

Option pricing for a partially observed pure jump price process

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Abstract. Financial asset price movements are considered, where the risky asset price is a marked point process. Let its dynamics depend on an underlying event arrivals process which is assumed to be a marked point process unobserved by the market agents. Taking into account the presence of catastrophic events, the possibility of common jump times between the risky asset price process and the arrivals process is allowed. The equivalent martingale measures are characterized. The arbitrage-free pricing of a European contingent claim is identified as the conditional expectation with respect to the observations under the minimal martingale measure.

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1 Introduction

This paper studies the problem of the arbitrage-free pricing of a European contingent claim under the minimal martingale measure in a financial market where the assets prices are modeled by market point processes.

In the economic literature, stochastic models of financial markets mainly consider continuous paths processes, quote [18] among others. Anyway, the form of the real data, suggests, as in [17], that the prices are piecewise constant and jump at irregularly spaced random times in reaction to trades or to significant new information.

Intraday information on financial asset price quotes and the increasing amount of studies on market microstructure lead many authors to believe that pure jump processes may be more suitable for modeling of the observed price or quantities related to the price, [11], [10], [9], [17], and references therein. Often, they believe that models that consider continuous trajectory processes, even if the presence of jumps are allowed, does not take into account the discreteness in the data and could lead to wrong conclusions.

In order to describe the amount of information received by the traders related to intraday market activity, the activity of other markets, macroeconomics factors or microstructure rules, some of these references introduce exogenous stochastic factors. In all these cases, the local characteristics of the price process, such as the jump-intensity and the jump-size distribution, depend on a latent process whose behaviour is described in different ways by different authors.

This is the framework of this paper that considers a stylized financial market with a single risky asset and a bond. All over the paper the price of the risky asset is assumed to be discounted with respect to the price of the bond and for the sake of simplicity it will be denoted by price process.

In order to link informations released and the behaviour of trading activity, as suggested by the economic literature (as in [4] and [13]), this paper assumes that the local characteristics of the price process depend on the whole history of an exogenous marked point process. Moreover, as in [13], the markovianity of this processes is assumed, but as an additional generalization, herein, they may have common jump times. This means that trading activity may affect the law of the exogenous process and the possibility of catastrophic events is also allowed.

The main interest of this paper relies on the fact that the price process has a jump behavior which implies incompleteness of the market. As a consequence a set may exist with an infinite number of risk-neutral measures.

Chosen a risk-neutral probability measure the no-arbitrage price of a contingent claim can be defined by the conditional expectation under this measure, as in the case of complete markets. Many choices have been discussed in the literature. Let us quote the minimal martingale measure, [1], [8], [17], and [18], the mean-variance martingale measure [5] and [19], the minimal entropy martingale measure, [9], [12], and [14].

This paper will focus, in particular, on the minimal martingale measure, introduced in [8].

However, in this note, recalling that the risky asset price has a pure jump behavior then some of the results proven in [8] cannot be applied. Anyway, sufficient conditions for existence and uniqueness of the minimal martingale measure are given following [1] and some of its properties are also derived.

Finally, the paper deals with the problem of the valuation of a contingent claim. Assuming that the exogenous process cannot be observed by the agents,

the arbitrage-free price of a contingent claim has to be defined. To this end taking into account the discussion performed in [16] and [3], the arbitrage-free price of a European contingent claim under restricted information is identified as its conditional expectation with respect to the observations under the minimal martingale measure.

Observing that under suitable assumptions the minimal martingale measure preserves the Markovianity of the model, following the innovation method the Kushner-Stratonovich equation for the filter is written down. Existence and uniqueness for its solution is proven and with a classical procedure the computation of this conditional expectation with respect to the observations reduces to the computation of an ordinary expected value.

The paper is organized as follows. In Section 2 the model and some of its properties are described. Section 3 is devoted to the characterization of the risk-neutral measures. Recall that, since the market is incomplete, this discussion is not trivial. Even if the result given in [8] cannot be used as a consequence of the jump behavior of the price process, in Section 4 the minimal martingale measure is determined explicitly and some of its properties are derived. Section 5 is devoted to the evaluation of a contingent claim, assuming that the exogenous process is not observable. Last section is devoted to some conclusions.

2 The Model

On a filtered probability space, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, where $\{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions, let us consider a market model with a single risky asset S and a non-risky asset. The price of the risky asset in units of the numeraire is a process S having the form $S_t = S_0 \exp\{Y_t\}$, with $S_0 \in \mathbb{R}^+$.

The logreturn price Y is supposed to be an \mathbb{R} -valued marked point process, with $Y_0 = 0$, characterized by the sequence $\{\tau_n^Y, Y_{\tau_n^Y} - Y_{\tau_n^Y-}\}_{n \geq 0}$, where $\{\tau_n^Y\}_{n \geq 0}$, is a nondecreasing nonexplosive sequence of stopping times. This means that the jump times of Y , $\{\tau_n^Y\}_{n \geq 0}$, are positive random variables, such that $\{\tau_n^Y \leq t\} \in \mathcal{F}_t$, $\forall t$, and

$$\tau_0^Y = 0, \quad \tau_n^Y < +\infty \quad \Rightarrow \quad \tau_n^Y < \tau_{n+1}^Y, \quad \lim_{n \rightarrow +\infty} \tau_n^Y = +\infty.$$

The dynamics of the logreturn price depend on an exogenous marked point process X , describing arrivals of news to the market, characterized by the sequence $\{\tau_n, X_{\tau_n} - X_{\tau_n-}\}_{n \geq 0}$ where again $\{\tau_n\}_{n \geq 0}$ is a nondecreasing nonexplosive sequence of stopping times. The process X takes values in a finite set $\mathcal{X} \subset \mathbb{R}$, with initial condition $X_0 = 0$, and let us assume that it admits only non-negative jump sizes.

