# Measuring and hedging the basis risk
# by Functional Demographic Models

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**Abstract.** Longevity phenomenon is a relevant aspect for insurance companies which are obliged to quantify the impact of uncertainty of mortality trend on issued products, in order to manage the risk derived from it. Recently, significant tools have been developed for transferring longevity risk to the capital markets, bringing additional capacity, flexibility and transparency to complement existing insurance solutions. In particular, hedging longevity risk with index-based longevity hedges can have several advantages. Nevertheless, the difference between the insurer's mortality experience based on annuitant mortality and the hedged standardized index based on reference population mortality give rise to the so-called basis risk. The presence of basis risk means that hedge effectiveness will not be perfect and that, post implementation, the hedged position will still have some residual risk. The present paper seeks to contribute to that literature by setting out a framework for quantifying the basis risk. In particular we propose a model that measure the population basis risk involved in a longevity hedge, in the functional demographic model setting. Moreover, while most existing models are designed for a

single population the research objective is to model mortality of two populations, in order to align with the hedging purpose. Finally, longevity hedging strategies are developed by involving mortality-linked securities.

## 1   Introduction

During the past century in many developed countries remarkable improvements in human life expectancy have been observed, although future demographic patterns are uncertain and difficult to be predicted accurately. The uncertainty affecting such trends is referred to as longevity risk, i.e. the risk to which a pension fund or life insurance company could be exposed as a result of higher-than-expected payout ratios. Longevity risk is due to the increasing life expectancy trends among policyholders and pensioners, and it can result in payout levels that are higher than those a company or fund originally accounts for.

The Global Financial Stability Report of the International Monetary Fund [12] highlights as typical assumption for pension liability valuations, based on updated data, are unable to account for future developments in longevity. In fact in some countries these valuations consider some future increases exceeding current life expectancy table but these increases are still much smaller than those occurred in the past.

This means that pensions need to be paid much longer than expected, raising the value of plan sponsors' obligation to their members. Significant underestimations of past longevity improvements have made plan sponsors more aware of the threat of longevity risk (see [14]). For these reasons the longevity phenomenon has to be taken into account from insurance companies which are obliged to quantify the impact of uncertainty of mortality trend on issued products, in order to manage the risk derived from it.

Longevity risk cannot be managed by pension funds and insurance companies through diversification as it moves in the same direction for all the policyholders. On the other hand, traditional reinsurance is not a viable solution due to the excessive costs and the limited capacity of the reinsurance market. As a consequence, transferring longevity risk to capital markets through securitisation appears as the more appealing alternative: some plans began to transfer their longevity risk exposures to the capital market. In this way the longevity has become a new asset class for different stakeholders like Annuity Providers, Life Insurance Companies, Pension Buyout Funds, Insurance Linked Securities investor, and others. The development of this market is still at an early stage but this kind of investors are very interested in it since they consider longevity as an attractive investment opportunity primarily because not correlated to non-life, credit and market risks.

Recently, significant tools for transferring longevity risk to the capital markets have been developed, bringing additional capacity, flexibility and transparency to complement existing insurance solutions (see [7]).

The market today presents two main types of mortality-linked contracts: the first is represented by contracts whose characteristic is to be linked to the actual mortality experience of the pension plan's own in this way they allow the pension plan to create a perfect hedge of longevity risk. Despite this, these contracts present several disadvantages, first of all their poor liquidity and burden arising from their close link with the hedgers' own risk features.

The second type of contract is represented by standardized contracts based on the mortality experience of a certain national population. In these contracts the reference mortality rate is determined by a LifeMetrics Index.

As widely shown in literature (for instance in [2], [6] and [16]), hedging longevity risk with index-based longevity hedges can have several advantages, but the risk is that of being unable to obtain a perfect hedge of longevity risk due to the differences in mortality experiences between the *exposed population* (e.g., the population of members of a pension plan or the beneficiaries of an annuity portfolio) and the *hedging population* associated with the hedging instrument (i.e., the population that determines the payoff on the hedge).

The residual risk is called population basis risk. Such risk exists due to, for example, differing profiles of socioeconomic group, lifestyle and geography (see [14]). The presence of basis risk does not mean that the hedge is invalidate but only not perfect, therefore, post implementation, the hedged position will still have some residual risk. The hedging level obtained has to be evaluated in comparison with the risk deriving from the initial unhedged position. To this aim it is fundamental to arrange a model allowing to quantify the basis risk in order to minimize it through a correct calibration of the hedging instrument (see [7]). As explained in [14]: ≪*having a sound method for measuring basis risk will encourage pension plans to use standardized longevity securities for hedging purposes. This will, in turn, stimulate a greater demand, facilitating the development of the market for standardized longevity securities*≫.

Several authors have explored the basis risk between populations associated with annuity portfolios and life insurance portfolios. [8] found empirical evidence of a (partial) natural hedge operating between such portfolios, implying that the basis risk between them is relatively small. [5] provided a calculation of the risk reduction between hypothetical annuity and life insurance portfolios using historical mortality experience data: the results suggest significant benefits in terms of reduction in risk and economic capital. [18] explored the basis risk associated with longevity swaps in a more qualitative fashion but draws similar conclusions. [14] developed a stochastic model for measuring population basis risk. In particular, they consider several variants of the Lee-Carter model [13], which has been widely used in actuarial science and other areas. They illustrate the proposed basis risk model with a hedge, formed by JPMorgan's q-forward contracts, for a hypothetical pension plan in Canada, for which a Life Metrics Index is not available.

The present paper seeks to contribute to that literature by setting out a framework for quantifying the basis risk. In particular, we propose a model measuring the population basis risk involved in a longevity hedge, in the Functional Demographic Model (FDM) setting. [11] suggests that the FDM forecast accuracy is arguably connected to the model structure, combining functional data analysis, nonparametric smoothing and robust statistics. The decomposition of the fitted curve via basis functions represents the advantage, since they capture the variability of the mortality trend, by separating out the effects of several orthogonal components. More in detail, while most existing models are designed for a single population, aim of this contribution is to model mortality of two populations in order to align with the hedging purpose.

This paper is organised as follows: Section 2 introduces the Functional Demographic Models and its variant for the basis risk. In Section 3, longevity hedging strategies involving mortality-linked securities are presented, while Section 4 introduces the key-q-duration and its role in defining the hedging portfolio. The longevity hedging strategies effectiveness is estimated with regard to pension annuities by graphical and numerical analyses in Section 5. Finally, concluding remarks are provided in Section 6.

