Multiple yield-curve dynamics: a parsimonious approach*

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Abstract. Classical interest-rate models are formulated to satisfy by construction no-arbitrage relationships, which allow to hedge forward-rate agreements in terms of zero-coupon bonds. In practice, these no-arbitrage relationships might not hold, as market players realized after summer 2007 when the recent crisis began. In the literature proposals to accommodate these facts can be found. Yet, the calibration of such models would typically require market quotes for all yield curves. At present, this is not possible since most of the quotes are missing or extremely illiquid. Here, thanks to a suitable extension of the HJM framework, we propose a parsimonious model based on observed rates that deduces the dynamics of the money-market yield curves from a single family of Markov processes. Initial yield curves are recovered by means of a bootstrapping algorithm based on forward rate spreads. Then, we detail a stochastic-volatility specification of the model.

Keywords: Yield curve dynamics, multi-curve framework, Cheyette model, HJM framework, interest rate derivatives, basis swaps, counterparty risk, liquidity risk.

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1 Introduction

Classical interest-rate models were formulated to satisfy by construction no-arbitrage relationships, which allow to hedge forward-rate agreements in terms

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of zero-coupon bonds. As a direct consequence, these models predict that forward rates of different tenors are related to each other by strong constraints. In practice, these no-arbitrage relationships might not hold. An example is provided by basis-swap spread quotes, which are significantly non-zero, while they should be equal to zero if such constraints held.

This is what happened starting from summer 2007, with the raising of the credit crunch, where market quotes of forward rates and zero-coupon bonds began to violate the usual no-arbitrage relationships in a macroscopic way, under both the pressure of a liquidity crisis, which reduced the credit lines, and the possibility of a systemic break-down suggesting that counterparty risk could not be considered negligible any more. The resulting picture, as suggested by [18], describes a money market where each forward rate seems to act as a different underlying asset. On the other hand, there are empirical studies on the nature of Libor rates' dynamics which analyse both the effect of credit and liquidity risks.

In a European Central Bank working paper (see [12]), the authors compare the spread of the Euribor over the general collateral repo-rate to the spread of banking-sector credit default swaps of the same tenor during the crisis period. They found that there is evidence of a large, persistent and time varying component of the Euribor-Eurepo spread that cannot be explained by counterparty credit risk.

As an example we could compare (see [23]) the historical series of Euribor-Eurepo spread for a rate tenor of one year and of a synthetic index composed by senior one-year CDS spread of a basket of twelve European banks representative of the Libor panel. Surely the two series have some common qualitative characteristics. Yet, we find a sharp rise in the Euribor-Eurepo spread in September 2008. This peak is only found three-four months later in the CDS spread series, confirming that a liquidity crisis needs time to evolve as credit crisis. Hence, counterparty risk is only one of the Libor dynamics driving factors, as discussed in [17].

In the recent literature, some authors try to build fully consistent dynamical models for multiple yield curves. For instance, we can mention [14] and [21, 22], which adopt a market model view, and [10] which extend the HJM model by introducing different (correlated) HJM models, one for each yield curve. In general, these authors describe the multiple yield-curve dynamics in term of replicated single-curve models, leaving to the calibration procedure the problem to fix all the correlation parameters.

The hypothesis of introducing different underlying assets may lead to over-parametrization issues that affect the calibration procedure. Indeed, the presence of swap and basis-swap quotes on many different yield curves is not sufficient, as the market quotes swaption premia only on few yield curve. For instance, even if the Euro market quotes one-, three-, six- and twelve-month swap contracts, liquidly traded swaptions are only those indexed to the three-month (maturity one-year) and the six-month (maturities from two to thirty years) Euribor rates. Swaptions referring to other Euribor tenors or to Eonia are not actively quoted. A similar line of reasoning holds also for caps/floors and other interest-rate options.

A review of interest-rate derivatives regularly quoted by the Euro money market can be found in [26].

Thus, a different approach is followed in [23], where the authors construct a parsimonious extension of the HJM framework where a unique family of Markov processes is able to drive all yield curves' dynamics. In particular, an uncertain parameter model specification is presentd and calibrated to cap and floors to price multi-curve derivatives. Here, we consider a more ambitious model specification: a multi-curve stochastic-volatility parsimonious HJM model. Model calibration to the whole volatility cube of swaptions' quotes is discussed in [24].