In order to describe the joint dynamics of the processes X and Y , let us define

$$N_t = \sum_{n \geq 0} \mathbf{1}_{\{\tau_n^Y \leq t\}}, \quad (1)$$

the point process counting the jump times of Y up to time t . Let N admit a (P, \mathcal{F}_t) -intensity λ_t , whose structure similar to that given in [4] and [13], is

$$\lambda_t = a(t) + be^{-kt}Z_t := \lambda(t, Z_t).$$

Thus, λ is a deterministic measurable function of the time t and of the process

$$Z_t := z_0 + \int_0^t e^{ks} dX_s,$$

which is a non homogeneous pure jump process, taking values in a suitable $\mathcal{Z} \subseteq \mathbb{R}^+$, and having the same jump times of X . The jump sizes of Z are given by $Z_t - Z_{t-} = e^{kt}(X_t - X_{t-})$.

The constants b and k are real positive parameters, z_0 is assumed to be strictly positive and $a(\cdot)$ is a measurable \mathbb{R}^+ -valued deterministic function, verifying

$$0 \leq a(t) \leq \bar{a} < +\infty.$$

A more explicit expression for λ_t is given by

$$\lambda_t = a(t) + bz_0e^{-kt} + b \sum_{i \geq 0} (X_{\tau_i} - X_{\tau_{i-1}}) e^{-k(t-\tau_i)} \mathbf{1}_{\{\tau_i \leq t\}}. \quad (2)$$

The latter shows that λ_t is strictly positive and in addition that λ_t is bounded, since for Λ suitable positive constant and $\forall t$,

$$\lambda_t \leq \bar{a} + bz_0 + bX_t < \Lambda < +\infty. \quad (3)$$

There is a natural and intuitive interpretation of Equation (2). When some news reaches the market a sudden increases in the trading activity takes place represented by the positive jump size of X at a random time τ_n . Successively a progressive normalization of the market occurs with a speed expressed by k .

Finally, the function $a(\cdot)$ describes the activity of the market in absence of perturbations. An adequately choice of $a(\cdot)$ allows us to take into account deterministic features like seasonalities. The constant b compares the effect of the arrivals of the news and that of the seasonalities.

On one hand, the model proposed in [13] has been generalized by allowing the possibility of common jump times between the latent process X and the logreturn process Y , thus X and Y are strictly correlated.

On the other hand the model has been simplified by assuming the following structure.

At first, let us introduce the counting processes

$$N_t^1 = \sum_{n \geq 0} \mathbf{1}_{\{\tau_n^Y \leq t\}} \mathbf{1}_{\{Y_{\tau_n^Y} - Y_{\tau_n^Y-} > 0\}} \quad \text{and} \quad N_t^2 = \sum_{n \geq 0} \mathbf{1}_{\{\tau_n^Y \leq t\}} \mathbf{1}_{\{Y_{\tau_n^Y} - Y_{\tau_n^Y-} < 0\}},$$

so that

$$N_t = N_t^1 + N_t^2.$$

Moreover, denoting by $\{\tau_n^X\}_{n \geq 0} \subseteq \{\tau_n\}_{n \geq 0}$ the sequence of stopping times at which $X_{\tau_n^X} \neq X_{\tau_n^X-}$ and $Y_{\tau_n^X} = Y_{\tau_n^X-}$, we define

$$N_t^0 = \sum_{n \geq 0} \mathbf{1}_{\{\tau_n^X \leq t\}}$$

and we assume that N^0 admits a (P, \mathcal{F}_t) -intensity given by $\lambda_t^0 := \lambda_0(t, X_{t-}, Z_{t-})$, where $\lambda_0(t, x, z)$ is a bounded non-negative measurable function,

$$\lambda_0(t, x, z) \leq A.$$

Furthermore, for $i = 1, 2$, the point process N^i admits a (P, \mathcal{F}_t) -intensity $\lambda_t p_t^i$, where $\lambda_t := \lambda(t, Z_{t-})$, and $p_t^i := p_i(t, X_{t-}, Z_{t-})$, $i = 1, 2$, with $p_i(t, x, z)$ strictly positive measurable functions such that

$$p_1(t, x, z) + p_2(t, x, z) = 1.$$

Then, let us assume the existence of the measurable functions $\xi(t, x, z)$, $\eta_1(t)$, $\eta_2(t)$, such that

- (a) $\xi : [0, T] \times \mathcal{X} \times \mathcal{Z} \longrightarrow \mathbb{R}^+ \cup \{0\}$,
and for any $t \in [0, T]$, $x \in \mathcal{X}$, $z \in \mathcal{Z}$, $x + \xi(t, x, z) \in \mathcal{X}$,
- (b) for $i = 1, 2$, $\eta_i : [0, T] \longrightarrow \mathbb{R}^+$ is such that,
 $0 < \underline{\eta} \leq \eta_i(t) \leq \bar{\eta}$, for some real constants $\bar{\eta}$ and $\underline{\eta}$.