## 2 Functional Demographic Models for the basis risk

The Functional Demographic Model (from herein FDM) has been introduced by [11] for forecasting functional time series. In particular, it is applicable to mortality and fertility data. The approach is based on the properties of functional data paradigm (see [17]) and nonparametric smoothing to decrease the randomness in the observed data.

The Functional Demographic Model is a generalization of the Lee Carter model. Nevertheless, the Lee Carter does not assume smoothness. In the FDM approach the log death rates are modeled by an underlying smooth function of the age $x$ and the amount of the noise which varies with $x$. It allows for capturing the increasing variance for higher ages generally observed into the data. Instead the Lee Carter classical version presents a homoskedastic Gaussian error structure which is denied by the general drop in mortality over time. Likewise the approach is robust for outlying years due to the wars and epidemics, as for instance 1918 when the World War I and the Spanish flu pandemic caused an unusual number of deaths between 15-50 years old. Indeed the procedure throughout the authors derive the set of the basis functions which explain the dynamics of the smooth function is a combination of the weighted Principal Component Analysis and RAPCA algorithm of Hubert [9]. It allows for obtaining robust principal components.

In the context of mortality, the authors define the functional time series $\{x_\iota, y_t(x_\iota)\}$, $t = 1, ..., n$, $\iota = 1, ..., p$, where $y_t(x)$ describes the logarithm of the observed mortality rate for age $x$ and time $t$ which is expressed as follows:

$$y_t(x_\iota) = f_t(x_\iota) + \sigma_t(x_\iota)\epsilon_{t,\iota}. \tag{1}$$

In the formula (1), $f_t(x)$ denotes a smooth function, $\sigma_t(x_\iota)$ the noise which varies with $x$, with $\epsilon_{t,\iota}$ an iid standard normal random variable. A nonparametric smoothing is then implemented for estimating $f_t(x)$ for $x \in [x_1, x_p]$ from $\{x_\iota,$ $y_t(x_\iota)\}$ for $\iota = 1, ..., p$, by using the weighted penalized regression splines as in [10]. A basis function expansion is used for decomposing the fitted curves:

$$f_t(x) = \mu(x) + \sum_{k=1}^{K} \beta_{t,k}\phi_k(x) + \epsilon_t(x), \tag{2}$$

where $\mu(x)$ is a measure of location of $f_t(x)$, $\phi_k(x)$ is a set of orthonormal basis functions and $\epsilon_t(x) \approx N(0, \nu(x))$.

For the estimation of the formula (2), the orthonormal basis set is obtained via principal components. For a given K the basis functions we take into account are those minimize the Mean Integrated Squared Error. The best fit to the estimated curves is obtained by the basis set.

In particular, the parameters $\mu(x, 1)$, $\mu(x, 2)$, $\phi_k(x, 1)$, $\phi_k(x, 2)$ are estimated by performing the aforementioned procedure separately on each single populations; otherwise the $\beta_{t,k}$ are obtained considering the matrix that simultaneously contains the log death rates of both populations.

In order to obtain robust principal components we preserve the two-step algorithm introduced by Hyndman and Hullah [11] composed by the weighted principal component and the RAPCA algorithm (as shown in Hubert, et al. [9]). The central death rates is approximately binomially distributed, so that the variance of the its logarithm, the observational error variance, is computed via a Taylor approximation as shown in Hyndman et al. 2007. The coefficients $\beta_{t,k}$, for $k = 1, ..., K$, are the corresponding principal component scores. The coefficients $\beta_{t,k}$ are forecasted for $t = n + 1, ..., n + h$, by possibly non-stationary ARIMA models. At this stage, the formula (2) is used to project $f_t(x)$ for $t = n+1, ..., n+ h$. Once obtained $f_t(x)$ the $y_t(x)$ are derived from the expression (1). As widely described in Section 1, in longevity hedges the basis risk arises. We propose a stochastic framework for measuring basis risk, based on the aforementioned FDM. The insurance company has to manage the longevity risk related to the population of insured of the in-force portfolio having a specific future trend of longevity phenomenon, different from one of the population of individuals associated with the hedging instrument. The main alternatives (as in [14]) are the following:

 – to represent the mortality of two populations using two models not related to each other;
 – to use a stochastic model that projects mortality of two populations simultaneously.

In the first case the statistical dependence is ignored, contrary to the second one, where the dependence between the two populations is taken into account. Here we propose both of the alternatives, related to the FDM, originally designed to represent a single population only. In the independent modelling, we estimate

the smooth function $f_t(x)$, age $x$ in year $t$, for population $i$, $i = 1, 2$, $f_t(x, i)$, $i = 1, 2$, according to the formula below:

$$f_t(x, i) = \mu(x, i) + \sum_{k=1}^{K} \beta_{t,k}(i)\phi_k(x, i) + \epsilon_t(x, i), \tag{3}$$

where $\mu(x, i)$ is a measure of location of $f_t(x, i)$ for the i-th population, $\{\phi_k(x, i)\}$ is a set of orthonormal basis functions for the $i$-th population $\epsilon_t(x, i) \approx N(0, \nu(x, i))$ is the population-specific error term, and the coefficients $\beta_{t,k}(i)$ for $k = 1, ..., K$ represents the principal component scores for the $i$-th population. As documented in literature (see [14]) the resulting estimates diverge in the log run. To avoid the problem, a joint FDM model is introduced here. In particular, let us assume that both populations are jointly driven by a single index. In other words two driving forces are indeed the same, as follows:

$$f_t(x, 1) = \mu(x, 1) + \sum_{k=1}^{K} \beta_{t,k}\phi_k(x, 1) + \epsilon_t(x, 1), \tag{4}$$

$$f_t(x, 2) = \mu(x, 2) + \sum_{k=1}^{K} \beta_{t,k}\phi_k(x, 2) + \epsilon_t(x, 2). \tag{5}$$

Specifically $\mu(x, 1)$ and $\mu(x, 2)$ respectively represent an age specific parameter indicating the average mortality level at age $x$ of the first and the second population. Furthermore the principal component score $\beta_{t,k}$ is common to both populations, where $\phi_k(x, i), i = 1, 2$ is the specific set of basis functions for the $i$-th population, which represent an age specific parameter indicating the sensitivity of the log death rates to the changes of $\beta_{t,k}$, as well as the error term with $i$ specific for the $i$-th population. The main attractive features of the above representation is the parsimony for considering together two trajectories of the populations under consideration and the convergence of long-term forecast.