The structure of the paper is the following: in Section 2 we review the fundamental money-market concepts that underlie the construction of a multi-curve framework; in Section 3 we describe the parsimonious HJM framework able to handle many yield curves; in Section 4, we detail the stochastic volatility version of the model; finally, Section 5 reviews our contributions and hints for further developments.

2 Discounting and forwarding with multiple yield curves

In order to motivate our modelling choices, it is useful to summarize the changes that occurred because of the credit crunch and the crucial issues a multi-curve framework should face. In this section we start by assuming the existence of a risk-neutral measure, and discussing why, for practical purposes, the risk-free yield curve is often approximated with the one derived from overnight rate indexed quotes, such as Overnight Indexed Swaps (OIS). Finally, we introduce Libor (risky) rates. Then, in the next Section, we continue by introducing the model dynamics, and we derive the relevant dynamical features of the model.

2.1 Risk-free rates and market rates

First of all we assume that the market is arbitrage free, hence postulating the existence of a risk-neutral measure. Under this measure every (risk-free) tradable asset instantaneously increases its value at the risk-free rate r_t. Furthermore, we introduce (risk-free) zero-coupon bond prices and instantaneous forward rates as

$$\begin{aligned} P_t(T) &:= E_t\left[-\int_t^T r_u du \right], \qquad (1) \\ f_t(T) &:= E_t^T[\, r_T \,], \end{aligned}$$

where the first expectation is taken under risk-neutral measure, and the last expectation is taken under a measure whose numeraire is $P_t(T)$ (hereafter simply T-forward measure).

As usual, we wish to link our risk-free rates to market quotes. In classical single-curve interest-rate models, zero-coupon bond prices observed at time $t = 0$ form a term structure

$$T \mapsto P_0(T),$$

which can be made consistent with a selection of quotes (deposits, futures and interest-rate swaps). However, since the beginning of the crisis, many of them have been carrying a relevant amount of credit and/or liquidity risk and cannot be considered as belonging to the risk-neutral economy. Thus, the subset of the instruments to bootstrap the risk-free term structure from has to be carefully chosen, see, for instance, [26] and references therein. A closer look at the Euro money market makes clear that quoted instruments are indexed on three reference indices[1]:

- Eonia is an effective rate calculated from the weighted average of all overnight unsecured lending transactions undertaken in the interbank market;
- Euribor(s) are offered rates at which Euro interbank term deposits of different maturities are traded by one prime bank to another one;
- Eurepo(s) are offered rates at which Euro interbank secured money market transactions are traded.

Eonia and Euribor rates are unsecured, so that they incorporate the default risk of the counterparty of the transaction, while Eurepo rates are secured and free of credit risk. Thus, Eurepo rates could seem the natural proxy for risk-free rates[2]. The main issue with Eurepo is that the longest quoted instrument has a maturity of one year. Longer maturities Euro money market deals are only indexed on Euribor and Eonia indices. In particular, we find Eonia swap contracts up to thirty years. Because of the plurality of available Eonia swaps and of the reduced credit/liquidity exposure on overnight deposits, to many extent Eonia based quotes are the best available proxy for Euro market risk-free rates. This point has been stressed by many authors, and we refer to [14] for more detailed arguments. Our modelling choices, however, will not be bound to a specific choice/identification for the risk-free curve.

We can now consider the Libor rates. It is a common habit to refer to unsecured deposit rates over the period $[S, T]$ as Libor rates $L(S, T)$. In this paper we follow this nomenclature and we reserve the term Euribor for the index used as reference rate for deposits in the Euro area. As usual we associate to Libor rates the corresponding forward rates $F_t(T, x)$ defined as

$$F_t(T, x) := E_t^T[\, L(T - x, T)\,] \; . \tag{2}$$

Forward rates $F_t(T, x)$ are by construction martingales under the T-forward measure and each of them represents the par rate seen at t for a swaplet accruing over $[T - x, T]$ and paying at T a fixed rate in exchange for $L(T - x, T)$. Libor rate $L(T - x, T)$ fixes, according to market conventions, with a settlement lag $\hat{\delta} > 0$, such that the corresponding forward undergoes the fixing condition $F_{T-x-\hat{\delta}}(T, x) \equiv L(T - x, T)$. In the remainder of the paper, for sake of simplicity,

[1] See Eonia, Euribor and Eurepo pages of European Banking Federation site at `http://www.euribor-ebf.eu`.

