Tackling equity release mortgage risks[⋆]

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Abstract. An equity release mortgage (ERM) is a loan available to elderly people, secured on the borrower's home. The debt matures interests until the borrower either dies or sells the house or goes into long-term care permanently. At maturity the debt is repaid by selling the property and the lenders may face a loss if the housing market has an insufficient performance. Despite the simplicity of the product payoff, many sources of uncertainty underlie an ERM contract: longevity risk, which affects the maturity date, the evolution of housing market and the dynamics of interest rates and inflation, which impact on the value of the property. As a consequence, an ERM is a non trivial hybrid derivative whose valuation is made more delicate by the lack of liquid markets for longevity and housing derivatives. We describe a simplified yet comprehensive framework to value such a contract.

Keywords. Longevity risk, real estate risk.

M.S.C. classification. 91G99.
J.E.L. classification. G13, G19.

1 Introduction

An equity release mortgage (ERM) is a loan available to elderly people, secured on the borrower's home. The loan may be issued in two forms: either as a lump sum at inception or as an annuity over a period of time specified in the contract. The choice between the two issuing options affects the payoff of the contract, but not the general modeling framework. Therefore, in this paper, in order to keep the notation as simple as possible, we choose to present the first case.

Key features of the contracts are the following:

[⋆] This paper expresses the views of its authors and does not represent the institutions where the authors are working or have worked in the past

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- at the *issue date* t_0, the borrower is advanced a lump sum of money:

  $$H_0 \cdot LTV \,,$$

  where $H_0 \equiv H(t_0)$ is the property value at issue and LTV represents the so called *loan-to-value* ratio, i.e. the percentage of the total appraised value of the house;
- interest on the initial sum is assumed to be compounded at a fixed rate K_H, i.e. at the generic time t, the debt is:

  $$H_0 \cdot LTV \cdot (1+K_H)^{t-t_0} \;; \tag{1}$$

- the contract maturity corresponds to the time τ when the borrower dies, or sells the house or goes into long-term care permanently. This time does not necessarily coincides with the repayment time, which may happen at a later time due to delays of various nature. We define the payment time in full generality as

  $$\tau_{\max} \equiv \tau + \Delta \,,$$

  where $\Delta \geq 0$ indicates the delay.
- The amount to be repaid corresponds to the minimum between the advanced sum and its accrued interest till maturity τ and the sale proceeds of the property. This quantity represents the payoff of the ERM contract and is indicated as:

  $$\Pi^{\mathrm{ERM}}(\tau) \equiv \min\left[H_0 \; LTV \; (1+K_H)^{\tau-t_0} \,,\, H(\tau)\right] \,. \tag{2}$$

The *fair value* of the contract is determined by equating the present value of payoff (2) to the lump sum advanced at inception:

$$H_0 \cdot LTV = \mathbf{PV}\left(\min\left[H_0 \; LTV \; (1+K_H)^{\tau-t_0} \,,\, H(\tau)\right]\right) \,. \tag{3}$$

Typically, the issuer has two options: either it fixes the rate K_H at a given level and solves (3) numerically in order to determine the loan-to-value which makes the contract *par*, or, vice-versa, it fixes the loan-to-value and searches for the fair rate K_H.

Despite the simplicity of the product payoff, many sources of uncertainty underlie an ERM contract: longevity risk, which affects the maturity date, the evolution of housing market and the dynamics of interest rates and inflation, which impact on the value of the property. As a consequence, an ERM is a non trivial hybrid derivative whose valuation is made more delicate by the lack of liquid markets for longevity and housing derivatives. International markets on equity release products (US, UK, Korea, Australia etc...) have been the subject of a growing interest starting from the early nineties. Existing literature includes among the others Szymanosky [1], Chinloy and Megbolugbe [2], Ma and Deng [3], Wang et al. [4], Chen et al. [5], Sherris and Sun [6] and Li et al. [7], Yang [8], [9] and Alai et al. [10].

In this paper, we present a simplified yet comprehensive framework to value such contract. In particular, longevity is valued as in the Lee Carter model [11] (as in many of the aforementioned references) corrected with a risk premium in the spirit of LLMA (Longevity Pricing Framework) [14]. Real estate risk is the risk referring to the housing market, which varies sensibly according to the geographical area and to the type of property (see the literature cited above). It is indirectly related to inflation risk and for many countries only partial data are available. We focus on the Italian market which is characterized by scarce data with strong seasonality features. In this case, a description of the market based on econometric approaches (which apply to mature markets such as the UK and US ones, see for example [7] or Australia [10]) does not appear to be satisfactory. On the other hand, the illiquid nature of the market poses problems to the straightforward application of models originally developed in different contexts: for example, modeling the house price process as a geometric Brownian motion under the risk neutral measure, as suggested in [4] and references therein, raises the problem of how to determine/estimate the parameters of the dynamics.