Finally, setting $\xi_t := \xi(t, X_{t-}, Z_{t-})$, the processes involved in this model will be represented as

$$\begin{aligned} X_t &= \int_0^t \xi_u [dN_u^0 + dN_u], \\ Y_t &= \int_0^t \eta_1(u) dN_u^1 - \int_0^t \eta_2(u) dN_u^2, \\ Z_t &= z_0 + \int_0^t e^{ku} \xi_u [dN_u^0 + dN_u]. \end{aligned}$$

The discussion performed in [7] about the structure of the generator of the pure jump processes allows us to claim that the joint generator of the process (X, Y, Z) , for $f(t, x, y, z)$ belonging to a suitable class of real-valued measurable functions, $t \geq 0$, $x \in \mathcal{X}$, $y \in \mathbb{R}$ and $z \in \mathcal{Z}$, is given by

$$\begin{aligned} Lf(t, x, y, z) &= \frac{\partial}{\partial t} f(t, x, y, z) + \\ &+ L_t^0 f(t, x, y, z) + L_t^1 f(t, x, y, z) + L_t^2 f(t, x, y, z), \end{aligned} \quad (4)$$

where

$$L_t^0 f(t, x, y, z) = \lambda_0(t, x, z) \left[f(t, x + \xi(t, x, z), y, z + e^{kt} \xi(t, x, z)) - f(t, x, y, z) \right],$$

and, for $i = 1, 2$,

$$L_t^i f(t, x, y, z) = \lambda(t, z) \cdot p_i(t, x, z) \cdot \left[f(t, x + \xi(t, x, z), y + (-1)^{i-1} \eta_i(t), z + e^{kt} \xi(t, x, z)) - f(t, x, y, z) \right].$$

Proposition 1. *In the framework of this paper, (X, Y, Z) is a Markov process.*

Last Proposition follows by Theorem 7.3 in [7], since the generator

$$L_t = L_t^0 + L_t^1 + L_t^2,$$

is bounded. Then the Martingale Problem associated with the operator L and initial condition $(X_0 = 0, Y_0 = 0, Z_0 = z_0)$, is well posed. In addition, its solution is a Markov process with trajectories in $D_{\{\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}\}}[0, T]$.

Lemma 1. *The stock price process S_t is a special (P, \mathcal{F}_t) -local semimartingale with canonical decomposition*

$$S_t = S_0 + A_t^S + M_t^S, \quad (5)$$

where A_t^S is a locally bounded variation predictable process and M_t^S is a locally square integrable (P, \mathcal{F}_t) -local martingale given by

$$A_t^S = \int_0^t \lambda_u S_{u-} \left[e^{\eta_1(u)} p_u^1 + e^{-\eta_2(u)} p_u^2 - 1 \right] du, \quad (6)$$

$$M_t^S = \int_0^t S_{u-} (e^{\eta_1(u)} - 1) [dN_u^1 - \lambda_u p_u^1 du] + \int_0^t S_{u-} (e^{-\eta_2(u)} - 1) [dN_u^2 - \lambda_u p_u^2 du], \quad (7)$$

respectively, and

$$\langle M^S \rangle_t = \int_0^t \lambda_u S_{u-}^2 \left[(e^{\eta_1(u)} - 1)^2 p_u^1 + (e^{-\eta_2(u)} - 1)^2 p_u^2 \right] du. \quad (8)$$

Proof. By Itô formula, the stock price process is such that

$$\begin{aligned} S_t &= S_0 + \int_0^t S_{u-} (e^{\eta_1(u)} - 1) dN_u^1 + \int_0^t S_{u-} (e^{-\eta_2(u)} - 1) dN_u^2 = \\ &= S_0 + \int_0^t S_{u-} (e^{\eta_1(u)} - 1) [dN_u^1 - \lambda_u p_u^1 du] + \int_0^t S_{u-} (e^{\eta_1(u)} - 1) \lambda_u p_u^1 du + \\ &\quad + \int_0^t S_{u-} (e^{-\eta_2(u)} - 1) [dN_u^2 - \lambda_u p_u^2 du] + \int_0^t S_{u-} (e^{-\eta_2(u)} - 1) \lambda_u p_u^2 du. \end{aligned} \quad (9)$$

Thus, (6) and (7) follows. Moreover, M_t^S is a locally square integrable (P, \mathcal{F}_t) -local martingale since

$$\int_0^t \lambda_u S_u^2 \left[(e^{\eta_1(u)} - 1)^2 p_u^1 + (e^{-\eta_2(u)} - 1)^2 p_u^2 \right] du < +\infty \quad P - a.s..$$

For $\bar{f}(y) = S_0 e^y$ and recalling that

$$\langle M^S \rangle_t = \int_0^t [L_u(\bar{f}(Y_{u-})^2) - 2\bar{f}(Y_{u-})L_u\bar{f}(Y_{u-})] du,$$

substituting we get the thesis. \square

3 Equivalent Martingale Measures

From now on, let T be a fixed finite horizon and by a little abuse of notations let $\mathcal{F}_t := \sigma\{X_s, Y_s, 0 \leq s \leq t\}$, for $t \leq T$.

This section characterizes the equivalent martingale measures, that is the set of probability measures Q , equivalent to P , under which S is a (Q, \mathcal{F}_t) -local martingale. The main tool for the characterization of the risk-neutral measures is a suitable version of the Girsanov Theorem.

The choice of the internal filtration allows us to claim that, for M (P, \mathcal{F}_t) -local martingale, any probability measure Q equivalent to P is a solution to the exponential equation $\mathcal{L}_t = 1 + \int_0^t \mathcal{L}_{s-} dM_s$, [6] and [15], whose unique solution is given by $\mathcal{L}_t = e^{M_t - \frac{1}{2}\langle M^c \rangle_t} \prod_{s \leq t} (1 + M_s - M_{s-}) e^{-(M_s - M_{s-})}$.

Furthermore, since $\mathcal{L}_t = (1 + M_t - M_{t-}) \mathcal{L}_{t-}$, if M is a (P, \mathcal{F}_t) -local martingale such that $M_t - M_{t-} > -1$, then \mathcal{L} is a (P, \mathcal{F}_t) -local martingale and a strictly positive supermartingale, which is a (P, \mathcal{F}_t) -martingale when $\mathbb{E}[\mathcal{L}_T] = 1$. In this last case, the measure Q defined by the Radon-Nykodim derivative $\frac{dQ}{dP} \Big|_{\mathcal{F}_T} = \mathcal{L}_T$ is a probability measure equivalent to P .