[2] See for instance [12] where the Euribor-Eurepo spread is used as an indicator of credit risk.

we will assume $\hat{\delta} = 0$, the extension to the case of non zero settlement lags being straightforward.

It is useful for later use to introduce also the risk-free linearly compounding forward rates

$$E_t(T,x) := \frac{1}{x}\left(\exp\left\{\int_{T-x}^{T} f_t(u)du\right\} - 1\right) . \tag{3}$$

Notice that we consider overnight deposits as being almost risk-free, while the longer the tenor, the greater will be the credit charge on unsecured deposit rates. By pushing this analogy further we can interpret Libor rates as microscopic rates at the same level of the short-rate, and model, in the HJM spirit, Libor forwards $F_t(T,x)$ and risk-free instantaneous forward rates $f_t(T)$.

Let us stress that even if we could formally define Libors for any possible value of x, in practice only a finite set of tenors are liquidly traded and used as underlying of derivative contracts. For instance, the Euro currency option market is mainly based on one-, three-, six-, and twelve-month Libors.

We conclude this Section by describing how to bootstrap the initial yield curves, which we embed within the dynamical framework proposed in the next Section.

2.2 Bootstrapping the initial yield curves

In our multiple curve framework we have one risk-free discounting curve and many forwarding curves, one for each quoted Libor rate tenor:

$$T \mapsto P_0(T) \,, \quad T \mapsto F_0(T,x) \,, \; x \in \{1\text{m}, 3\text{m}, 6\text{m}, 12\text{m}\}$$

The problem of bootstrapping different curves corresponding to rates of different tenors has been addressed for example in [1] where the authors solve the problem in terms of different yield-curves coherent with market quotes of basic derivatives. By following [26] we review here an independent approach to bootstrap the term structures, which focuses on the spread between forward rates of different tenor, see equation (2). We bootstrap the yield curves by interpolating on such spreads. In particular our aim is to produce smooth curves of forward rates and basis spreads.

As a practical case, let us focus on the Euro area market, describing the set of available instruments and sketching the calibration procedure.
Let us start by listing the market instruments we are about to use, which we group according to the tenor of the interbank rate to which they are indexed to:

- **Overnight Indexation:** Eonia fixing, OIS from one to thirty years;
- **1m Indexation:** Euribor one-month fixing, swaps from one to thirty years paying an annual fix rate in exchange for the Euribor 1m rate (some of these swaps may be substituted with one-vs-three-months basis-swaps);

- **3m Indexation:** Euribor three-months fixing, Short Futures, FRA rates up to one year, swaps from one to thirty years paying an annual fix rate in exchange for the Euribor 3m rate (some of these swaps may be substituted with at-the-money cap strikes or with three-vs-six-months basis-swaps);
- **6m Indexation:** Euribor six-months fixing, FRA rates up to one year and a half, swaps from one to thirty years paying an annual fix rate in exchange for the Euribor 6m rate (some of these swaps may be substituted with at-the-money cap strikes);
- **12m Indexation:** Euribor twelve-months fixing, FRA rates up to two years, six-vs-twelve-months basis-swaps from two to thirty years.

Notice that the payoffs of these instruments must be calculated without resorting to the usual non-arbitrage relationships. In particular for FRA, IRS and Basis Swaps we get relevant modifications, see also [21] and [26].

We continue by considering the risk-free discounting curve. Since the quotes for Eonia and OIS depend only on this curve, we can bootstrap it by means of the usual techniques, for instance we choose to employ the monotone cubic interpolation based on Hermite polynomials (see [15] and [1] for a review of bootstrapping techniques).

Once the discounting curve is known, we derive from it a curve of 1d forward rates obtained from equation (3). Then, starting from this curve, we obtain the forwarding curves corresponding to the different rate tenors.

We start from the six-months tenor, which corresponds in the Euro area to the family of most liquid instruments. We take the following steps:

1. we define the rate difference $y_{6\text{m}/1\text{d}}(t) := F_0(t, 6\text{m}) - E_0(t, 1\text{d})$;
2. we bootstrap the curve of the y's to match the six-months-tenor market quotes by using as interpolation scheme the monotone cubic interpolation based on Hermite polynomials;
3. we get the curve of the six-months F's by inverting the definition of the y's.

