \frac{1}{2} - \frac{\delta_R}{2}) + (2^j a + \frac{\delta_L}{2} + N - 1) = 2^j(a-b+1) + 2N + \frac{\delta_L+\delta_R-3}{2}$ functions, which we have to select from those functions that belong to the space $V_{j+1}[a,b]$ and correspond to the middle part of $[a,b]$. Let us consider those indexes k for which $g_{m-2k} \neq 0$ which implies $m \in S_{j+1,I}$. It follows from the previous consideration that an inequality $0 \le m-2k \le 2N-1$ holds for all these indexes k. Besides they belong to the set $S_{j,I}[a,b] \equiv \left\{k : 2^{j+1}a + 2\delta_L \le 2k \le 2^{j+1}b - 2(2N-1) - 2\delta_R\right\}$. Combining the above inequalities we arrive at the following relations $2^{j+1}a + 2\delta_L \le 2k \le m \le 2^{j+1}b - 2N + 1 - 2\delta_R$. Now to describe the collection of indexes corresponding to the set $\Phi_{j+1,k}$ we have to add $2(N-1)$ to the left side and to subtract the same number from the right side of the inequalities. Hence $2^{j+1}a + 2\delta_L + 2(N-1) \le k \le 2^{j+1}b - 2(2N-1) + 1 - 2\delta_R$. Finally, we have to take into account the following two conditions, namely, the left side of the last inequality ought to be greater then the left side of that in the definition of $S_{j+1,I}$ and the right side ought to be less than the right one: $2^{j+1}a + \delta_L \le 2^{j+1}a + 2(N-1) - 1 + 2\delta_L$ and $2^{j+1}b - 2(2N-1) + 2 - 2\delta_L \le 2^{j+1}b - (2N-1) - \delta_L$. The index j has to satisfy an inequality $2^j(b-a) \ge \delta_L + \delta_R + 3(N-1) - 1$.

Since the functions $\Phi_{j+1,k}$ can be represented in the form

$$\Phi_{j+1,k} = \sum_l < \Phi_{j+1,k}, \Phi_{j,l} > \Phi_{j,l} + \sum_l < \Phi_{j+1,k}, \Psi_{j,l} > \Psi_{j,l} =$$

$$= \sum_l \overline{h}_{k-2l}\Phi_{j,l} + \sum_l \overline{g}_{k-2l}\Psi_{j,l} = h_k\Phi_{j,0} + h_{k-2}\Phi_{j,1} + h_{k-4}\Phi_{j,2} + ... \qquad (2)$$

we describe only those relations which correspond to the left endpoint. Here we use an extra notation δ_l arriving in the second index of the initial function Φ of the series in the following way: the function $\Phi_{j,\delta_L-\delta_l}$ corresponding to such a value of index δ_l is an initial function Φ in (2). Then for $\delta_l = 1$ we have

$$\Phi_{j+1,2^{j+1}a+2\delta_L+2(N-1)-1} = h_{2^{j+1}a+2N-1}\Phi_{j,\delta_L-\delta_l} + h_{2^{j+1}a+2N-3}\Phi_{j,\delta_L-\delta_l+1} + ... =$$

$$= h_{2^{j+1}a+2N-1}\Phi_{j,\delta_L-1} + h_{2^{j+1}a+2N-3}\Phi_{j,\delta_L} + ...$$

$\delta_l := 1 :$

$$\Phi_{j+1,2^{j+1}a+2\delta_L+2N-4} = h_{2^{j+1}a+2N-2}\Phi_{j,\delta_L-\delta_l} + h_{2^{j+1}a+2N-4}\Phi_{j,\delta_L-\delta_l+1} + ... =$$

$$= h_{2^{j+1}a+2N-2}\Phi_{j,\delta_L-1} + h_{2^{j+1}a+2N-4}\Phi_{j,\delta_L} + ...$$

$\delta_l := 2:$

$$\Phi_{j+1,2^{j+1}a+2\delta_L+2N-5} = h_{2^{j+1}a+2N-1}\Phi_{j,\delta_L-\delta_l} + h_{2^{j+1}a+2N-3}\Phi_{j,\delta_L-\delta_l+1} + ... =$$

$$= h_{2^{j+1}a+2N-1}\Phi_{j,\delta_L-2} + h_{2^{j+1}a+2N-3}\Phi_{j,\delta_L-1} + ...$$

...