## 3  The hedging strategies

A life annuity provider has to pay the benefits to its annuitants on the basis of the realized mortality rates. So he is exposed to the risk that realized mortality rates are smaller than expected. This situation implies an extension of the average period of the annuity payment and an increase in actuarial liabilities. Therefore, the annuity provider interested to hedge longevity risk could build an hedging strategy based on a portfolio of q-forward contracts.

A q-forward is a zero-coupon swap that involves the exchange at the maturity date of a fixed (at time 0) amount, for a random amount that is proportional to a mortality index (LifeMetrics Index) for a fixed population (the reference population) in some future time (the reference year). The fixed payment is proportional to the forward mortality rate for the reference population and is set so that q-forward value is zero at inception (see [5] for further details on the

mechanism of q-forwards). Therefore, the annuity provider could play the role of floating payer (i.e. the q-forward seller) receiving a fixed payment from the instrument, while an investor could be a q-forward buyer.

An effective hedging strategy could consist in calibrating the portfolio so that it contains a suitable mixture of q-forwards linked to mortality rates of different ages and time horizons, in order to maximize the degree of longevity risk reduction. The annuity provider entering in a portfolio of q-forwards to hedge the longevity risk has to pay a risk premium to the investor that takes on the risk. Since investors (fixed-rate payers) require compensation to bear longevity risk, the forward mortality rate must be lower than the expected mortality rate. Therefore, the risk premium can be found as the difference between the expected mortality rate, $q^e$, and the forward mortality rates, $q^f$.

Different pricing models have been proposed in the literature. For example, the LifeMetrics technical document [5] refers to Sharpe ratio to price the q-forward. According to this approach the forward mortality rate is determined as follows:

$$q^f = (1 - SR \cdot T \cdot \sigma_q)q^e, \tag{6}$$

where $T$ is the time to maturity, $SR$ is the required annualized Sharpe ratio, and $\sigma_q$ is the standard deviation of changes in the mortality rate.

We are interested in hedging the longevity risk associated with a portfolio of life annuities consisting of a cohort of one annuitant paying a fixed amount at the end of each year, through a portfolio of q-forwards. The hedge effectiveness is analysed by considering the present value of unexpected cash flows of the insurance portfolio.

Let $q(x, t, i)$ be the probability that an annuitant from population $i$, survived to age $x$, dies between ages $x$ and $x + 1$ in the year $(t, t + 1)$. We define as $\boldsymbol{q_i}$ the stochastic vector of death probabilities for the cohort of interest belonging to population $i$ and $E(\boldsymbol{q_i})$ the best estimate of $\boldsymbol{q_i}$. Let $V$ be the present value (at time 0) of all cash flows payable to the annuitant until the maximum attainable age and $H$ be the present value of cash flows generated by the q-forwards in the hedging portfolio depending on the death probabilities $\boldsymbol{q_2}$. In the case of no hedge, the present value of unexpected cash flows is equal to:

$$X = V(\boldsymbol{q_1}) - V(E(\boldsymbol{q_1})). \tag{7}$$

While, in the case of longevity hedge for the annuity provider, the present value of the unexpected cash flows becomes:

$$X^* = V(\boldsymbol{q_1}) - V(E(\boldsymbol{q_1})) - H(\boldsymbol{q_2}). \tag{8}$$

Note that if the q-forwards reference population is the same of the insurance portfolio (i.e. $\boldsymbol{q_2} = \boldsymbol{q_1}$), basis risk is absent. The hedging strategy is considered effective if the random variable $X^*$ is significantly less variable than $X$. As in [14], we use two measures of longevity hedge effectiveness provided by the hedging strategy. The first one is evaluated by the difference between $VaR$ at a 95%

confidence level of the random variable $X$ and $X^*$, respectively. The second one is the longevity risk reduction index, $R$:

$$R = 1 - (\sigma^2(X^*))/(\sigma^2(X)), \tag{9}$$

where $\sigma^2(X)$ and $\sigma^2(X^*)$ are the variances of $X$ and $X^*$, respectively. The higher the value of $R$ the greater the hedge effectiveness. To obtain an optimal hedge we need a mixture of a number of q-forwards with different terms equal to the maximum number of years that the annuitants could be alive. However, as the market of longevity derivatives is far to be complete, trading is limited to a small number of q-forwards. This problem has been addressed in the papers [3], [6], [14] and [15]. While [3] propose a hedging strategy based on q-forwards under the assumption that death rates at different ages are independent of one another, [15] define a hedging strategy taking into account the property of age dependence. This property arises from the consideration that a shock to a mortality rate is often accompanied by shocks to the mortality rates at neighboring ages. Following [15] we aim to find a specified set of q-forwards to hedge the annuity provider liability. As these liabilities depend on the entire mortality curve, we explore the property of age dependence in our data, that is potentially important when only few q-forwards are used to replicate the annuity provider liability. Before exploring the age dependence through mortality reduction factors, we firstly determine the key mortality rates, i.e. key points on the mortality curve well representing the mortality age profile of the population.

## 4 Key-q duration

### 4.1 Determining the key mortality rates

Let $j$ be the number of key points and let $x_1, x_2, ..., x_j$ be the ages corresponding to the key rates on the mortality curve. We consider $j$ consecutive age groups $X_1, ..., X_j$, to find the key mortality rates for the reference population in the age range we are interested in. For each age group key rates are determined by considering the following model (the same model is used in [15]):

$$\ln(m_{x,t}) = \mu(x) + \sum_{h=1}^{j} \kappa_t^{(h)} I(x \in X_h) + e_{x,t}, \tag{10}$$

subject to the identifiability constraint $\sum_t \kappa_t^{(h)} = 0$ for all $h$. Where $\mu(x)$ is the mean of $\ln(m_{x,t})$ respect to time $t$, $\kappa_t^{(h)}$ is a time varying stochastic factor for the age group $h$, and $I$ is the indicator function. The error term is indicated by $e_{x,t}$.

Such a model is consistent with a FDM with only one coefficient $\beta_t$ assuming $\phi(x) = \phi^{(h)}$ for $x \in X_h$. Therefore, we can write $\kappa_t^{(h)} = \beta_t \phi^{(h)}$. We use the simplified representation given by equation (10) in order to determine a small number $j$ of age groups satisfying a fixed value of explanation ratio (ER). The

explanation ratio explains the proportion of variance in the mortality rates and it is defined as:

$$ER = 1 - \frac{\sum_{x,t}(e_{x,t})^2}{\sum_{x,t}[\ln(m_{x,t}) - \mu(x)]^2}. \tag{11}$$

Parameters of the model are fitted by the least squares method. For a fixed $j$, parameter $\kappa_t^{(h)}$ for the generic age group $h$ is obtained by:

$$\kappa_t^{(h)} = E[\ln(m_{x,t}) - \mu(x)] \quad \text{for } x \in X_h, \tag{12}$$

where $E[.]$ is the expectation operator with respect to $x$. This method to find the key age groups could be improved by optimizing the $ER$ value both under the group width and the starting age of each group.