Notice that we choose to bootstrap the rate differences y, instead of directly acting on the rates F, so that the interpolation scheme can produce a smoother basis between the six-months and the 1d forward rates.

Once we know the six-months curve, we can proceed in a similar way (interpolation on rate differences) to obtain the curves corresponding to the other tenors. We define the spreads $y_{x/\bar{x}}(t) := F_0(t, x) - F_0(t, \bar{x})$, where $x, \bar{x} \in \{1\text{m}, 3\text{m}, 6\text{m}, 12\text{m}\}$, and we consider the liquidity of the underlying instruments to select which rate difference we want to bootstrap, and

4. we obtain the three-months curve using as starting point the six-months curve, since the market quotes the three-vs-six-months basis-swaps;
5. we obtain the one-month curve using as starting point the three-months curve, since the market quotes the one-vs-three-months basis-swaps;
6. we obtain the twelve-months curve using as reference the six-months curve, since the market quotes the six-vs-twelve-months basis-swaps.

The results of this procedure are shown in Fig. 1.

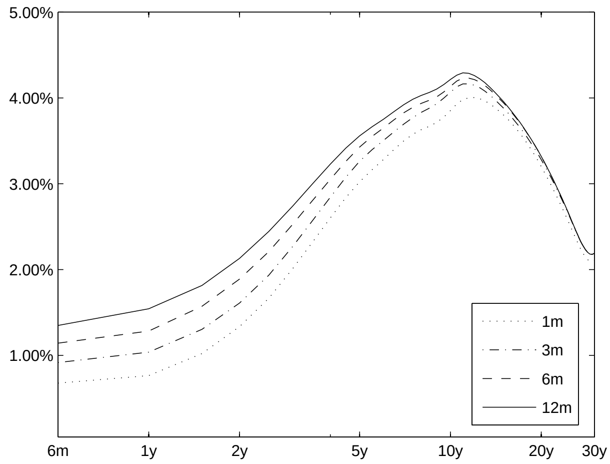

Fig. 1. Curve bootstrap in a multi-curve framework with monotone cubic interpolation based on Hermite polynomials. Market data observed on 14 June 2010

3 The parsimonious HJM framework

Our goal is to extend the classical framework by [16] to include curves associated to different tenors by modelling forward Libor rates by means of a common family of (Markov) processes. Here, we follows [23]. In the literature other authors proposed generalizations of the HJM framework, see for instance [8], [2], [5], [3], or [9]. In particular, recent papers [14], [10] extended the HJM framework to incorporate multiple yield-curves and to deal with foreign currencies.

Our approach differs from the previous ones mainly on two relevant points. First, we model only observed rates as in Libor market model approaches, avoiding the introduction of quantities such as "forecasting curve bonds" or "forecasting curve instantaneous rates". Second, we consider a common family of processes for all the yield curves of a given currency, so that we are able to build parsimonious yet flexible models.

3.1 HJM generalized dynamics

As a consequence of the discussion of previous sections, and in order to keep the model as simple as possible, let us summarize the basic requirements the model must fulfill:

i) existence of a risk free curve, with instantaneous forward rates $f_t(T)$;
ii) existence of Libor rates, typical underlying of traded derivatives, with associated forwards $F_t(T, x)$;

iii) no arbitrage dynamics of the $f_t(T)$ and the $F_t(T,x)$ (both being T-forward measure martingales);
iv) possibility of writing both the $f_t(T)$ and the $F_t(T,x)$ as function of a common family of Markov processes.

While the first two requisites are related to the set of financial quantities we are about to model, the last two are conditions we impose on their dynamics, and will be granted by a befitting choice of model volatilities.

According to requirements i) and ii) we model risk-free forward instantaneous rates $f_t(T)$ and (risky) forward Libor rates $F_t(T,x)$, for which we choose, under the $T-$forward measure, the following SDE[3].

$$
\begin{aligned}
df_t(T) &= \sigma_t^*(T) \cdot dW_t \\
\frac{dF_t(T,x)}{k(T,x)+F_t(T,x)} &= \Sigma_t^*(T,x) \cdot dW_t
\end{aligned}
\tag{4}
$$

with

$$
\sigma_t(T) := \sigma_t(T;T,0)\,, \quad \Sigma_t(T,x) := \int_{T-x}^{T} \sigma_t(u;T,x)du\,,
$$

where we introduced the family of volatility (row) vector stochastic processes $\sigma_t(u;T,x)$, the (row) vector of independent Brownian motions W_t, and the set of determinstic shifts $k(T,x)$, such that $k(T,x) \approx 1/x$ if $x \approx 0$.