The end of such a chain of relations depends on the parity of the last value of δ_l. If this last value of δ_l is even, i.e $\delta_l = 2t_L$, then the last relations have the form

...

$\delta_l := 2N-2:$

$$\Phi_{j+1,2^{j+1}a+\delta_L+1} = h_{2^{j+1}a+2N-1}\Phi_{j,\delta_L-\delta_l} + h_{2^{j+1}a+2N-3}\Phi_{j,\delta_L-\delta_l+1} + ... =$$

$$= h_{2^{j+1}a+2N-1}\Phi_{j,\delta_L-2N+2} + h_{2^{j+1}a+2N-3}\Phi_{j,\delta_L-2N+3} + ...$$

$\delta_l := 2N-2:$

$$\Phi_{j+1,2^{j+1}a+\delta_L} = h_{2^{j+1}a+2N-2}\Phi_{j,\delta_L-\delta_l} + h_{2^{j+1}a+2N-4}\Phi_{j,\delta_L-\delta_l+1} + ... =$$

$$= h_{2^{j+1}a+2N-2}\Phi_{j,\delta_L-2N+2} + h_{2^{j+1}a+2N-4}\Phi_{j,\delta_L-2N+3} + ...$$

If the last value of δ is odd, i.e. $\delta_l = 2t_L - 1$, then the last relations have the form

...

$\delta_l := 2N-1:$

$$\Phi_{j+1,2^{j+1}a+\delta_L+1} = h_{2^{j+1}a+2N}\Phi_{j,\delta_L-\delta_l} + h_{2^{j+1}a+2N-2}\Phi_{j,\delta_L-\delta_l+1} + ... =$$

$$= h_{2^{j+1}a+2N}\Phi_{j,\delta_L-2N+1} + h_{2^{j+1}a+2N-2}\Phi_{j,\delta_L-2N+2} + ... =$$

$\delta_l := 2N-1:$

$$\Phi_{j+1,2^{j+1}a+\delta_L} = h_{2^{j+1}a+2N-1}\Phi_{j,\delta_L-\delta_l} + h_{2^{j+1}a+2N-3}\Phi_{j,\delta_L-\delta_l+1} + ... =$$

$$= h_{2^{j+1}a+2N-1}\Phi_{j,\delta_L-2N+1} + h_{2^{j+1}a+2N-3}\Phi_{j,\delta_L-2N+2} + ...$$

Now using the orthogonality relations

$$\sum_k h_k h_{k-2l} =< \Phi_{j,0}, \Phi_{j,l} >= \delta_{0l},$$

$$\sum_k h_k g_{k-2l} =< \Phi_{j,0}, \Psi_{j,l} >= 0,$$

we can select coefficients h_k for which our functions belong to the $V_j[a,b]$. We multiply the first equality in sequence by h_1 and the second one by h_0. Then we arrive at the following expression:

$$h_1\Phi_{j+1,2^{j+1}a+2\delta_L+2(N-1)-1} + h_0\Phi_{j+1,2^{j+1}a+2\delta_L+2N-4} =$$

$$= (h_1 h_{2^{j+1}a+2N-1} + h_0 h_{2^{j+1}a+2N-2})\Phi_{j,\delta_L-1}+$$

$$+(h_1 h_{2^{j+1}a+2N-3} + h_0 h_{2^{j+1}a+2N-4})\Phi_{j,\delta_L} + ...$$

Since the almost all items in the right side belong to $V_j[a,b]$, it suffices to consider the remainder:

$$(h_1 \Phi_{j+1,2^{j+1}a+2\delta_L+2(N-1)-1} + h_0 \Phi_{j+1,2^{j+1}a+2\delta_L+2N-4})|_{[a,b]} =$$

$$(h_1 h_{2^{j+1}a+2N-1} + h_0 h_{2^{j+1}a+2N-2})\Phi_{j,\delta_L-1}|_{[a,b]} \,(\mathrm{mod}\, V_j[a,b]) = 0 \,(\mathrm{mod}\, V_j[a,b]).$$