On the other hand, setting $p_s^0 := \lambda_s^0 / \lambda_s$, any (P, \mathcal{F}_t) -local martingale M_t admits the representation

$$M_t = M_0 + \sum_{i=0,1,2} \int_0^t g_s^i [dN_s^i - \lambda_s p_s^i ds] \quad (10)$$

where g_s^i , for $i = 0, 1, 2$, are (P, \mathcal{F}_s) -predictable processes. Under the assumption that

$$\sum_{i=0,1,2} \int_0^t |g_s^i| \lambda_s^i ds < +\infty \quad P - a.s. \quad \text{or} \quad \mathbb{E} \left[\sum_{i=0,1,2} \int_0^t |g_s^i| \lambda_s^i ds \right] < +\infty,$$

M_t is a (P, \mathcal{F}_t) -local martingale or a (P, \mathcal{F}_t) -martingale, respectively. In this last case, necessarily, M_t is uniformly integrable, since we are working on a finite horizon.

Remark 1. By (10), $M_t - M_{t-} = \sum_{i=0,1,2} g_t^i (N_t^i - N_{t-}^i)$. Therefore, if t does not coincide with a jump time of X or of Y , obviously $M_t - M_{t-} > -1$. Otherwise, at any jump time of the process (X, Y, Z) ,

$$M_t - M_{t-} > -1 \quad \Longleftrightarrow \quad g_t^i > -1 \quad \text{for } i = 0, 1, 2. \quad (11)$$

In this last case,

$$\mathcal{L}_T = \prod_{i=0,1,2} \exp \left\{ \int_0^T \log(1 + g_s^i) dN_s^i - \int_0^T g_s^i \lambda_s p_s^i ds \right\} \quad (12)$$

is the density of Q with respect to P .

Next, we are going to determine the semimartingale structure of the price process under the measure Q . Note that under the assumption

$$\sum_{i=0,1,2} \int_0^T (g_t^i + 1) \lambda_t p_t^i dt < +\infty \quad P - a.s., \quad (13)$$

the process N^i , for $i = 0, 1, 2$ admits (Q, \mathcal{F}_t) -intensity given by $(g_t^i + 1) \lambda_t p_t^i$.

Proposition 2. *Under the condition $g_t^i > -1$ for $i = 0, 1, 2$ and under the condition (13) the price of the risky asset, S , is a special (Q, \mathcal{F}_t) -local semimartingale, such that*

$$S_t = S_0 + A_t^Q + M_t^Q$$

where

$$A_t^Q = \int_0^t \lambda_u S_{u-} \left[(e^{\eta_1(u)} - 1)(g_u^1 + 1)p_u^1 + (e^{-\eta_2(u)} - 1)(g_u^2 + 1)p_u^2 \right] du,$$

and

$$\begin{aligned} M_t^Q &= \int_0^t S_{u-} (e^{\eta_1(u)} - 1) [dN_u^1 - (g_u^1 + 1)\lambda_u p_u^1 du] + \\ &+ \int_0^t S_{u-} (e^{-\eta_2(u)} - 1) [dN_u^2 - (g_u^2 + 1)\lambda_u p_u^2 du]. \end{aligned}$$

The proof is similar to that of Lemma 1, taking into account the (Q, \mathcal{F}_t) -intensities of the processes N^i for $i = 1, 2$.

Remark 2. Recalling Lemma 1, if $A_t^S = 0$, P -a.s., the original measure P is a risk-neutral measure. With the same procedure, Q is risk-neutral if and only if

$$A_t^Q = 0, \quad P - a.s., \quad (14)$$

and M_t^Q is a (Q, \mathcal{F}_t) -local martingale. If

$$\int_0^t \lambda_u S_{u-} \left[(e^{\eta_1(u)} - 1)(g_u^1 + 1)p_u^1 + (1 - e^{-\eta_2(u)}) (g_u^2 + 1)p_u^2 \right] du < +\infty \quad P\text{-a.s.}$$

then M_t^Q is a (Q, \mathcal{F}_t) -local martingale which turns to be a (Q, \mathcal{F}_t) -martingale if

$$\mathbb{E} \left[\int_0^t \lambda_u S_{u-} \left\{ (e^{\eta_1(u)} - 1)(g_u^1 + 1)p_u^1 + (1 - e^{-\eta_2(u)}) (g_u^2 + 1)p_u^2 \right\} du \right] < +\infty.$$

Let us note that if the price process is strictly increasing, or strictly decreasing, (14) cannot be satisfied and the model does not admit any equivalent martingale measure.

Next, in the set of the risk-neutral measures, when it is not empty, one can find an element preserving the Markovianity of the process (X, Y, Z) assuming the existence of the real valued measurable deterministic functions $g^i(t, x, y, z)$, $i = 0, 1, 2$, such that

$$g_t^i = g^i(t, X_{t-}, Y_{t-}, Z_{t-}). \quad (15)$$

In this case, defining

$$L^Q f(t, x, y, z) = \frac{\partial}{\partial t} f(t, x, y, z) + L_t^Q f(t, x, y, z), \quad (16)$$

where, setting $\lambda_i^Q(t, x, z) = \lambda(t, z)p_i(t, x, z)(1 + g^i(t, x, y, z))$ for $i = 0, 1, 2$,

$$\begin{aligned} L_t^Q f(t, x, y, z) = & \\ = & \lambda_0^Q(t, x, z) [f(t, x + \xi(t, x, z), y, z + e^{kt}\xi(t, x, z)) - f(t, x, y, z)] + \\ & + \lambda_1^Q(t, x, z) [f(t, x + \xi(t, x, z), y + \eta_1(t, x, z), z + e^{kt}\xi(t, x, z)) - f(t, x, y, z)] + \\ & + \lambda_2^Q(t, x, z) [f(t, x + \xi(t, x, z), y - \eta_2(t, x, z), z + e^{kt}\xi(t, x, z)) - f(t, x, y, z)], \end{aligned}$$

for any bounded, real valued, measurable function f , under (13) with g_t^i given by (15), we have that the process

$$M_f^Q(t) = f(t, X_t, Y_t, Z_t) - f(0, x_0, 0, z_0) - \int_0^t L^Q f(s, X_s, Y_s, Z_s) ds$$

is a (Q, \mathcal{F}_t) -local martingale.