Under this parametrization the risk free curve dynamics is the very same as in an ordinary HJM framework, where instantaneous forward rates have non-shifted evolutions, while linearly compounding forward rates $E_t(T,x)$, which can be identified with the forwards of single-period OIS rates, have a shifted dynamics, the shifts being equal to the inverse of the tenor

$$
\frac{dE_t(T,x)}{1/x+E_t(T,x)} = \left(\int_{T-x}^{T} \sigma_t^*(u)du \right) \cdot dW_t\,.
$$

This is of fundamental importance, as discussed in [22], to ensure that differentiating on both side the compounding relationship

$$
(1+xE_t(T,x))(1+x'E_t(T+x',x')) = (1+(x+x')E_t(T+x',x+x'))\,,
$$

the dynamics of all simple rates is of the same type for all $x, x', T > 0$.

Moreover, we would like to ensure both a formal and substantial analogy between the dynamics of the simple risk free forward rates $E_t(T,x)$ and of the simple risky forward rates $F_t(T,x)$, and we believe that, when small tenors are chosen (e.g. $x = 1$ day), the two rates should behave essentially the same way. This is the reason why we identified the volatility of risk free instantaneous forward rates with $\sigma_t(T;T,0)$, and required the shifts to satisfy the boundary condition for small tenors.

[3] See the appendix for vector and matrix notation.

In literature, direct modelling of shifted forward rates is also considered in [11] (see also references therein), and in [25].

By means of the change of numeraire technique we have that

$$dW_t^{(T)} = dW_t^{(rn)} - d\left\langle W^{(rn)}, \log P(T)\right\rangle_t = dW_t^{(rn)} + \left(\int_t^T \sigma_t(u;u,0)du\right)dt$$

where $W^{(T)}$ and $W^{(rn)}$ are standard Brownian motions under $T-$forward and risk-neutral measure, respectively. It is then straightforward to write the dynamics of forward Libor rates and instantaneous risk-free rates under the risk neutral measure as

$$\frac{dF_t(T,x)}{k(T,x)+F_t(T,x)} = \Sigma_t^*(T,x)\cdot\left[\left(\int_t^T \sigma_t(u;u,0)du\right)dt + dW_t\right] \tag{5}$$
$$df_t(T) = \sigma_t^*(T)\cdot\left[\left(\int_t^T \sigma_t(u;u,0)du\right)dt + dW_t\right],$$

W_t being a risk-neutral measure multidimensional standard Brownian motion.

3.2 Constraints on the volatility process

Let us analyse more in detail the dynamics of the shifted forward Libors under risk-neutral measure. By integrating the SDE over the time period $[0,t]$ we get

$$\ln\left(\frac{k(T,x)+F_t(T,x)}{k(T,x)+F_0(T,x)}\right) =$$
$$\int_0^t \Sigma_s^*(T,x)\cdot\left[dW_s - \frac{1}{2}\Sigma_s(T,x)ds + \left(\int_s^T \sigma_s^*(u;u,0)du\right)ds\right].$$

To ensure the tractability and a Markovian specification of the model, we extend the single-curve HJM approach of [7], [4], [6], and [27] by setting

$$\sigma_t(u;T,x) := h_t\cdot q(u;T,x)g(t,u)$$
$$g(t,u) := \exp\left\{-\int_t^u \lambda(y)dy\right\}$$
$$q(u;u,0) = 1,$$

where h is a matrix adapted process, q is a diagonal matrix deterministic function (i.e. $q^{ij} = q^i 1_{i=j}$) and λ is a deterministic array function. The condition on q when $T = u$ ensures that the standard HJM for risk free rates fulfills the usual Ritchen-Sankarasubramanian's separability condition. By plugging the expression for the volatility into equation (5), it is possible to work out the expression ending up with the representation

$$\ln\left(\frac{k(T,x)+F_t(T,x)}{k(T,x)+F_0(T,x)}\right) = \tag{6}$$
$$G^*(t,T-x,T;T,x)\cdot\left(X_t + Y_t\cdot\left(G_0(t,t,T) - \frac{1}{2}G(t,T-x,T;T,x)\right)\right),$$