## 4.2 Mortality reduction factors analysis and mortality key rate shifts

Once the key mortality rates are determined, we analyse the age dependence of mortality rates on the reference population in order to find a suitable function of mortality shift. To model mortality shift we perform an analysis of the historical mortality reduction factors in the age range and the calendar years we consider. Following the notation of [14] we define $R(x,t)$ as the reduction factors at age $x$ in the year $t$ as:

$$R(x,t) = 1 - \frac{q_{x,t+1}}{q_{x,t}}. \tag{13}$$

In order to analyse the property of age dependence in mortality, we study the trend of the correlation coefficient, $\rho(x,y)$, of the crude and smoothed reduction factors:

$$\rho(x,y) = \frac{\sum_t [R(x,t) - \overline{R}(x)][R(y,t) - \overline{R}(y)]}{\sqrt{\sum_t [R(x,t) - \overline{R}(x)]^2}\sqrt{\sum_t [R(y,t) - \overline{R}(y)]^2}}, \tag{14}$$

where $\overline{R}(x)$ is the average of $R(x,t)$ respect to time t. According to the analysis of the correlation between reduction factors calculated at different ages, we can model the mortality shift. In [15] results on sample correlation show that the dependence between two mortality rates lessen when the age width increases suggesting the following specification for the shift at age $x$ associated with a change in the $j$-th key mortality rate:

$$sh(x, j, \delta(j)) = \begin{cases} 0 & if \ x \leq x_{j-1} \\ \delta(j)\frac{x - x_j}{x_j - x_{j-1}} & if \ x_{j-1} < x \leq x_j \\ \delta(j)\frac{x_{j+1} - x}{x_{j+1} - x_j} & if \ x_j < x \leq x_{j+1} \\ 0 & if \ x \geq x_{j+1} \end{cases} \tag{15}$$

where the change in the $j$-th key mortality rate is constant and set to: $\delta(j) = 0.001$ (10 basis point). For $j = 1$ $sh(x, 1, \delta(1))$ is set to $\delta(1)$ for $x \leq x_j$, while for $j = n$ $sh(x, n, \delta(n))$ is equal to $\delta(n)$ for $x \geq x_n$.

### 4.3 Portfolio hedging

We assume that the q-forwards used to hedge the portfolio are based on the key mortality rates chosen as shown in Section 4.1. Other sets of key mortality rates can be defined. For instance, if a hedger decides to use two q-forwards only, then two key mortality rates could be specified. The choice of key mortality rates also depends on the availability of the associated q-forwards because the strategy requires q-forwards that are linked to the key mortality rates chosen. Let $q$ and $\tilde{q}$ be the original and the shifted mortality curves, respectively. Following [15] we define the key q-duration $(KQD)$ as the portfolio's price sensitivity to a shift in a key mortality rate. Then the $j$-th key q-duration of a security is given by:

$$KQD = (P(\boldsymbol{q}), j) = \frac{P(\tilde{\boldsymbol{q}}) - P(\boldsymbol{q})}{\delta(j)}, \tag{16}$$

where $P(\boldsymbol{q})$ is the price of the security on the basis of the mortality curve $\boldsymbol{q}$. The KQD for a q-forward with 100 monetary unit as notional, can be computed analytically:

$$KQD(F_j(\boldsymbol{q}), j) = -100(1 + r)^{T_j}, \tag{17}$$

where $r$ is the interest rate at which the cash flows are discounted and $F_j(\boldsymbol{q})$ is the present value of the payoff from the q-forward with a notional amount of \$1, for example if $i$ is the reference population of the q-forward, we can write:

$$F_j(\boldsymbol{q}) = 100(1 + r)^{T_j}(q^f(x_j, t_j, i) - q(x_j, t_j, i)), \tag{18}$$

where $q^f$ is the forward mortality rate and $T_j$ is the time to maturity of the q-forward. Therefore, net payoff to the hedger at maturity is 100 times the difference between the fixed (forward) and realized mortality rates. While the KQD for the pension liabilities are approximated via (16) taking $\boldsymbol{q} = E(\boldsymbol{q})$ and $\delta(j) = 10$ basis points.
To construct a good hedging strategy for the life annuities portfolio $V$ and $H$ should have a similar sensitivity to the mortality curve
$bmq$. In light of these considerations, the following notional amount of q-forward linked to the $j$-th key mortality rate is needed:

$$w(j) = \frac{KQD(V(\boldsymbol{q}), j)}{KQD(F_j(\boldsymbol{q}), j)}. \tag{19}$$

Note that $KQD(V(\boldsymbol{q}), j)$ and $KQD(F_j(\boldsymbol{q}), j)$ are the key-q durations for the portfolio liabilities and for the q-forward, respectively (for further details on the key-q durations calculation we remind to the paper [15]). If q-forwards linked to population (1) are unavailable, while q-forwards linked to population (2) are available, the key-q duration of the q-forwards can be calculated as:

$$KQD(F_j(\boldsymbol{q_1}), j) = -100(1 + r)^{T_j} \frac{\partial q(x_j, t_j, 2)}{\partial q(x_j, t_j, 1)}, \tag{20}$$

where $\frac{\partial q(x_j,t_j,2)}{\partial q(x_j,t_j,1)}$ is the adjustment factor depending on the specified mortality model. And the formula of notional amount $w(j)$ becomes:

$$w(j) = \frac{KQD(V(\boldsymbol{q_1}),j)}{KQD(F_j(\boldsymbol{q_1}),j)} = \frac{KQD(V(\boldsymbol{q_1}),j)}{KQD(F_j(\boldsymbol{q_2}),j)} \frac{\partial q(x_j,t_j,1)}{\partial q(x_j,t_j,2)}. \tag{21}$$

Therefore, to calculate the number of q-forwards to hedge the portfolio we need to find an expression of the adjustment factor $\frac{\partial q(x_j,t_j,2)}{\partial q(x_j,t_j,1)}$.