The Martingale Problem for the operator L^Q given in (16) and initial condition $(0, 0, 0, z_0)$ is well posed since such is the Martingale Problem for the operator L given in (4) with the same initial conditions. This implies that the

process (X_t, Y_t, Z_t) is Markovian under the measure Q . We observe that, when (15) holds true, the condition

$$(e^{\eta_1(t,x,z)} - 1) [1 + g^1(t, x, y, z)] p_1(t, x, z) + (e^{-\eta_2(t,x,z)} - 1) [1 + g^2(t, x, y, z)] p_2(t, x, z) = 0 \quad (17)$$

provides a sufficient condition for (14).

4 Minimal Martingale Measure

The minimal martingale measure \widehat{P} , as observed in [17], was introduced in [8], to obtain hedging strategies, which are optimal in a suitable sense.

In [18], the author shows that the value process can be computed as the conditional expectation with respect to \widehat{P} and then a risk-neutral approach to option valuation is provided. The classical definition given in [18] is

Definition 1. *An equivalent martingale measure \widehat{P} is called minimal, if each (P, \mathcal{F}_t) -local martingale, R , strictly orthogonal to M^S , (7), is a $(\widehat{P}, \mathcal{F}_t)$ -local martingale.*

For any initial probability P , there exists at most one minimal martingale measure, (Theorem 2.1, [1]). Herein, for this model its existence is proven and its structure is described below.

The main tool is the following Theorem whose proof can be found in [20], Proposition 2. We recall this Theorem for sake of completeness.

Theorem 1. *Assuming that there exists a (P, \mathcal{F}_t) -predictable process c_u such that*

$$\begin{aligned} (i) \quad & A_t^S = \int_0^t c_u d \langle M^S \rangle_u, \\ (ii) \quad & \int_0^t |c_u|^2 d \langle M^S \rangle_u < +\infty \quad P - a.s., \\ (iii) \quad & 1 - c_t(M_t^S - M_{t-}^S) > 0 \quad P - a.s., \end{aligned}$$

then, $\widehat{\mathcal{L}}_t := \mathcal{E} \left(- \int_0^t c_u dM_u^S \right)$ is a (P, \mathcal{F}_t) -strictly positive local martingale.

When $\widehat{\mathcal{L}}_t$ is a (P, \mathcal{F}_t) -martingale, the probability measure \widehat{P} defined by

$$\widehat{\mathcal{L}}_t = \frac{d\widehat{P}}{dP} \Big|_{\mathcal{F}_t}$$

is the minimal martingale measure.

On the other hand, when $P(1 - c_t(M_t^S - M_{t-}^S) \leq 0) > 0$, the minimal martingale measure does not exist.

From now on, taking into account the particular structure of the model studied in this note, results stronger than in a more general setting can be obtained.

Recalling (6), (7) and (8), (i) implies that

$$c_t = \frac{(e^{\eta_1(t)} - 1) p_t^1 + (e^{-\eta_2(t)} - 1) p_t^2}{S_{t-} [(e^{\eta_1(t)} - 1)^2 p_t^1 + (e^{-\eta_2(t)} - 1)^2 p_t^2]}.$$

Note that c_t is a predictable process as required. Moreover, Condition (ii) of Theorem 1 is verified, since λ_u is bounded and

$$\int_0^t |c_u|^2 d \langle M^S \rangle_u \leq \int_0^t \lambda_u du < +\infty \quad \text{P-a.s.}$$

Condition (iii) of Theorem 1 is trivially satisfied if t is not a jump time of Y , recalling (7) and Remark 1. Otherwise, if t is a jump time of Y , (iii) becomes

$$\frac{(e^{\eta_1(t)} - 1) p_t^1 + (e^{-\eta_2(t)} - 1) p_t^2}{S_{t-} (e^{\eta_1(t)} - 1)^2 p_t^1 + (e^{-\eta_2(t)} - 1)^2 p_t^2} \cdot S_{t-} \cdot \left[(e^{\eta_1(t)} - 1) \mathbf{1}_{\{Y_t - Y_{t-} > 0\}} + (e^{-\eta_2(t)} - 1) \mathbf{1}_{\{Y_t - Y_{t-} < 0\}} \right] < 1,$$

that is always verified.

Summing up, $\widehat{\mathcal{L}}_t := \mathcal{E} \left(- \int_0^t c_u dM_u^S \right)$ is a (P, \mathcal{F}_t) -strictly positive local martingale. Setting

$$\widehat{M}_t := - \int_0^t c_u dM_u^S$$

the martingale \widehat{M}_t has the structure given in (10), with

$$\widehat{g}_t^0 = 0, \quad \widehat{g}_t^1 = -c_t S_{t-} (e^{\eta_1(t)} - 1), \quad \widehat{g}_t^2 = -c_t S_{t-} (e^{-\eta_2(t)} - 1). \quad (18)$$

Proposition 3. *The probability measure \widehat{P} is equivalent to P . It is a risk-neutral measure and coincides with the minimal martingale measure. Furthermore \widehat{P} preserves the Markovianity.*