In light of this and given the existence in Italy of a more developed market of inflation products, we propose to link the house price process to the inflation one, by means of a regression of housing data on the values of a properly chosen inflation index (modeling uncertainties embedded in the proxy can be overcome by considering different scenarios of volatility). Once the coefficients of the regression have been estimated, the information embedded in the inflation market (through the prices of quoted inflation instruments) is sufficient to determine the dynamics of the inflation index, and, as a consequence, of the house price process. We bring on a numerical analysis focusing on the sensitivity of an ERM to longevity risk. In particular, we show how an increase in the longevity may differently affect its present value according to the age of the borrower.

We conclude by noticing that, in some cases, realistic contracts might present legal issues of various nature, which could impact the repayment time of the loan and/or the value of the property when sold. These aspects may be dealt with a scenario analysis, without impacting the general modeling framework at the heart of this work.

The paper is organized as follows: section 2 sets the basis for pricing an ERM contract on general grounds, leaving the issues of modeling longevity risk and the other sources of randomness open. Section 3 explores in detail longevity risk while section 4 deals with the modeling of the housing market. Section 5 concludes with a numerical analysis of the contract.

2 ERM pricing framework

Given the payoff (2) and the repayment time $\tau_{\max}$, the general pricing formula for the ERM contract, at a given trade date[1] $t \geq t_0$ is given by:

$$V^{\mathrm{ERM}} = \mathbb{E}\left[D(t, \tau_{\max})\, \Pi^{\mathrm{ERM}}(\tau)\right] , \tag{4}$$

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[1] The trade date may in principle differ from the issue date t_0.

where expectations are taken under the risk-neutral measure $\mathbb{P}$ and $D(t, \tau_{\max})$ represents the discount factor from time $\tau_{\max}$ to the trade date t.

In order to value (4), it is convenient to rewrite the payoff in terms of two contributions:

$$\Pi^{\mathrm{ERM}}(\tau) = X_H(\tau, LTV) - [X_H(\tau, LTV) - H(\tau)]^+ \ , \tag{5}$$

where the short-hand notation has been introduced:

$$\begin{aligned} \chi_H(\tau) &\equiv (1 + K_H)^{\tau - t_0} \\ X_H(\tau, LTV) &\equiv H_0 \cdot LTV \cdot \chi_H(\tau) \,. \end{aligned} \tag{6}$$

The first term in (5) represents the payoff associated to the loan face value (LFV) while the second one is the payoff of a European put option on the mortgaged property, with strike equal to $X_H(\tau, LTV)$. We refer to it as a no-negative-equity guarantee (NNEG) contract, with payoff:

$$\Pi_H^{\mathrm{NNEG}}(\tau) \equiv [X_H(\tau, LTV) - H(\tau)]^+ \ . \tag{7}$$

The price of the ERM contract at the trade date t, therefore, becomes:

$$V^{\mathrm{ERM}} = V^{\mathrm{LFV}} - V^{\mathrm{NNEG}} \,, \tag{8}$$

where

$$\begin{aligned} V^{\mathrm{LFV}} &\equiv \mathbb{E}\left[D(t, \tau_{\max}) \cdot X_H(\tau, LTV)\right] \\ V^{\mathrm{NNEG}} &\equiv \mathbb{E}\left[D(t, \tau_{\max}) \cdot \Pi_H^{\mathrm{NNEG}}(\tau)\right] . \end{aligned} \tag{9}$$

In order to evaluate (8), we assume that longevity risk is independent of the other sources of risk. The two terms at the RHS of eq. (8) can be cast in the general form:

$$\mathbb{E}[Y(\tau)] \,. \tag{10}$$

First we show how to calculate such general expectation and then we specialize the result to the LFV and NNEG contributions.

Consider a person who is aged x at the issue date t_0 and whose maximum attainable age is denoted by ω. In order to model the stochastic variable τ, we think of a pool of borrowers and assume that all deaths occurring between two consecutive years $t_0 + k$ and $t_0 + k + 1$ are accounted for as if they had happened at time $\tau = t_0 + k + 1$. Under these assumptions, the expectation (10) assumes the form:

$$\mathbb{E}\left[Y(\tau)\right] = \sum_{k=0}^{\omega - x - 1} \mathbb{E}\left[Y(t_0 + k + 1)\right] \, {}_k p_x \, q_{x+k} \ , \tag{11}$$

where q_{x+k} is the probability that a borrower who is aged x at inception dies in the time interval $[t_0 + k, t_0 + k + 1)$, given that he/she has survived to age $x + k$ (conditional death probability) and ${}_k p_x$ is the probability that the same borrower survives to age $x + k$ years (survival probability). Such quantities are the building blocks of mortality/survival tables, whose construction will be dealt with in section 3.