Hence:

$$(h_{3-2N-\delta_L}\Phi_{j+1,2^{j+1}a+2\delta_L+2N-3} + h_{4-2N-\delta_L}\Phi_{j+1,2^{j+1}a+2\delta_L+2N-4} + ...$$

$$+h_1\Phi_{j+1,2^{j+1}a+\delta_L+1} + h_0\Phi_{j+1,2^{j+1}a+\delta_L})|_{[a,b]} = 0 \,(mod\, V_j[a,b]).$$

The missing functions belonging to $W_j[a,b]$ and corresponding to the left endpoint have the following form:

$$\Psi^1_{j,L} = \Phi_{j+1,2^{j+1}a+2\delta_L+2N-3}|_{[a,b]} - proj_{V_j[a,b]}\Phi_{j+1,2^{j+1}a+2\delta_L+2N-3}$$

$$\Psi^2_{j,L} = \Phi_{j+1,2^{j+1}a+2\delta_L+2N-5}|_{[a,b]} - proj_{V_j[a,b]}\Phi_{j+1,2^{j+1}a+2\delta_L+2N-5} \qquad (3)$$

$$\cdots$$

The same construction can be realized for the right endpoint.

Then all such functions $\Psi^*_{j,L} \in W_j[a,b]$ are linear independent and also orthogonal to the wavelets $\{\psi_{jk}\}_{k\in S_{j,I}}$. To get an orthonormal basis for $W_j[a,b]$ we need to orthonormalize them. For this process let $\mathbf{X}^L_1$ and $\mathbf{X}^R_1$ are the mass matrices of $\Psi^*_{j,L}$ and $\Psi^*_{j,R}$ respectively, and I be the indentity matrix. Note, that $\mathbf{X}^L_1$ and $\mathbf{X}^R_1$ are symmetric positively defined. So we have

$$\mathbf{X}^L_1 = I - H^t_{LL}H_{LL}$$

$$\mathbf{X}^R_1 = I - H^t_{RR}H_{RR}.$$

Now we define functions $\psi^\alpha_{j,L}$ and $\psi^\alpha_{j,R}$ of $W_j[a,b]$, corresponding to the left and right end-points, respectively:

$$\begin{pmatrix} \psi_{j,2^j a+\delta_L/2+t_L}(x) \\ ... \\ ... \\ ... \\ \psi_{j,2^j a+\delta_L/2+N-2}(x) \end{pmatrix} = \left(\mathbf{X}^L_1\right)^{-1/2} \begin{pmatrix} \Psi^1_{j,L} \\ ... \\ ... \\ ... \\ \Psi^{N-1-t_L}_{j,L}(x) \end{pmatrix} \qquad (4)$$

$$\begin{pmatrix} \psi_{j,2^j b-N+1-\delta_R/2}(x) \\ \ldots \\ \ldots \\ \ldots \\ \psi_{j,2^j b-1-\delta_R/2-t_R}(x) \end{pmatrix} = \left(\mathbf{X}_1^R\right)^{-1/2} \begin{pmatrix} \Psi_{j,R}^1 \\ \ldots \\ \ldots \\ \ldots \\ \Psi_{j,R}^{N-1-t_R}(x) \end{pmatrix}. \tag{5}$$

Now we can write an orthonormal basis of $W_j[a,b]$:

$$\{\psi_{j,2^j a+\delta_L/2+t_L}, \ldots, \psi_{j,2^j a+\delta_L/2+N-2}, \ldots, \psi_{j,2^j b-N+1-\delta_R/2}, \ldots, \psi_{j,2^j b-1-\delta_R/2-t_R}\}.$$