Proof. For a real constant $K > 0$, depending on η and $\bar{\eta}$,

$$|\widehat{g}_t^1| \vee |\widehat{g}_t^2| \leq K, \quad (19)$$

and, for all $t \geq 0$, $\widehat{\mathcal{L}}_t$, the solution to

$$\begin{aligned} \widehat{\mathcal{L}}_t &= 1 + \int_0^t \widehat{\mathcal{L}}_{s-} d\widehat{M}_s = \\ &= 1 + \int_0^t \widehat{\mathcal{L}}_{s-} \widehat{g}_s^1 [dN_s^1 - \lambda_s p_s^1 ds] + \int_0^t \widehat{\mathcal{L}}_{s-} \widehat{g}_s^2 [dN_s^2 - \lambda_s p_s^2 ds], \end{aligned}$$

is a strictly positive supermartingale, [1], [6]. Consequently, [2], $\mathbb{E}[\widehat{\mathcal{L}}_t] = 1$, since

$$\mathbb{E} \left[\int_0^t \widehat{\mathcal{L}}_{s-} |\widehat{g}_s^1| \lambda_s p_s^1 ds + \int_0^t \widehat{\mathcal{L}}_{s-} |\widehat{g}_s^2| \lambda_s p_s^2 ds \right] \leq K \Lambda \int_0^t \mathbb{E} [\widehat{\mathcal{L}}_{s-}] ds < +\infty.$$

The last claim is true because (15) is a consequence of (18) and (13) follows by (19) since

$$\mathbb{E} \left[\sum_{i=0,1,2} \int_0^T (1 + \widehat{g}_t^i) \lambda_t p_t^i dt \right] \leq \mathbb{E} \left[\int_0^T (\lambda_t^0 + \lambda_t K) dt \right] \leq T \Lambda (1+K) < +\infty \quad \square$$

Under \widehat{P} , the generator of the Markov process (X, Y, Z) is then given by

$$\begin{aligned} \widehat{L}F(t, x, y, z) &= \\ &= \frac{\partial}{\partial t} F(t, x, y, z) + \widehat{L}_t^0 F(t, x, y, z) + \widehat{L}_t^1 F(t, x, y, z) + \widehat{L}_t^2 F(t, x, y, z), \end{aligned} \quad (20)$$

where, setting, for $i = 1, 2$,

$$\widehat{\lambda}_0(t, X_t, Z_t) = \lambda_0(t, X_t, Z_t) \quad \text{and} \quad \widehat{\lambda}_i(t, X_t, Z_t) = \lambda_t (1 + \widehat{g}_t^i) p_t^i,$$

we get

$$\begin{aligned} \widehat{L}_t^0 F(t, x, y, z) &= L_t^0 F(t, x, y, z) = \\ &= \lambda_0(t, x, z) \left[f(t, x + \xi(t, x, z), y, z + e^{kt} \xi(t, x, z)) - f(t, x, y, z) \right] \end{aligned}$$

and

$$\begin{aligned} \widehat{L}_t^i F(t, x, y, z) &= \lambda_i(t, x, z) \cdot \\ &\cdot \left[F(t, x + \xi(t, x, z), y + (-1)^{i-1} \eta_i(t), z + e^{kt} \xi(t, x, z)) - F(t, x, y, z) \right]. \end{aligned}$$

5 Pricing

Let us consider a European contingent claim with maturity T whose payoff is given by $H(S_T)$, referred to as the option and such that is a random variable belonging to $\mathcal{L}^2(\Omega, \mathcal{F}_t^S, P)$. From now on the exogenous process X is assumed to be unobserved by the market agents, hence this section deals with the problem of pricing, that is to determine the value of the payoff at each time $t \in [0, T]$ in order to avoid arbitrage opportunities, in a partially observed model.

Let us define the price of the claim as the expectation conditioned to the observations, under a suitable martingale measure. As suggested by the considerations developed in [16], here the minimal martingale measure is chosen.

In order to compute $\mathbb{E}^{\widehat{P}}[H(S_T)|\mathcal{F}_t^Y] = \mathbb{E}^{\widehat{P}}[\mathbb{E}^{\widehat{P}}[H(S_T)|\mathcal{F}_t]|\mathcal{F}_t^Y]$, since \widehat{P} preserves the Markovianity of the process (X, Y, Z) , there exists a measurable function $h(t, x, y, z)$ such that

$$\mathbb{E}^{\widehat{P}}[H(S_T)|\mathcal{F}_t] = h(t, X_t, Y_t, Z_t), \quad (21)$$

where h solves the system

$$\begin{cases} \widehat{L}h(t, x, y, z) = \frac{\partial}{\partial t}h(t, x, y, z) + \widehat{L}_t h(t, x, y, z) = 0 \\ h(T, x, y, z) = H(S_0 e^y). \end{cases} \quad (22)$$

Next proposition shows that the latter is a linear system with final data which can be solved by classical recursive methods.

Proposition 4. *Assuming that $H(\cdot) \leq H$, the problem (22) admits a unique measurable bounded solution which is absolutely continuous with respect to t . Then, for any (x, y, z) and for a.a. t there exists $\frac{\partial}{\partial t}h(t, x, y, z)$ and is bounded.*

Proof. Let

$$\begin{aligned} \alpha(t, x, z) &:= \widehat{\lambda}_0(t, x, z) + \widehat{\lambda}_1(t, x, z) + \widehat{\lambda}_2(t, x, z) = \\ &= \lambda_0(t, x, z) + \lambda_t(t, z)(1 + \widehat{g}^1(t, x, y, z))p^1(t, x, z) + \\ &\quad + \lambda_t(t, z)(1 + \widehat{g}^2(t, x, y, z))p^2(t, x, z) \end{aligned}$$

and

$$\begin{aligned} G(t, x, y, z, h) &:= \widehat{\lambda}_0(t, x, z) h(t, x + \xi(t, x, z), y, z + e^{kt}\xi(t, x, z)) + \\ &\quad + \widehat{\lambda}_1(t, x, z) h(t, x + \xi(t, x, z), y + \eta_1(t, x, z), z + e^{kt}\xi(t, x, z)) + \\ &\quad + \widehat{\lambda}_2(t, x, z) h(t, x + \xi(t, x, z), y - \eta_2(t, x, z), z + e^{kt}\xi(t, x, z)), \end{aligned}$$

substituting (22) can be written as

$$\begin{cases} \frac{\partial}{\partial t}h(t, x, y, z) - \alpha(t, x, z)h(t, x, y, z) + G(t, x, y, z, h) = 0 \\ h(T, x, y, z) = H(S_0 e^y). \end{cases} \quad (23)$$