## 5   Numerical application

In this paper we consider a portfolio of life annuities consisting of a cohort of 1 annuitant aged $x=60$ at time 0 (set to beginning of year 2008) paying a 1\$ at the end of each year. We are interested in hedging the longevity risk associated with this portfolio for 40 years, subject to the same mortality as the female Australian population, through a portfolio of q-forwards. We fix 100 as the maximum age for the annuitant. The mortality data refer to the period 1958-2007 and to ages 60-100. Moreover, we suppose that q-forwards are available only for the Australian total population. We refer to the female and total Australian populations as populations (1) and (2), respectively.

   We report below in Fig. 1 and Fig. 2 the parameter estimation of the independent FDM modeling and in Fig. 3 the joint FDM coefficients. The choice of the FDM setting instead of the Lee Carter scheme used forhedging the basis risk by [14] in the augmented common factor model (ACF) is furthermore confirmed by better performances of the former model, as shown in Table 1. The models are compared in terms of error measures ME, MSE, MPE and MAPE.

**Table 1.** Error measures - Australian female and total population

| Females | ME | MSE | MPE | MAPE | Total | ME | MSE | MPE | MAPE |
|---------|-----|-----|-----|------|-------|-----|-----|-----|------|
| FDM | 0.00000 | 0.00106 | 0.00041 | 0.01153 | FDM | 0.00000 | 0.00067 | 0.00037 | 0.01021 |
| ACF | 0.00003 | 0.00001 | 0.01355 | 0.05524 | ACF | 0.00000 | 0.00005 | 0.01410 | 0.03454 |

Now, we determine the key mortality rates as explained in Section 4.1. The following four age groups provide an $ER=0.9659$ (an $ER=1$ indicates a perfect fit): $X_1 = [60 - 70); X_2 = [70 - 80); X_3 = [80 - 90); X_4 = [90 - 100]$. The corresponding key mortality rates are the central value of each age group, i.e. they are in the following ages: $x_1 = 65; x_2 = 75; x_3 = 85; x_4 = 95$.