The loan face value is characterized by:

$$Y(\tau) = D(t, \tau_{\max}) \cdot X_H(\tau, LTV) \, .$$

Therefore, according to eq. (11), its price is given by:

$$V^{\mathrm{LFV}} = H_0 \; LTV \sum_{k=0}^{\omega-x-1} P(t, \overline{k}_{\max}) \; \chi_H(\overline{k}) \; {}_k p_x \; q_{x+k} \; . \tag{12}$$

where:

$$\begin{aligned} \overline{k} &\equiv t_0 + k + 1 \\ \overline{k}_{\max} &\equiv t_0 + k + 1 + \Delta \end{aligned} \tag{13}$$

and $P(t, \overline{k}_{\max})$ is the price of a zero coupon bond traded at time t and maturing at time $\overline{k}_{\max}$. By means of an analogous analysis, the NNEG price is given by the sum of a succession of put options' prices, indexed to different maturities $\overline{k}$, each put being characterized by underlying $H(\overline{k})$, strike $X_H(\overline{k}, LTV)$ and payment time $\overline{k}_{\max}$ years. It follows that the NNEG contract price is given by:

$$V^{NNEG} = \sum_{k=0}^{\omega-x-1} {}_k p_x \; q_{x+k} \; \textsc{Put}\left(\overline{k}, \overline{k}_{\max}, \{H, r\}, X_H(\overline{k}, LTV)\right) \, , \tag{14}$$

where $\{H, r\}$ compactly denotes all the relevant information about housing and interest rate risks. In order to proceed to the evaluation of the embedded put option we need to model the housing and interest rate markets. This issue will be dealt with in detail in section 4.

3 Longevity risk

Longevity risk is the risk due to the unexpected increase of life expectancy. Key variables are:

- *base mortality rates:* they refer to the most recent set of period mortality rates for the population of lives which is available at issue time t_0. In the best case scenario, the most updated ones are taken at time t_0, but in general mortality rates could refer to an earlier time[2], which by convention we associate to a date $\bar{t} = 0$. Therefore, we denote these rates by $q_x(0)$;
- *projected mortality rates:* they refer to expected mortality rates taken at different times t in future. We denote them by $q_x(t)$;
- *mortality improvements:* they refer to relative changes in mortality rates with respect to consecutive years $\delta_x(t)$;

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[2] For instance, the Italian Institute for Statistics and Demographics – Istat – publishes the mortality rates with a 2-3 years delay.

– *risk premium λ:* it represents the cost that a hedge provider would charge to take on longevity risk from a party.

Projected mortality rates (so called Best Estimate mortality rates, $q_x^{BE}(t)$) are obtained starting from experience data following the work by Lee-Carter (see [11] and [13]). The model describes the log of a time series of age-specific death rates as the sum of two contributions: a component which does not depend on time but varies with age, and a term given by the product of two factors: a time-varying parameter taking into account the general level of mortality and an age-specific component that describes the sensitivity of mortality at each age to changes in the general level thereof. This model is fitted to historical data and the resulting estimate of the time-varying component is dealt with standard tools of time series analysis yielding forecasts of the general level of mortality. The actual age-specific rates are derived using the estimated age effects.

BE mortality tables, having been derived starting from historical data, represent quantities under the empirical measure. However, since we are interested in pricing the ERM contract, we need to express them under the risk neutral measure. If a liquid market dealing in longevity products (e.g. longevity bonds, swaps etc...) were available, we could extract the risk premium to be applied to the historical measure in order to get the risk neutral one. Unfortunately, such market is only in its infancy and no liquid quotes are available. Therefore, in order to estimate the risk premium λ we follow the line suggested by the Life and Longevity Market Association (LLMA) [14]. From BE mortality tables we define mortality improvements as the relative difference between the mortality rates associated to two consecutive years:

$$\delta_x(t) = 1 - \frac{q_x^{BE}(t)}{q_x^{BE}(t-1)}\,, \qquad t = 1, 2, \dots$$

A risk premium λ is added to mortality improvements, in order to get the risk-neutral mortality rates[3]:

$$q_x^{RN}(t) = q_x^{RN}(t-1) \cdot (1 - \delta_x(t) - \lambda)\,, \qquad t = 1, 2, \dots \tag{15}$$

Given this framework, survival probabilities can be expressed through the recursive formula:

$${}_tp_x = \prod_{k=1}^{t} (1 - q_{x+k-1}(k-1))\,, \qquad t = 1, 2, \dots \tag{16}$$

which holds both for BE and risk neutral survival probabilities.