where we have defined the stochastic process X_t

$$X_t^i := \sum_{k=1}^{N} \int_0^t g_i(s,t) \left(h^*_{ik,s} dW_{k,s} + (h^*_s h_s)_{ik} \left(\int_s^t g_k(s,y) dy \right) ds \right) , i = 1, \dots, N$$

and the auxiliary matrix process Y_t

$$Y_t^{ik} := \int_0^t g_i(s,t)(h^*_s h_s)_{ik} g_k(s,t) ds \quad i,k = 1, \dots, N$$

with $X_0^i = 0$ and $Y_0^{ik} = 0$, as well as the vectorial deterministic functions

$$G_0(t, T_0, T_1) := \int_{T_0}^{T_1} g(t,y) dy$$
$$G(t, T_0, T_1; T, x) := \int_{T_0}^{T_1} q(y; T, x) g(t,y) dy \, .$$

3.3 Dynamics of state variables

Equation (6) is the analogous of standard HJM reconstruction formula and is the main result of our paper. Let us notice that it returns a reconstruction formula for forward Libor rates, while standard HJM one is based on bonds. This important feature is consistent with the requirement of a model capable to directly describe market relevant quantities.

Thanks to our assumption we are fully able to describe instantaneous risk-free forward rates and forward Libor rates once we know the state variables $\{X_t, Y_t\}$, which satisfy, under the risk neutral measure, the following coupled (S)DE

$$dX_t^i = \sum_{k=1}^{N} \left(Y_t^{ik} - \lambda_i(t) X_t^i \right) dt + h_t^* \cdot dW_t$$
$$dY_t^{ik} = \left[(h_t^* h_t)_{ik} - (\lambda_i(t) + \lambda_k(t)) Y_t^{ik} \right] dt.$$

Let us notice that forward Libor diffusion pre-factors[4] $G(t, T-x, T; T, x)$ depend on the $q(u; T, x)$. This flexibility is a desirable feature, as it allows for a locally tuned dynamics for forward Libor rates, as we show in the next section.

4 A multi-curve stochastic-volatility parsimonious HJM model

In this section we describe a particular stochastic-volatility model, within the multiple-curve parsimonious HJM framework, and we calibrate it to the swaption

[4] Actually, starting from (6), and switching to the terminal Q^T measure, we have

$$dF_t(T,x) = [\kappa(T,x) + F_t(T,x)] G^*(t, T-x, T; T, x) \cdot h_t^* \cdot dW_t \, .$$

volatility cube. We consider a relatively simple specification to limit the number of parameters. In order to deal with swaption smile we can add a stochastic volatility process to our model by extending the filtration to include also the information generated by the volatility process. A popular choice for standard HJM framework is to model the matrix process h_t by means of a square-root process (see for instance [29] and reference therein).

4.1 Brownian motion settings

We consider N Markov driving factors (X^i) to model the yield curves, and we add one more factor to model the volatility process h_t. In particular, in the following numerical section we fix $N = 2$. The driving factors are specified in term of a vector of standard Brownian motions $\{W^0, W^1, \ldots, W^N\}$ under risk neutral measure. Such driving factors are correlated by means of a correlation matrix ρ, which can be decomposed in term of a (lower) triangular matrix R such that $\rho = RR^*$. We define a $(1 + N)$-dimensional correlated (risk neutral measure) Brownian motion $\{Z^0, Z^1, \ldots, Z^N\}$ by

$$Z^i = \sum_{j=1}^{N} R^{ij} W^j$$

such that

$$d\langle Z^i, Z^j \rangle_t = \rho^{ij} dt\,.$$

4.2 Dynamics of forward Libor rates

Under T-forward measure we have from equation (4)

$$dF_t(T,x) = (k(T,x) + F_t(T,x))\, \Sigma_t^*(T,x) \cdot dW_t\,, \tag{7}$$

where we now define the rate shift as

$$k(T,x) \doteq \frac{e^{-\gamma x}}{x}\,,$$

with γ a deterministic constant. Notice that for small tenors, namely for $x \approx 0$ with have $k(T,x) \approx 1/x$.