Consequently, (23) is equivalent to

$$\begin{aligned} h(t, x, y, z) &= H(S_0 e^y) \exp \left\{ - \int_t^T \alpha(u, x, z) du \right\} + \\ &\quad + \int_t^T G(s, x, y, z, h) \exp \left\{ - \int_t^s \alpha(u, x, z) du \right\} ds, \end{aligned} \quad (24)$$

since, differentiating both sides of (24) with respect to t , one can obtain an equation that, joint with (24) reproduces (23).

The equation (24) has a unique bounded solution. In fact, if h_1, h_2 are two bounded solutions, setting

$$\Delta_h(t) = \sup_{x,y,z} |h_1(t, x, y, z) - h_2(t, x, y, z)|$$

and recalling (19), we get that

$$\Delta_h(t) \leq \int_t^T \alpha(s, x, z) \Delta_h(s) ds \leq \Lambda(3 + 2K) \int_t^T \Delta_h(s) ds$$

and the assertion follows by a slight modification of the Gronwall Lemma.

Finally, the existence of a bounded solution absolutely continuous with respect to t is obtained by a classical recursive method. Defining, recursively,

$$h_0(t, x, y, z) = H(S_0 e^y) e^{-\int_t^T \alpha(s, x, z) ds} + \int_t^T H(S_0 e^y) e^{-\int_t^s \alpha(u, x, z) du} \alpha(s, x, z) ds$$

and, for $j \geq 0$,

$$h_{j+1}(t, x, y, z) = H(S_0 e^y) e^{-\int_t^T \alpha(u, x, z) du} + \int_t^T G(s, x, y, z, h_j) e^{-\int_t^s \alpha(u, x, z) du} ds,$$

then

$$\|h_1 - h_0\| \leq T \Lambda H (3 + 2K) [2 + T \Lambda (3 + 2K)]$$

and

$$\begin{aligned} |h_{j+1}(t, x, y, z) - h_j(t, x, y, z)| &\leq \frac{\Lambda^j (3 + 2K)^j}{j!} (T - t)^j \|h_1 - h_0\| \leq \\ &\leq \frac{\Lambda^j (3 + 2K)^j}{j!} T^j \|h_1 - h_0\| \end{aligned}$$

and the conclusion by standard arguments. \square

Thus, the pricing problem reduces to a filtering problem, i.e. the computation of

$$\mathbb{E}^{\widehat{P}}[H(S_T) | \mathcal{F}_t^Y] = \mathbb{E}^{\widehat{P}}[h(t, X_t, Y_t, Z_t) | \mathcal{F}_t^Y],$$

and one can deal with this problem by the classical innovation method.

The conditional law of the process (X, Z, Y) , given the σ -algebra $\mathcal{F}_t^S = \mathcal{F}_t^Y$, is characterized by introducing the filter, which is the cadlag version of this conditional law.

For any bounded measurable F , the filter, $\pi_t(F) = \mathbb{E}^{\widehat{P}}[F(t, X_t, Z_t, Y_t) | \mathcal{F}_t^Y]$, satisfies a stochastic differential equation known as the Kushner-Stratonovich equation [2]. Following a procedure analogous to that presented in [13], the

filtering equation is written down and strong uniqueness of its solutions is proven.

Recall that $N_t := N_t^1 + N_t^2$ and that the process Y_t can be represented as

$$Y_t = \int_0^t \eta_1(u) dN_u^1 - \int_0^t \eta_2(u) dN_u^2,$$

then $\mathcal{F}_t^Y = \mathcal{F}_t^{N^1} \vee \mathcal{F}_t^{N^2}$.

The main result of this section is given by Theorem 2 below.

Theorem 2. *The $(\widehat{P}, \mathcal{F}_t)$ -intensity of N^i , is given by the process*

$$\widehat{\lambda}_t^i = \widehat{\lambda}_i(t, X_{t-}, Z_{t-}), \quad i = 1, 2.$$

For any bounded measurable function F defined on $\mathcal{X} \times \mathbb{R} \times \mathcal{Z}$, the Kushner-Stratonovich equation can be written as

$$\begin{aligned} \pi_t(F) &= \pi_0(F) + \int_0^t \pi_s \left(\widehat{L}_s F(s, \cdot, Y_{s-}, \cdot) \right) ds + \\ &+ \sum_{i=1,2} \int_0^t \pi_{s-} \left(\widehat{\lambda}_i(s, \cdot, \cdot) \right)^+ \Psi_{s-}^i(F) \left(dN_s^i - \pi_{s-} \left(\widehat{\lambda}_i(s, \cdot, \cdot) \right) ds \right) \end{aligned} \quad (25)$$

where

$$\Psi_{s-}^i(F) = \pi_{s-} \left(\widehat{\lambda}_i(s, \cdot, \cdot) F \right) - \pi_{s-} \left(\widehat{\lambda}_i(s, \cdot, \cdot) \right) \pi_{s-}(F) + \pi_{s-} \left(\widehat{L}_s F(s, \cdot, Y_{s-}, \cdot) \right),$$

and $a^+ := \frac{1}{a} \mathbf{1}_{\{a>0\}}$. Moreover, the filtering equation has a unique strong solution.