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[3] We notice that in $t = 0$, risk-neutral, best estimate and base mortality rates coincide:

$$q_x^{RN}(0) = q_x^{BE}(0) = q_x(0)\,.$$

3.1 The role of gender in longevity risk

Historical data which allow to estimate and forecast mortality and survival tables are available for female (F) and male (M) populations only. However, ERM contracts are often subscribed by couples (C). Therefore, it is necessary to model their joint longevity features. The probability that a couple survives till time t is defined by:

$$\mathbb{P}(\tau^C \geq t) = \mathbb{P}(\tau^F \geq t \vee \tau^M \geq t)\,,$$

where τ^F, τ^M and τ^C represent the (random) mortality times for female, male and couple. The RHS can be expressed in terms of the probability that *both* members of the couple die before date t i.e.

$$\mathbb{P}(\tau^F \geq t \vee \tau^M \geq t) = 1 - \mathbb{P}(\tau^F < t \wedge \tau^M < t)\,.$$

Therefore, the probability of the couple surviving till time t is given by:

$$\mathbb{P}(\tau^C \geq t) = 1 - \mathbb{P}(\tau^M < t \,|\, \tau^F < t)\; \mathbb{P}(\tau^F < t)\,. \tag{17}$$

Eq. (17) encodes the information about the dependence structure between the members of the couple. For example, the event that the male component of the couple dies given that the female component has already died, $\{\tau^M < t \,|\, \tau^F < t\}$, may not be independent of the event related to the death of the female component, $\{\tau^F < t\}$. If it were, then $\mathbb{P}(\tau^M < t \,|\, \tau^F < t) = \mathbb{P}(\tau^M < t)$. We propose a simple (though approximated) way of taking into consideration possible effects of such dependencies by introducing a parameter[4] $\vartheta \sim 1$, which mimics an “efficient” dependence as follows:

$$\mathbb{P}(\tau^C > t) = 1 - \vartheta\, [1 - {}_tp_x^M]\; [1 - {}_tp_x^F]\,, \tag{18}$$

where, ${}_tp_x^M$ and ${}_tp_x^F$ are single members’ survival probabilities (male and female).

4 Real estate risk

Real estate risk is the risk referring to the housing market. Given the assumption of independence of longevity risk from real estate and interest rate risk, real estate risk appears in the NNEG formula given by eq. (14), only through the value of:

$$\textsc{Put}\left(\overline{k}, \overline{k}_{\max}, \{H, r\}, X_H(\overline{k}, LTV)\right)\,, \tag{19}$$

where we recall that $\overline{k}$ is the maturity of the put option under consideration, $\overline{k}_{\max}$ is the repayment time of the loan, $\{H, r\}$ encodes housing and interest rate risks and $X_H(\overline{k}, LTV)$ is the strike. Before delving into the technical details of the calculation of (19), we present a list of issues related to the modeling of real estate risk:

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[4] For example, $\vartheta = 1$ corresponds to independence, while empirically $\vartheta \gtrsim 1$ is usually needed.

1. it is connected to inflation risk, but inflation indices such as the HCPI[5] and the FOI[6] do not explicitly take it into consideration. However, these indices partially reflect, though in an indirect way, the behavior of the housing market, through e.g. costs of house maintenance, renovations etc...
2. there are no consensus models apt to describe the housing market (which also deeply depends on the geographical region). Possible approaches include:
   (a) the *econometric approach* (e.g. [7]): though applied to some mature markets (e.g. the UK and US markets) it presents serious drawbacks to its application in the case of an Italian ERM market:
      - the number of historical data are scarce (in the range of 20-30 observations);
      - historical data present strong seasonality (with cycles of about 7-8 year periods);
      - in order to price an ERM, the housing market should be evaluated under the risk-neutral measure and no hint on how to move from the historical to the risk-neutral measure is available;

   (b) *regression* of housing data on inflation (e.g. for the Italian market, the FOI index). It has the advantage that, once the regression coefficients are obtained, one can price directly under the risk-neutral measure. This idea, combined with different scenarios of volatility, allows to take into account modeling uncertainties which are embedded in the proxy;
3. legal issues: once the default event has occurred, the average time needed by the bank to sell the house is about 4-5 years.

Since we are interested in the Italian market, where no liquid housing indices are available, we opt for the *regression* approach (b), and we sketch the main steps in order to value (19). In other more mature markets (e.g. UK and US) direct methods based on housing indices quotes are preferred.