   Once the key mortality rates are determined, we analyse the age dependence of mortality rates on the Australian female population in order to find a suitable function of mortality shift. We show in Fig. 4 the values of $R(x,t)$ calculated both on the crude and smoothed mortality rates, where the latter are obtained by applying 1-dimension P-splines functions for graduation.
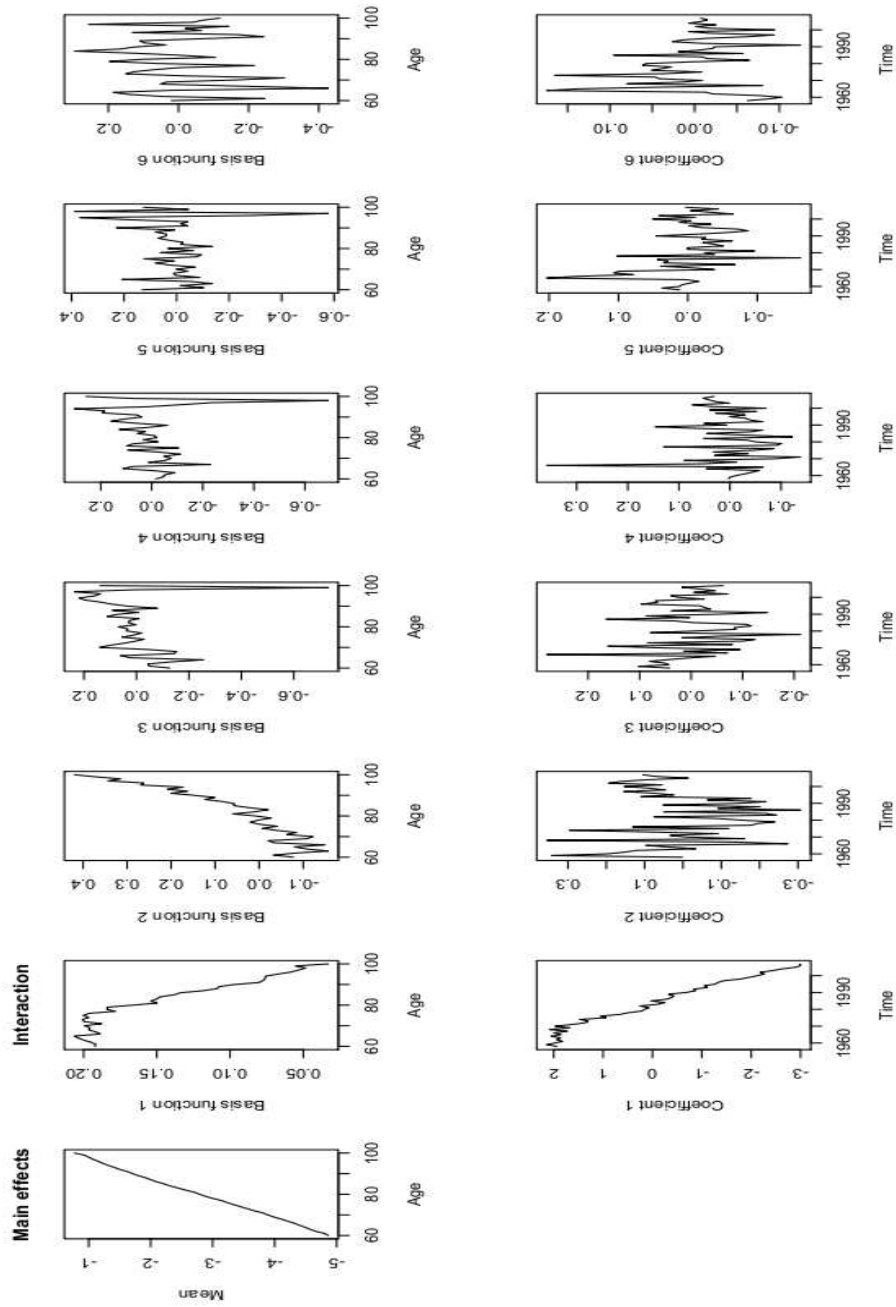
**Fig. 1.** Basis functions of FDM and associated coefficients - Australian female population
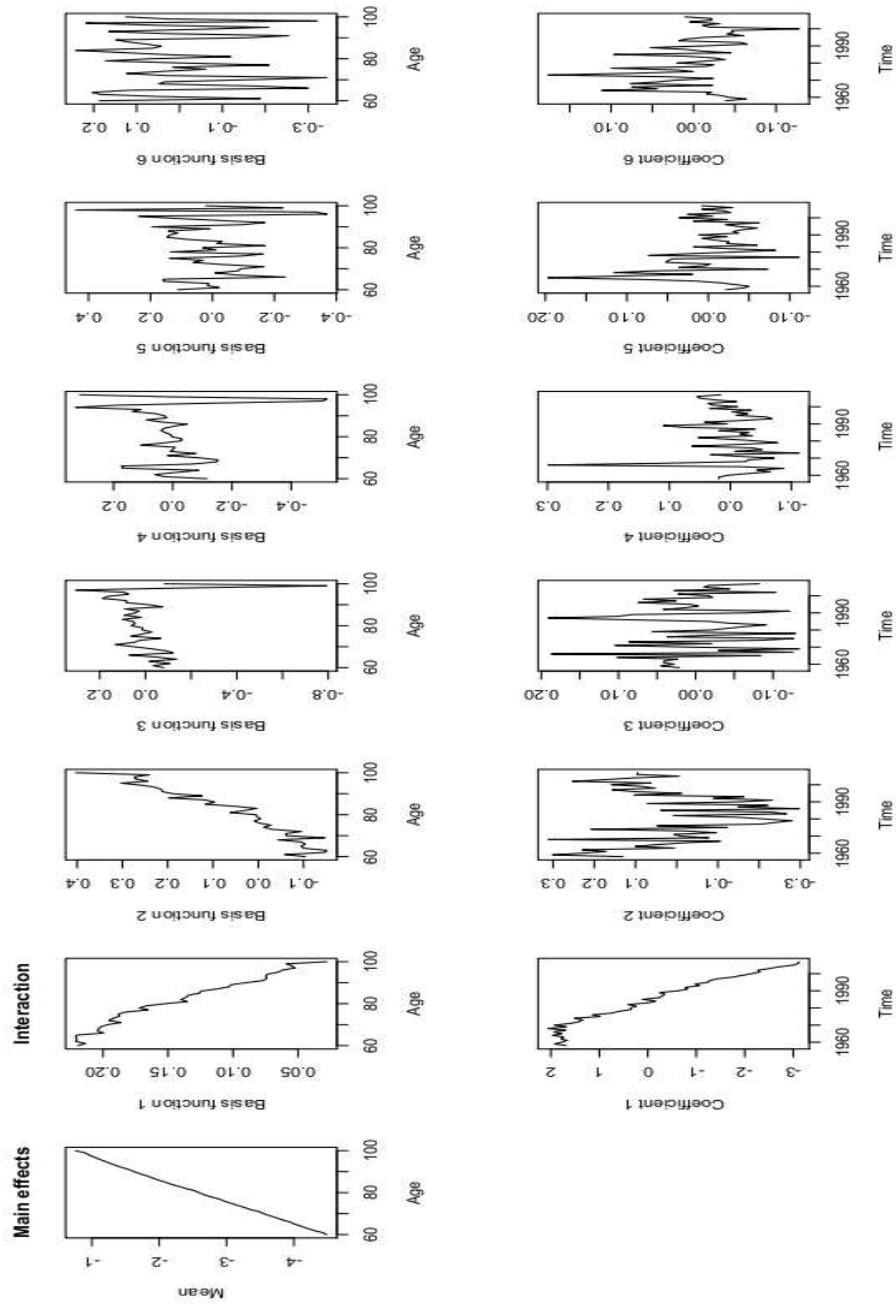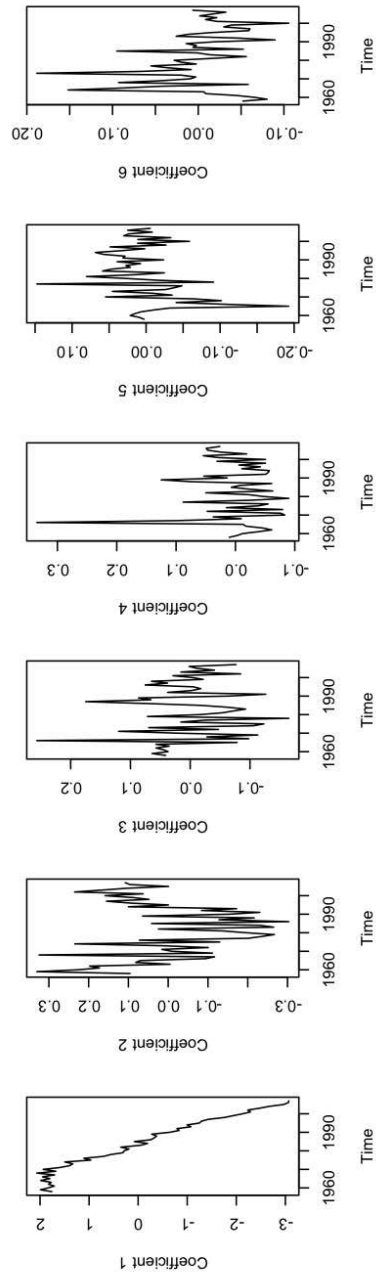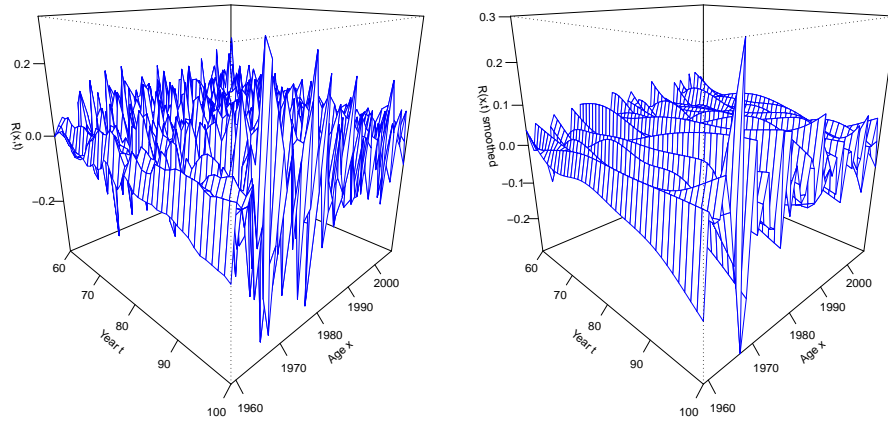
**Fig. 2.** Basis functions of FDM and associated coefficients - Australian total population

**Fig. 3.** Joint FDM coefficients - Australian population

**Fig. 4.** $R(x, t)$ - Crude and smoothed rates

Further, we calculate the mean respect to time $t$, $E_t[R(x, t)]$, and the volatility, $\sigma_x^2 = Var[R(x, t)]$, of both crude and smoothed reduction factors. Results are depicted in Fig. 5.