Proof. The first claim is obtained by taking into account the joint dynamics of X_t, Y_t, Z_t given in (20).

By applying the classical innovation method as described in [2] the Kushner-Stratonovich equation is written down. In particular, the last term in $\Psi_{s-}^i(F)$, which arises when common jump times between the state and the observations are allowed, is related with $\langle F(X, Y, Z), N^i \rangle_{>t}$, $i = 1, 2$.

As far as the strong uniqueness of the solutions of (25), observe that at any jump time $t = \tau_k^Y$, the filter is uniquely determined by the knowledge of π_{t-} . In fact, for $Y_t - Y_{t-} > 0$, $\pi_{t-}(\lambda_1(t, \cdot, \cdot)) \neq 0$, or for $Y_t - Y_{t-} < 0$, $\pi_{t-}(\lambda_2(t, \cdot, \cdot)) \neq 0$, and

$$\begin{aligned} \pi_t(F) &= \pi_{t-}(F) + \\ &+ \pi_{s-} \left(\widehat{\lambda}_1(s, \cdot, \cdot) \right)^+ \Psi_{t-}^1(F) \mathbb{I}_{\{Y_t - Y_{t-} > 0\}} + \pi_{s-} \left(\widehat{\lambda}_2(s, \cdot, \cdot) \right)^+ \Psi_{t-}^2(F) \mathbb{I}_{\{Y_t - Y_{t-} < 0\}} = \\ &= \frac{\pi_{s-} \left(\widehat{\lambda}_1(s, \cdot, \cdot) F + \widehat{L}_s^1 F(s, \cdot, Y_{s-}, \cdot) \right)}{\pi_{s-} \left(\widehat{\lambda}_1(s, \cdot, \cdot) \right)} \mathbb{I}_{\{Y_t - Y_{t-} > 0\}} + \\ &\quad + \frac{\pi_{s-} \left(\widehat{\lambda}_2(s, \cdot, \cdot) F + \widehat{L}_s^2 F(s, \cdot, Y_{s-}, \cdot) \right)}{\pi_{s-} \left(\widehat{\lambda}_2(s, \cdot, \cdot) \right)} \mathbb{I}_{\{Y_t - Y_{t-} < 0\}}. \end{aligned}$$

For $t \in [\tau_k^Y, \tau_{k+1}^Y)$, the behaviour of the filter is defined by the equation

$$\pi_t(F) = \pi_{\tau_k^Y}(F) + \int_{\tau_k^Y}^t \{\pi_s(L_s^0 F) + \pi_s(\widehat{\lambda}(s, \cdot))\pi_s(F) - \pi_s(\widehat{\lambda}(s, \cdot)F)\} ds, \quad (26)$$

where

$$\widehat{\lambda}(t, X_t, Z_t) = \lambda(t, Z_{t-}) [1 + \widehat{g}_t^1 p_t^1 + \widehat{g}_t^2 p_t^2].$$

For any two solutions π_t^1 and π_t^2 of (26), such that $\pi_{\tau_k^Y}^1(F) = \pi_{\tau_k^Y}^2(F)$, there exists a suitable positive constant C depending on $\|F\| = \sup_{t,x,y,z} |F(t, x, y, z)|$, such that

$$|\pi_t^1(F) - \pi_t^2(F)| \leq C \int_{\tau_k^Y}^t \|\pi_s^1 - \pi_s^2\| ds,$$

where $\|\cdot\|$ denotes the bounded variation norm of the signed measure $\pi_s^1 - \pi_s^2$. The last inequality guarantees uniqueness for $t \in [\tau_k^Y, \tau_{k+1}^Y)$ since (26) is Lipschitz with respect to the bounded variation norm and the thesis follows by induction. \square

Finally a representation for the filter via a classical linearized method can be performed, as for example in [13], showing that the computation of the filter between two consecutive jump times can be reduced to the evaluation of an ordinary expectation.

More precisely, for $t \in [\tau_k^Y, \tau_{k+1}^Y)$,

$$\pi_t(F) = \frac{\int_{\mathcal{X} \times \mathcal{Z}} \mathbf{E}_{s,x,z} \left[F(X_t, Z_t) \exp \left\{ - \int_s^t \widehat{\lambda}(u, X_u, Z_u) du \right\} \right] \Big|_{s=\tau_k^Y} \pi_{\tau_k^Y}(dx, dz)}{\int_{\mathcal{X} \times \mathcal{Z}} \mathbf{E}_{s,x,z} \left[\exp \left\{ - \int_s^t \widehat{\lambda}(u, X_u, Z_u) du \right\} \right] \Big|_{s=\tau_k^Y} \pi_{\tau_k^Y}(dx, dz)}$$

where $P_{s,x,z}$ denotes the law of the process (X, Z) , whose dynamics is defined by the operator L_t^0 with initial condition (s, x, z) . Technical details on this procedure can be found in [13] and references therein.

6 Conclusions

This paper studies a financial model in which the asset prices have a jump behavior depending on an exogenous stochastic factor, supposed unobservable. The market is incomplete and an infinite number of risk-neutral measures may exist.

The contribution of this paper is twofold. First aim is the characterization of the equivalent martingale measures and in particular of the minimal martingale measure. Then, given an European contingent claim, the value of the payoff at

each time is determined in order to avoid arbitrage opportunities, under the minimal martingale measure.

The first aim requires a change of probability measure obtained by a Girsanov type change of measure. This procedure characterizes the class of the risk-neutral measures.

The dynamics of the price process guarantee that this class is not empty. By Proposition 2 in [20], the minimal martingale measure is given and in our setting we get an explicit expression of it.

For the second aim, following the existing literature, the problem reduces to evaluate the expectation conditioned to the observations, under a suitable martingale measure, which is chosen to be the minimal martingale measure. This brings us to a filtering problem for which an explicit solution is given.

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