4.1 House modeling as a regression on inflation

Consider the payoff associated to the NNEG contract, as introduced in eq. (7), i.e.:

$$\Pi_H^{\mathrm{NNEG}}(\tau) = [X_H(\tau, LTV) - H(\tau)]^+ \ .$$

Given that the modeling framework for the inflation market has already been studied and developed, a solution to the problem of modeling the housing market could be to find a relation between house prices and inflation indices and to apply the pricing methodology set up for inflation. For instance, if a linear regression on FOI were used, this payoff could be reduced to a Zero Coupon (ZC) put option on an inflation index $I(t)$, with strike $X_{ZC}(\tau)$, i.e.:

$$\Pi^{\mathrm{ZCI}}(\tau) = [X_{ZC}(\tau) \ - \ I(\tau)\,]^+ \, . \tag{20}$$

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[5] Harmonized Consumer Price Index (Eurostat, European Commission).

[6] Indice dei prezzi al consumo per le Famiglie di Operai e Impiegati, i.e. Consumer Price Index for families of workers and clerks (Istat, Italian Institute of Statistics).

In order to investigate the relation between house prices and the FOI index, we consider historical data of residential property's prices, the sample consisting of 23 yearly observations (see Fig. 1), spanning the time interval 1988-2010 (data are normalized to 100 at the end of the observation period).

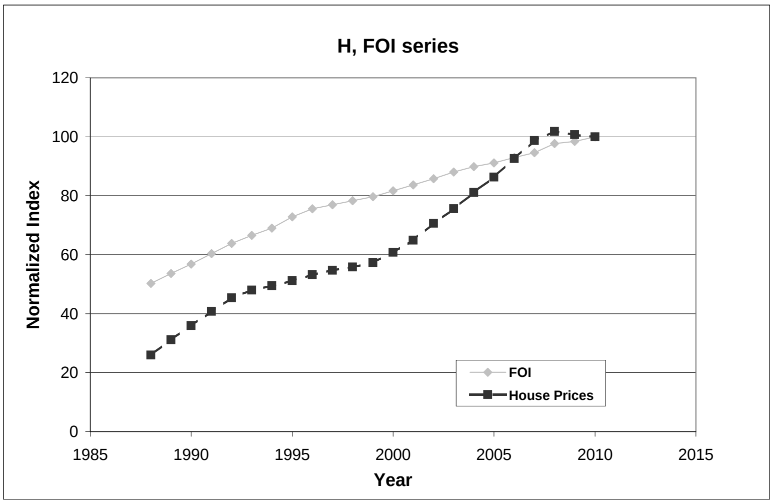

Fig. 1. Historical data of FOI index and house prices (nominal values) for the period 1988-2010.

House prices present a cyclical behavior with periods of about 7-8 years. Though inflation values also present a seasonality, the effect is not so pronounced and the period coincides with the solar year. The simplest idea would consist in representing the dependence of $H(t)$ on $I(t)$ by means of a linear regression i.e.:

$$H(t) = \alpha_1 \, I(t) + \alpha_2 \, . \tag{21}$$

The result of the regression is shown in Fig. 2.
The coefficients of the regression read explicitly:

$$\alpha_1 = 1.5318 \qquad\qquad \alpha_2 = -55.947 \, . \tag{22}$$

In this simplified framework, discrepancies due to the linear approximation could be taken into account by simulating different scenarios of volatility, which generate a "cone" around the regression line, able to capture extreme behaviors.

Alternatively, a periodic adjustment can be added to the linear behavior in the inflation index, such that eq. (21) can be recast as follows:

$$\begin{aligned} H(t) &= \alpha_1 \, I(t) + \alpha_2 - \alpha_3 \cdot \sin \left[\frac{\pi \, (t - \widehat{t})}{P} \right] \\ &= \alpha_1 \, I(t) + \alpha_2^{eff}(t) \end{aligned} \tag{23}$$

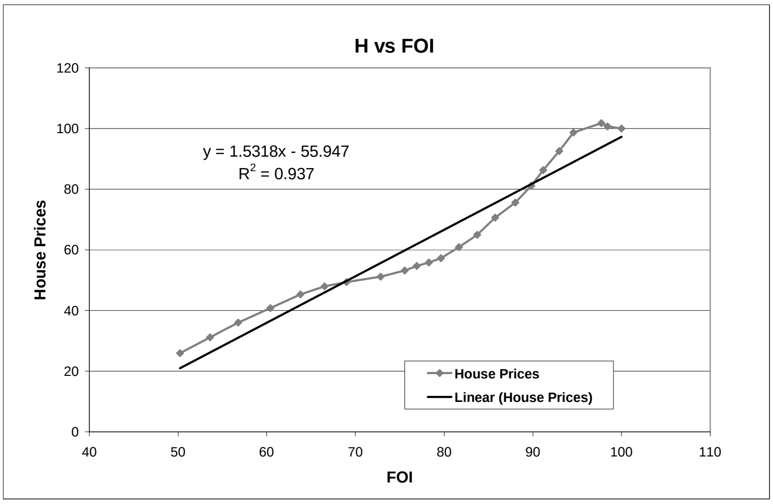

Fig. 2. Linear regression of house prices $H(t)$ on the FOI index series $I(t)$ for the period 1980-2010.