We can define the basic elements of $W_j[a,b]$ by following relations:

$$\tilde{\psi}_{jk} = \begin{cases} \Psi_{j,L}^{\alpha}, & \text{if } k = k_{j,l} - \alpha \text{ for } \alpha = 1, \ldots, N-1-t_L, \\ \Psi_{jk}, & \text{if } k_{j,l} \le k \le k_{j,r}, \\ \Psi_{j,R}^{\alpha}, & \text{if } k = k_{j,r} + \alpha \text{ for } \alpha = 1, \ldots, N-1-t_R, \end{cases}$$

where $k_{j,l}$ is the smallest integer $\ge 2^j a + N - 1 + \delta_L/2$ and $k_{j,r}$ is the largest one $\le 2^j b - N - \delta_R/2$.

We can summarize all above said in the following theorem.

Theorem 1. *The space $W_j[a,b], j \ge j_1$ is a linear span of the functions $\Psi_{jk}, 2^j a + N - 1 + \delta_L/2 \le k \le 2^j b - N - \delta_R/2$, where the functions $\Psi_{j,L}^{\alpha}, 0 \le \alpha \le N-1-t_L$ (3), and $\Psi_{j,R}^{\alpha}, 0 \le \alpha \le N-1-t_R$ are defined in (4) and (5), respectively.*

Considering the missing functions in $W_j[a,b]$ at the left and right endpoint separately we can build a block matrix G of the wavelet coefficients (see [1], [8]) as $\Psi_j = G\Phi_{j+1}$, where

$$\Psi_j = \begin{pmatrix} \Psi_{j,2^j a+\delta_L/2+t_L} \\ \cdots \\ \cdots \\ \cdots \\ \Psi_{j,2^j a+\delta_L/2+N-2} \\ \cdots \\ \cdots \\ \cdots \\ \Psi_{j,2^j b-N+1-\delta_R/2} \\ \cdots \\ \cdots \\ \cdots \\ \Psi_{j,2^j b-1-\delta_R/2-t_R} \end{pmatrix}, \Phi_{j+1} = \begin{pmatrix} \varphi_{j+1,2^{j+1}a+\delta_L-N+1} \\ \cdots \\ \cdots \\ \cdots \\ \varphi_{j+1,2^{j+1}a+\delta_L} \\ \cdots \\ \cdots \\ \cdots \\ \varphi_{j+1,2^{j+1}b-(2N-1)-\delta_R} \\ \cdots \\ \cdots \\ \cdots \\ \varphi_{j+1,2^{j+1}b-N-\delta_R} \end{pmatrix}.$$

Our construction of matrix G is same what we have for H:

$$G = \begin{pmatrix} G_{LL} & G_{LI} & G_{LR} \\ G_{IL} & G_{II} & G_{IR} \\ G_{RL} & G_{RI} & G_{RR} \end{pmatrix}.$$

In terms of matrix H, we can rewrite that

$$G_{LL}=(I-H^t_{LL}H_{LL})^{1/2}$$

$$G_{LI}=-(I-H^t_{LL}H_{LL})(H^t_{LL}H_{LI})$$

$$G_{RI}=-(I-H^t_{RR}H_{LL})(H^t_{RR}H_{RI})$$

$$G_{RR}=(I-H^t_{RR}H_{RR})^{1/2}.$$

We can note that $H^tH+G^tG=I$. Under the interior orthogonality some matrices will be zero matrixes and the general shape of matrix G will be as

$$G=\begin{pmatrix} G_{LL} & G_{LI} & 0 \\ 0 & G_{II} & 0 \\ 0 & G_{RI} & G_{RR} \end{pmatrix}.$$

The entries of the matrix G are mainly determined by the sets $\overline{\{(X-a)^\alpha_{j,L}\}}$ and $\overline{\{(X-b)^\alpha_{j,R}\}}$ corresponding at upper left and lower right endpoints. The matrix G can be obtained by using finite number of transformation (applied to the matrix H). Hence each of the spaces $V_j[a,b]$ has an orthonormal basis determined by dilation and translation of certain functions ϕ^i:

$$\phi_{jk}(x)=2^{j/2}\phi^i(2^jx-k),\; j,k\in\mathbb{Z}.$$

There are only finite number of such functions ϕ^i. Therefore the bases of $W_j[a,b]$ can be obtained as dilation and translation of one of them.