Then, we study the trend of the correlation coefficient, $\rho(x, y)$, of the crude and smoothed reduction factors. The values of $\rho(x, y)$ are shown in Fig. 6 and Fig. 7, the first reporting the correlation coefficient between the key age group $X_i$ and the key age $y = x - j$, the second between all the age range $x = (60, 100)$ and the key age $y = x_j$. The obtained outcomes confirm that the dependence between two mortality rates lessen when the age width increases. Therefore, we model the mortality shift as in [15] (see expression (15)). Now, we consider the four key mortality rates at ages $x_1 = 65, x_2 = 75, x_3 = 85$ and $x_4 = 95$ for the cohort of interest, born in 1948, i.e.: $q(65, 2013, 1), q(75, 2023, 1), q(85, 2033, 1)$, and $q(95, 2043, 1)$. Note that the q-forwards used to hedge the portfolio are based on these key rates. We set the discount rate to $r = 2\%$ and we simulate 10,000 mortality scenarios according to Monte Carlo technique. We suppose that the forward mortality rates are the same as the corresponding best estimate mortality rates. This assumption, which implies zero risk premium, would affect the cost but not the performance of the longevity hedge. We firstly calculate the key-q duration of both the annuities portfolio and the q-forwards modelling the mortality projection as in Section 3. The values of key-q durations, as well as the notional amount $w(j)$, are expressed in Table 2 in the case where the basis risk is absent.

If the basis risk is present, i.e. the q-forwards reference population is the total Australian population, the vector of notional amounts becomes: $w = (1.5088, 0.9993, 0.5658, 0.3569)$. We calculate the distributions of the present value of unexpected cash flows $X$ and $X^*$ when there is no basis risk and when basis risk exists (see Fig. 8). We observe from Fig. 8 that the longevity hedge significantly
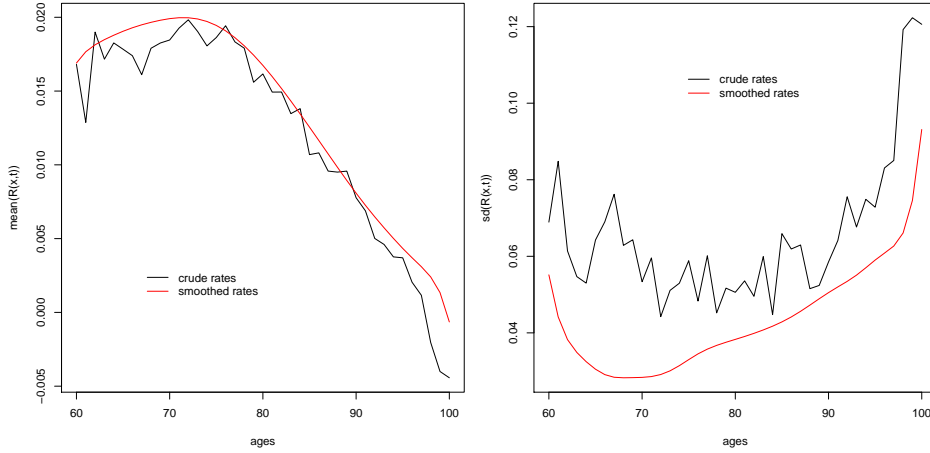
**Fig. 5.** Mean and standard deviation of $R(x,t)$ - Crude and smoothed rates

**Table 2.** Key-q durations of the Australian female population - No basis risk

|                      | j=1      | j=2      | j=3      | j=4      |
|----------------------|----------|----------|----------|----------|
| $KQD(V(\boldsymbol{q_1}),j)$ | -97.5566 | -59.4588 | -32.4796 | -18.3041 |
| $KQD(F_j(\boldsymbol{q_1}),j)$ | -90.5731 | -74.3015 | -60.9531 | -50.0028 |
| $w(j)$               | 1.0771   | 0.8002   | 0.5329   | 0.3661   |

reduces the dispersion of the unexpected cash flows, even if basis risk exists.

As previously mentioned, the hedge effectiveness provided by the hedging strategy is measured according to the amount of longevity risk reduction, $R$ and to the $VaR(X)$ and $VaR(X^*)$ at 95% confidence level. As shown in Table 3, without a longevity hedge the $VaR(X)$ is 0.70. When basis risk is absent, the longevity hedge can reduce the $VaR(X^*)$ to 0.27 with a value of longevity risk reduction $R = 86.70\%$, while when it is present the $VaR(X^*)$ is 0.36 and $R = 76.71\%$.
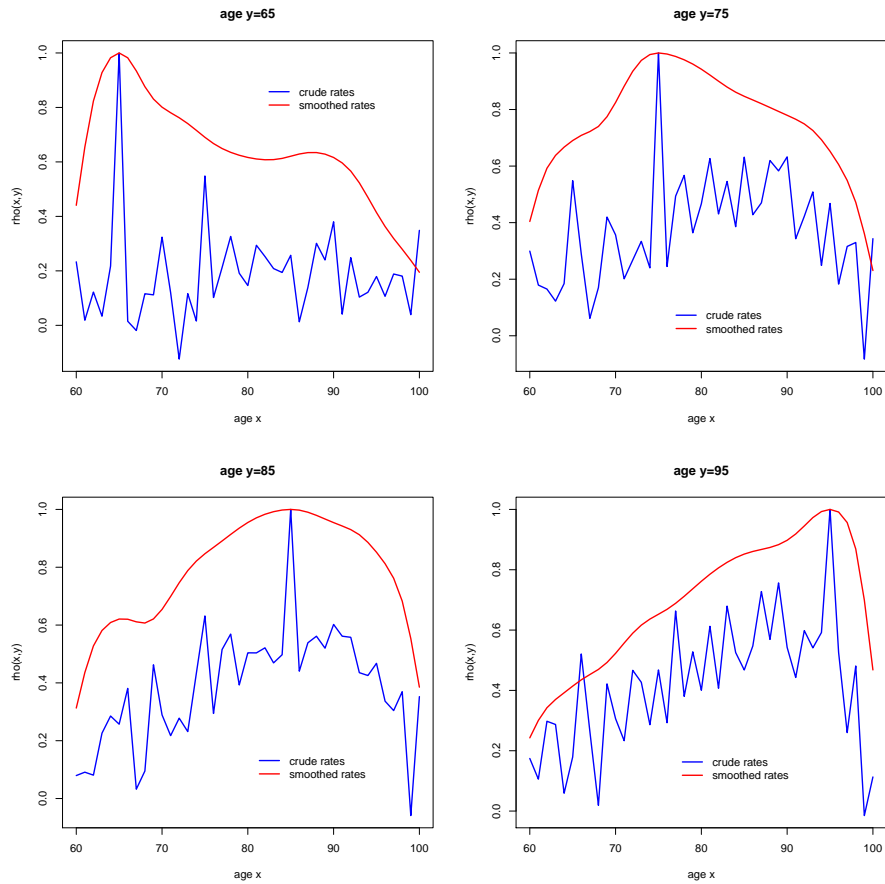
**Table 3.** Results of $VaR(X)$ and $VaR(X^*)$ at 95% confidence level and $R$