where

$$\alpha_2^{eff}(t) \equiv \alpha_2 - \alpha_3 \cdot \sin\left[\frac{\pi\,(t-\widehat{t})}{P}\right]$$

compactly indicates an efficient (time-dependent) parameter, taking into account the constant term and the oscillatory behavior, $\widehat{t} = 2010$ being the year of the last observation and P the oscillation period. By making explicit reference to the historical data, choosing $P = 7.5$ years (see Fig. 1), by means of a OLS regression the new coefficients assume the following values:

$$\alpha_1 = 1.561 \qquad \alpha_2 = -59.882 \qquad \alpha_3 = 7.761\,. \tag{24}$$

It is worth noticing that the new coefficients (24) are consistent with the linear approximation ones (22), i.e. the slope coefficient α_1 assumes almost the same value in both regressions. In other words, the functional form (23) describes oscillations around the linear proxy (21). A graphic representation of the proxy (23)-(24) is given in Fig. (3).

Given the regression (23) (which for $\alpha_3 = 0$ includes also the linear case), we obtain for the NNEG payoff an expression proportional to (20), where now the strike depends on the coefficients of the regression:

$$X_{ZC}(\tau) \equiv X^{\text{NNEG}}(\tau, LTV) = X_H(\tau, LTV) - \alpha_2^{eff}(\tau)\,. \tag{25}$$

Summarizing, the NNEG payoff to be priced is expressed in terms of inflation as follows:

$$\Pi_H^{\text{NNEG}} = \alpha_1\,\left[X^{\text{NNEG}}(\tau, LTV) - I(\tau)\right]^+ \equiv \alpha_1\,\cdot \Pi_I^{\text{NNEG}}\,. \tag{26}$$

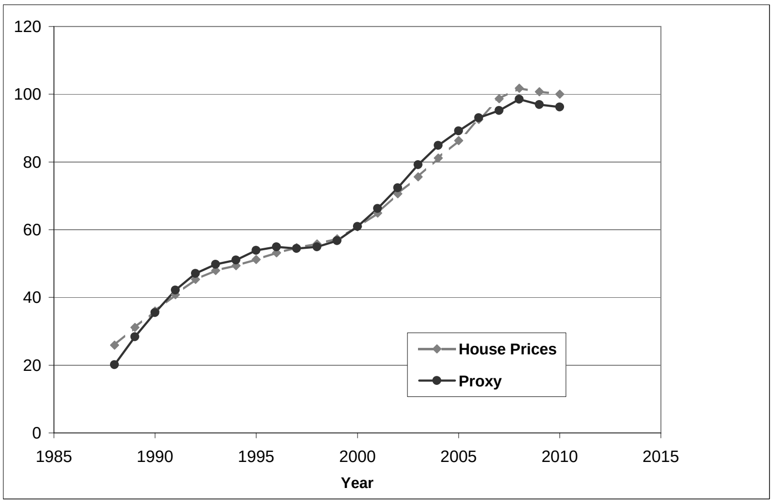

Fig. 3. Regression of house prices $H(t)$ on the FOI index series $I(t)$ for the period 1988-2010, according to eq.s (23)-(24).

Once the value of the strike for a given maturity τ is known, in order to price the embedded put option given in (19), the only thing which remains to be done is to model the inflation index $I(t)$.

4.2 Put price

The NNEG price becomes:

$$V^{\mathrm{NNEG}} = \mathbb{E}\left[D(t,\tau_{\max})\ \Pi_H^{\mathrm{NNEG}}\right] = \alpha_1 \cdot \mathbb{E}\left[D(t,\tau_{\max})\ \Pi_I^{\mathrm{NNEG}}\right] ,$$

where Π_I^{NNEG} represents the price of a put option on $I(t)$. The short rate process $r(t)$ and the inflation index $I(t)$ are assumed to follow, under the risk-neutral measure, respectively a one-factor Hull and White process and a log-normal dynamics:

$$\begin{aligned} dr_t &= a\ (\vartheta(t) - r_t) + \sigma\ dW_t \\ \frac{dI_t}{I_t} &= \mu(t)\ dt + \eta(t)\ dZ_t \end{aligned} \tag{27}$$

where

$$dW_t\, dZ_t = \rho(t)\, dt\, ,$$

and $\vartheta(t)$, $\mu(t)$, $\eta(t)$ and $\rho(t)$ are deterministic functions of time. Concerning the short rate process $r(t)$, $\vartheta(t)$ represents its long term average, a its speed of mean reversion and σ the corresponding volatility. The price of the put option in the NNEG contract is computed through a Black-Scholes formula.