The forward-rate diffusion term Σ can be expanded if we choose a form for the volatility process. In particular, we consider a square-root stochastic process given by

$$h_t \doteq \sqrt{V_t}\, \hat{\sigma} R\,,$$

where $\hat{\sigma}$ is a deterministic constant diagonal matrix, and V_t satisfies under risk-neutral measure

$$dV_t = \kappa(\theta - V_t)dt + \varepsilon\sqrt{V_t}\, dZ_t^0\,, \quad V_0 = v\,, \tag{8}$$

where v, κ, θ, and ε are positive constants, and $Z_t^0 := (RW_t)^0$. Furthermore, we define the g and q as given by

$$g^i(t,T) \doteq e^{-\lambda^i(T-t)} \,, \quad q^{ij}(u;T,x) \doteq e^{\eta^i x} 1_{i=j} \,, \quad 1 \le i,j \le N \,,$$

where λ and η are deterministic constant vectors.

Hence, the forward-rate diffusion term can be written as

$$\Sigma_t^*(T,x) \cdot dW_t = \sqrt{V_t} \sum_{i=1}^{N} e^{\eta^i x} G_0^i(t, T-x, T) \hat{\sigma}^{ii} \, dZ_t^i \,,$$

where we explicitly write the vector components. Furthermore, we can explicitly calculate the G_0 term, and we get

$$G_0(t, T_0, T_1) = \frac{1}{\lambda} \left(g(t,T_0) - g(t,T_1) \right) \,.$$

4.3 Dynamics of swap rates

Our framework allows us to derive an (approximated) expression for swap rates dynamics based on freezing techniques

Let us consider a swap with a x-tenor floating leg and a $\bar{x}$-tenor fixed one paying at times $\{T_{a+1}, \dots, T_b\}$ and $\{T_{\bar{a}+1}, \dots, T_{\bar{b}}\}$, respectively. The swap par rate equating the two legs is

$$S_t^{ab}(x,\bar{x}) := \frac{\sum_{k=a+1}^{b} \tau_k P_t(T_k) F_t(T_k, x)}{\sum_{k=\bar{a}+1}^{\bar{b}} \bar{\tau}_k P_t(T_k)}$$

where the quantities with a bar refer to the fix leg. We introduce the weights w as

$$w_{k,t}^{ab}(x,\bar{x}) := \frac{\tau_k P_t(T_k)}{\sum_{k=\bar{a}+1}^{\bar{b}} \bar{\tau}_k P_t(T_k)} \,,$$

and perform the usual freezing technique $w_{k,t}^{ab}(x,\bar{x}) \approx w_{k,0}^{ab}(x,\bar{x})$ to obtain the (approximated) dynamics of swap rates in swap measure, as shown in [28] or in [13].

$$\frac{dS_t^{ab}(x,\bar{x})}{S_t^{ab}(x,\bar{x}) + \psi^{ab}(x,\bar{x})} \approx \sqrt{V(t)} \sum_{j=a+1}^{b} \sum_{i=1}^{N} e^{\eta^i x} \delta_{ab}^j(x) G_0^i(t, T_{j-1}, T_j) \hat{\sigma}^{ii} \, dZ_t^i \quad (9)$$

where we define

$$\psi^{ab}(x,\bar{x}) := \frac{\sum_{j=a+1}^{b} P_0(T_j) e^{-\gamma \tau_j}}{\sum_{l=\bar{a}+1}^{\bar{b}} \bar{\tau}_l P_0(T_l)}$$

and

$$\delta_{ab}^j(x) := \frac{P_0(T_j)(e^{-\gamma \tau_j} + \tau_j F_0(T_j, x))}{\sum_{l=a+1}^{b} P_0(T_l)(e^{-\gamma \tau_l} + \tau_l F_0(T_l, x))} \,.$$