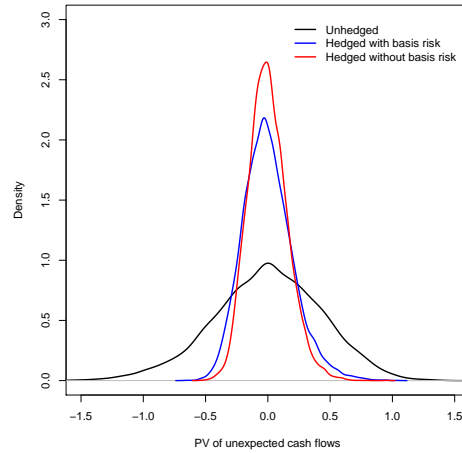|            | $\boldsymbol{q_2 = q_1}$ (No basis risk) | $\boldsymbol{q_2 \neq q_1}$ (Basis risk) |
|------------|------------------|--------------|
| $VaR(X)$   | 0.70             | 0.70         |
| $VaR(X^*)$ | 0.27             | 0.36         |
| $R$        | 86.70%           | 76.71%       |
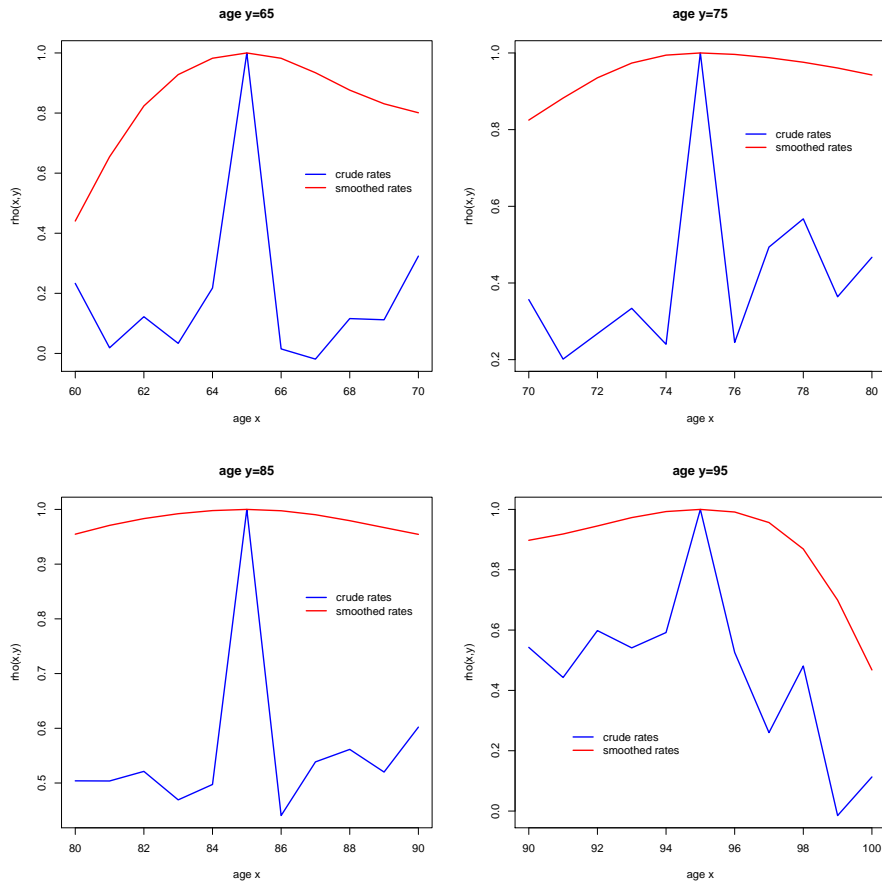
**Fig. 6.** $\rho(x,y)$ for $x = X_i$ and $y = x - j$. Crude and smoothed rates

**Fig. 7.** $\rho(x, y)$ for $x = (60, 100)$ and $y = x_j$. Crude and smoothed rates

**Fig. 8.** Simulated distribution of $X$ and $X^*$

## 6  Conclusions

The paper deals with the impact of basis risk in longevity hedging strategies. In order to measure basis risk we propose a mortality model for two populations in the framework of Functional Demographic Models, combining functional data analysis, nonparametric smoothing and robust statistics. In particular, the decomposition of the fitted mortality curve via basis functions represents the advantage, since they capture the variability of the mortality trend, by separating out the effects of several orthogonal components. Then hedging strategies based on a mixture of q-forward are presented together with measures suitable to evaluate hedging effectiveness. A perfect strategy require many q-forwards as the maximum numbers of years that the annuitants could be alive. As some of the required q-forwards may not be available, we evaluate as the property of age dependency could simplify the hedging problem. Actually we can obtain a good hedging with only few q-forwards written on specified mortality rates called key mortality rates. The strategies we consider are obtained constructing a portfolio of q-forward having a similar sensitivity to mortality curve than the annuity portfolio where the portfolio sensitivity to mortality curve is measured via the key-q duration. In order to measure the sensitivity to mortality curve the key q-duration for annuity portfolio and q-forward under the mortality model adopted is calculated.

The original nature of the proposed contribution relies in a flexible approach for quantifying the basis risk in longevity risk hedging. We introduce here a Joint Functional Demographic Model. In particular, we assume that the population of portfolio in-force as well as the reference population of the hedging instrument are jointly driven by a single index. The main attractive feature of the above representation is the parsimony for considering together two trajectories

of the populations under consideration and the convergence of long-term forecast. The framework is characterized by a strong versatility, being the Functional Demographic setting a generalization of the Lee-Carter model commonly used in mortality forecasting it allows to adapt to different demographic scenarios. Further studies will focus on the comparison between different schemes referable to the Functional Demographic setting.

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