5 Risk analysis

In this section we present the results obtained by applying the methodology illustrated so far. We assume a flat curve of interest rates (zero yield equal to 6.5%), fix the loan to value to $LTV = 25\%$ and, as far as longevity risk is concerned, consider the parameter $\lambda = 0\%$, if not otherwise stated. FOI data refers to February 2012.

5.1 Solving w.r.t the par fixed rate, K_H^{par}

First, we calculate the fixed rate which makes the contract par at inception K_H^{par} for populations of different age and gender and compare the results obtained assuming a linear regression of house prices on FOI index as given by eq. (21) and those obtained under the assumption of a cyclic regression (eq. (23)). Table 1 collects the numerical outcome and Fig. 4 displays a plot of the fixed rate K_H^{par} thus obtained *versus* age, for different genders.

Table 1. Fixed rate K_H^{par} obtained for populations of different age and gender, assuming a linear (eq. (21)) and a cyclic (eq. (23)) regression of house prices on inflation.

		Linear Regression			**Cyclic Regression**		
		Male	Female	Couple	Male	Female	Couple
Age	66	7.57%	7.75%	7.98%	7.44%	7.57%	7.72%
	68	7.45%	7.57%	7.69%	7.36%	7.44%	7.52%
	70	7.37%	7.44%	7.51%	7.30%	7.34%	7.38%
	72	7.32%	7.35%	7.38%	7.27%	7.28%	7.28%
	74	7.30%	7.30%	7.29%	7.26%	7.24%	7.22%
	76	7.30%	7.27%	7.24%	7.27%	7.23%	7.19%
	78	7.32%	7.27%	7.22%	7.31%	7.25%	7.19%
	80	7.37%	7.30%	7.22%	7.36%	7.29%	7.21%
	82	7.45%	7.36%	7.25%	7.45%	7.36%	7.25%
	84	7.56%	7.45%	7.31%	7.55%	7.45%	7.31%
	86	7.69%	7.57%	7.40%	7.69%	7.57%	7.40%

Fig. 4 highlights some important aspects:

- the par fixed rate is not a monotonic function of age and a *smile* behavior appears in both the linear and the cyclic regression case;
- crossings among curves associated to different genders occur, determining two separate regimes: one for people younger than the crossing age (corresponding approximately to 73-74 years) and one for people older than that.

5.2 Risk components

In order to understand the smile property, we try to isolate the effects of the different sources of risk involved. For simplicity, we consider a population of males ranging from 66 to 86 years.

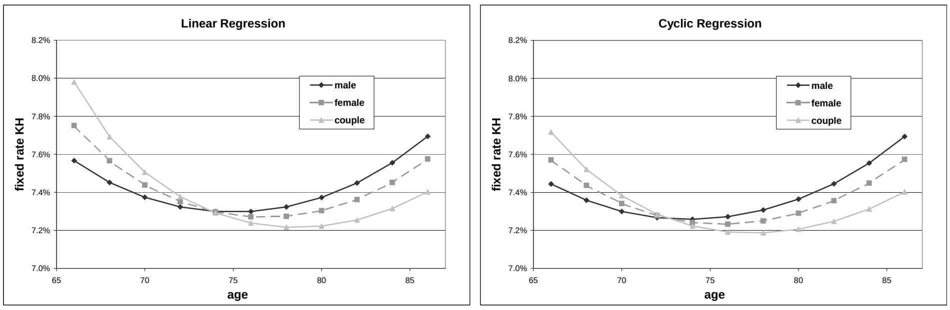

Fig. 4. Fixed rate K_H^{par} obtained for populations of different age and gender, assuming a linear (eq. (21)) and a cyclic (eq. (23)) regression of house prices on inflation.

Fig. 5 shows the par rate K_H^{par}, as a function of age, for contracts with different features:

- absence of the implicit put option (black line);
- volatility of the housing market close to zero, in order to study the effects of the *intrinsic value* of the put option, neglecting its *time value* (dark grey line);
- standard contract (light grey line).