The remaining question is to find such indexes for every interval and to remove those functions appeared more than one time. For instance, there are $2N-2$ wavelets whose support contains each endpoint. But we need only $N-1$ basic functions at each endpoint.

5 The structure of the spaces V_j and W_j corresponding to different intervals

Now we consider the concrete example of a "signal" defined on the union of two intervals $[0,2]$ and $[7,12]$. Our aim is to construct the spaces V_j and W_j for both of these intervals and to show how the incomparability of the length of intervals impacts on the structure of the spaces. For determines we put $\delta_L=\delta_R=1$. By applying the results of the previous section we obtain:
for interval $[0,2]$

$$S_{j,L}=\{k:-(2N-1)<k<1\},$$

$$S_{j,R}=\left\{k:2^{j+1}-2N<k<2^{j+1}\right\},$$

$$S_{j,I}=\{k: supp\ \phi_{j,k-1} \text{ and } supp\ \phi_{j,k+1}\subset(0,2)\}=\left\{k:1\le k\le 2^{j+1}-2N\right\},$$

$$V_j[0,2] = \overline{\{(X)^{\alpha}_{j,L}\}}_{0\le\alpha\le N-1} \bigcup \overline{\{\phi_{jk}\}}_{k\in S_{j,I}} \bigcup \overline{\{(X-2)^{\alpha}_{j,R}\}}_{0\le\alpha\le N-1};$$

for interval $[7,12]$

$$S_{j,L} = \left\{k : 7\cdot 2^j - (2N-1) < k < 7\cdot 2^j + 1\right\},$$

$$S_{j,R} = \left\{k : 12\cdot 2^j - 2N < k < 12\cdot 2^j\right\},$$

$$S_{j,I} = \{k : \mathit{supp}\ \phi_{j,k-1} \text{ and } \mathit{supp}\ \phi_{j,k+1} \subset (7,12)\} = \\ = \left\{k : 7\cdot 2^j + 1 \le k \le 12\cdot 2^j - 2N\right\},$$

$$V_j[7,12] = \overline{\{(X-7)^{\beta}_{j,L}\}}_{0\le\beta\le N-1} \bigcup \overline{\{\phi_{jk}\}}_{k\in S_{j,I}} \bigcup \overline{\{(X-12)^{\beta}_{j,R}\}}_{0\le\beta\le N-1}.$$

The basic elements for the spaces $V_j[0,2]$ and $V_j[7,12]$ are given by the formulas:

$$\varphi_{jk} = \begin{cases} \varphi^{\alpha}_{j,L}, & \text{if } k = 1-\alpha \text{ for } \alpha = 0,...,N-1; \\ \Phi_{jk}, & \text{if } k \in S_{j,I}; \\ \varphi^{\alpha}_{j,R}, & \text{if } k = 2^{j+1} - 2N \text{ for } \alpha = 0,...,N-1. \end{cases}$$

and

$$\varphi'_{jk} = \begin{cases} \varphi^{\beta}_{j,L}, & \text{if } k = 7\cdot 2^j + 1 - \beta \text{ for } \beta = 0,...,N-1; \\ \Phi_{jk}, & \text{if } k \in S_{j,I}; \\ \varphi^{\beta}_{j,R}, & \text{if } k = 12\cdot 2^j - 2N \text{ for } \beta = 0,...,N-1. \end{cases}$$

In our example, scaling function $\varphi(t)$ can be chosen to be characteristic function on the interval $[0,1)$, known as the box function:

$$\varphi(t) = \begin{cases} 1, \text{ for } 0 \le t < 1, \\ 0, \text{ otherwise.} \end{cases}$$

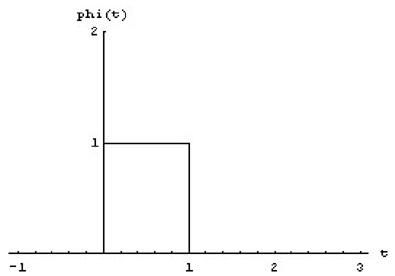

Fig. 1. The box function.