The volatility process entering the above dynamics must be expressed under swap measure, and it is given by

$$dV_t = \kappa_t^{ab}(\theta_t^{ab} - V_t)\, dt + \varepsilon\sqrt{V_t}\, dZ_t^0 \,, \tag{10}$$

where we define

$$\kappa_t^{ab} := \kappa + \frac{\varepsilon}{A_t^{ab}} \sum_{l=\bar{a}+1}^{\bar{b}} \bar{\tau}_l P_t(T_l) \sum_{i=1}^{N} G_0^i(t,t,T_l)\hat{\sigma}^{ii}\rho_{i0}$$

and

$$\theta_t^{ab} := \kappa\theta/\kappa_t^{ab} \,,$$

with the swap annuity (used as numeraire under swap measure) given by

$$A_t^{ab} := \sum_{l=\bar{a}+1}^{\bar{b}} \bar{\tau}_l P_t(T_l) \,.$$

With this choice we get shifted Heston dynamics for market rates, so that we can calculate option pricing with usual Fourier transform techniques as presented in [20]. See also [2] for direct an application on interest-rate derivatives. Once we are able to compute option pricing, we can calibrate our model to market quotes as in [24]. We report in Fig. 2 an example of calibration on at-the-money swaptions' volatilities for various expiries and tenors.

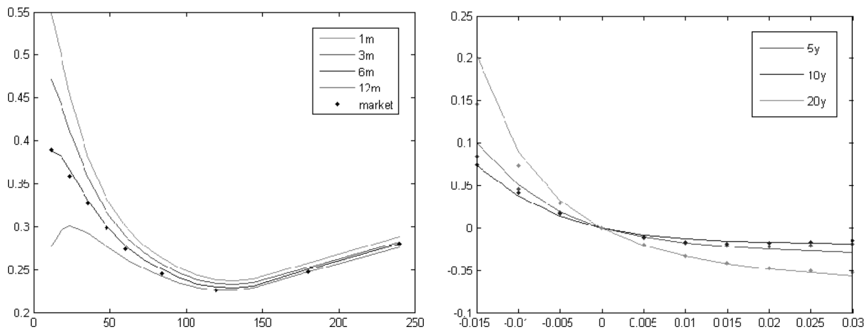

Fig. 2. Left panel: at-the-money swaption volatilities. On the horizontal axis swaptions' expiries, on the vertical axis swaptions' volatilities. The curves correspond to swaptions with tenor of 1y, 2y 5y and 10y. The dots are market quotes. Right panel: swaption volatility smile. On the horizontal axis differences between swaptions' strike rate and underlying forward swap-rate, on the vertical axis differences between swaptions' volatilities and corresponding swaptions' at-the-money volatilities. The curves correspond to swaptions with expiry of 5y, 10y and 20y and tenor of 5y. The dots are market quotes. Market data observed on 4 April 2012. See [24]

5 Conclusions and further developments

Interest-rate modelling requires a framework able to incorporate many initial yield curves, one for each Libor rate tenor plus one for discounting. Classical models may be extended in many ways, but, unfortunately, the market is too young to quote options on all tenors: only limited number of quotes are available and they are concentrated only in few tenors. Thus, a model, which allows a minimal extension of classical frameworks and, at the same time, allows for more complex dynamics when quotes will be available, is a relevant tool for both quants and practitioners.

In this paper we presented a methodology which allows to bootstrap market quotes of plain-vanilla interest rate instruments in order to obtain a set of initial forwarding term structures, one for each rate tenor, and we illustrated how this approach works in practice. Then, we introduced an extension of the HJM model which is able to describe the dynamics of the discounting yield curve and of market Libor rates of any tenor starting from a single family of Markov processes. Furthermore, we discussed a stochastic-volatility version of the model, and we derived the dynamics of market rates along with some calibration examples.

Our next step will be the study of more complex specifications of forward-rate volatilities to better incorporate basis dynamics, as soon as the market will start to quote swaption volatilities on more tenors.

Appendix: vector and matrix notation

When we consider a vector quantity v, we think it as a matrix with only one row, if a "column" vector is needed we use the transposition operator, namely v^*. Further, we introduce also the vector whose entries are all of ones and we name it 1.

Let us consider two matrix quantities a and b, whose elements are respectively a_{ij} and b_{ij} with $1 \leq i \leq n$ and $1 \leq j \leq m$. We define element-wise multiplication as the matrix ab with elements:

$$(ab)_{ij} := a_{ij} b_{ij}$$

and, in the same fashion, also multiplication by a vector v, whose elements are v_i with $1 \leq i \leq n$, or a scalar κ as

$$(va)_{ij} := v_i a_{ij} \ , \quad (\kappa a)_{ij} := \kappa a_{ij} \ ,$$

while index contraction as the matrix $a^* \cdot b$ with elements:

$$(a^* \cdot b)_{jk} := \sum_{i=1}^{n} a_{ij} b_{ik} \ .$$

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