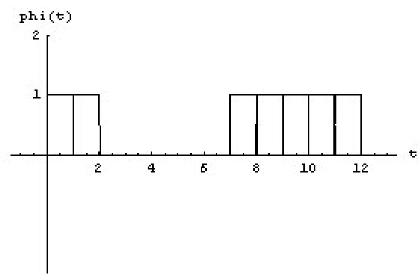

Fig. 2. Translations of the box function.

It can be noted that $\varphi(t)$ are satisfied for all conditions for multiresolution analysis: they are orthonormal since there is no overlap between the supports of functions $\varphi(t-k)$ and $\varphi(t-k')$ whenever $k \neq k'$ and the integral of $\varphi^2(t)$ is equal 1. Any function in V_0 can be written as a superposition of these box functions $f(t) = \sum_{k=-\infty}^{\infty} f(k)\phi(t-k)$. Therefore, the box function is the desired scaling function for the multiresolution analysis which is called the Haar multiresolution analysis.

Now we can use such functions to provide the multiresolution analysis on our set $[0,2]$ and $[7,12]$. Using the translations of the function $\varphi(t)$, we can cover all our set.

By fixing the level of j, for instance $j = 4$, one can carry out the same procedure for the scaling function $\varphi(2^j \cdot t)$ (see Fig. 3, Fig. 4). So the associated wavelet will be as

$$\phi_{4k}(t) = 4\phi(16t-k) = \begin{cases} 4, \text{ for } k/16 \leq t < (1+k)/16, \\ 0, \text{ otherwise.} \end{cases}$$

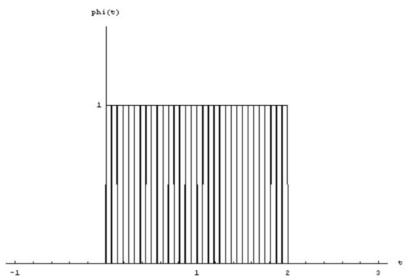

Fig. 3. Translations and dilations of the box function on the interval [0,2].

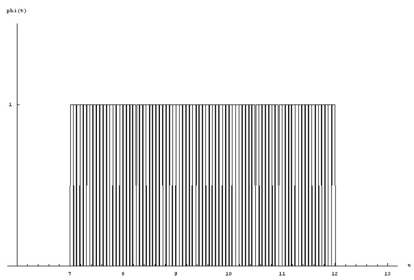

Fig. 4. Translations and dilations of the box function on the interval [7,12].

The orthonormal wavelet-basis for our intervals will be

$$\{\psi_{4,1}, ..., \psi_{4,N-3/2}, ..., \psi_{4,65/2-N}, ..., \psi_{4,30}\}$$

and

$$\{\psi_{4,113}, ..., \psi_{4,221/2+N}, ..., \psi_{4,385/2-N}, ..., \psi_{4,190}\}$$

respectively for interval $[0,2]$ and $[7,12]$. Suchwise we have for the first interval 30 basis functions and 78 for the second interval. Now we could summarize this in a form of propositon.

Proposition 1. *The basis of two intervals could be obtain as the combination of the basic functions from each intervals through the continuation its elements up to the functions defined on the whole real line* $\mathbb{R}$.

Remark 1. The traces of this functions on the second interval form the family which, generally speaking, has a nonempty intersection with the linear span of the wavelet-basis on the second interval. This hold to the necessity to select those family of functions from the union of two basis, which can engender an orthonormal basis on each interval. Exists the difference of the quantity of the corresponding to different intervals translations of a fixed basic function. By increasing the level of j we increase such difference. Eliminating of the extra-functions can be done algorithmically and this is the topic of the future work.

6 Conclusion

We discuss here the construction of the multiresolution corresponding and the wavelet spaces corresponding to a finite interval. The corresponding construction is generalized for the case of two intervals with rational endpoints. The main difficulties are outlined. The application to the study of financial time series is described.

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