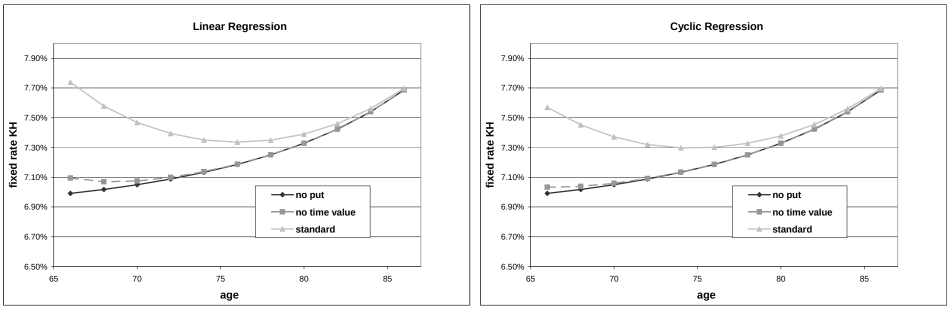

Fig. 5. Fixed rate K_H^{par} *vs* age, for a population of male individuals, assuming contracts with three different specifications: absence of the put component (black line), absence of *time value* (dark grey line) and standard contract (light grey line).

Both regression cases share similar features: for individuals aged above 74-75 years K_H^{par} tends to rise for all contract's specifications, converging eventually to a common value, for ages above 80. This suggests that as age increases, K_H^{par} becomes independent of the put value. Having assumed a flat curve for interest rates, such behavior is only ascribable to longevity risk. The housing market affects instead younger populations (till 75-80 years). The intrinsic value of the

Table 2. Fixed rate K_H^{par} *vs* age, for a population of male individuals, assuming contracts with three different specifications: absence of the put component, absence of *time value* and standard contract.

			Linear Regression		**Cyclic Regression**	
		No put	Standard	No time value	Standard	No time value
Age	66	6.99%	7.74%	7.09%	7.57%	7.03%
	68	7.02%	7.58%	7.07%	7.45%	7.04%
	70	7.05%	7.47%	7.08%	7.37%	7.06%
	72	7.09%	7.39%	7.10%	7.32%	7.09%
	74	7.13%	7.35%	7.14%	7.30%	7.13%
	76	7.19%	7.34%	7.19%	7.30%	7.19%
	78	7.25%	7.35%	7.25%	7.33%	7.25%
	80	7.33%	7.39%	7.33%	7.38%	7.33%
	82	7.42%	7.46%	7.42%	7.45%	7.42%
	84	7.54%	7.56%	7.54%	7.56%	7.54%
	86	7.69%	7.70%	7.69%	7.70%	7.69%

put option alone only creates a very mild smile effect (dark grey line), while the main contribution to the smile is given by the time value of the option (the gap between the light grey and the dark grey lines).

Summarizing, the behavior of K_H^{par} is predominantly affected by the housing market risk for ages below the crossing point and by longevity risk for ages above. In the following, we study in detail the roles of the housing market (through the value of the put option) and of longevity risk.

Role of the housing market The housing market affects the calculation of the par fixed rate (or the net present value) through the value of the implicit put option, namely the so called NNEG contribution of the ERM contract. Such value does not depend on the age of the population. A study of the *moneyness* properties of the put is showed in Fig. 6.
The evolution of the house price $H(t)$ is plotted for different values of the housing market volatility. Namely, taking as reference the market volatility σ^{mkt}, first row pictures have been obtained by using this value (i.e. $H(t)$ and $H_{\pm}(t) \equiv H(t) \pm \sigma^{mkt}$), second row pictures by using $0.5 \times \sigma^{mkt}$ (i.e. $H(t)$ and $H_{\pm}(t) \equiv H(t) + 0.5 \times \sigma^{mkt}$). On the same graphs, four different values of the fixed rate have been chosen, i.e. $K_H = 7\%, 8\%, 9\%, 10\%$. The first column refers to the linear regression case while the second to the cyclic one.

A comparison between the two different types of regression reveals that they have a quite similar behavior, the moneyness of the embedded put option increasing gradually 15 years from inception (in Fig. 6, approximately in year 2027), for a reasonable range of fixed rates K_H. This information must be combined with that embedded in the curves of the average matured loans (Fig. 7): in particular, loans subscribed by 85 years old people (and above) are much more likely to mature in the first 15 years while loans associated to younger people mature on average on a wider time span. It follows that ERM value for cohorts of 85 is almost insensitive to the embedded put option, which is OTM, while cohorts of younger people are affected to different extents by the moneyness of the put option. This is consistent with the general picture emerging from Fig. 5. As already mentioned, for people below 70-75 the time value of the option plays a major role, while the upward sloping curve in Fig. 5 for older people is only ascribable to longevity risk.

Role of longevity risk We conclude by studying the role of longevity risk on the value of the ERM contract. We consider three cohorts of male individuals aged 65, 75 and 85 and analyze the behavior of the net present value (NPV) of the contract as a function of the longevity risk premium λ, age and fixed rate. The results are collected in Table 3.

Some comments are in order:

- the NPV is an increasing function of the fixed rate K_H for every population and for every value of λ;
- for a given value of K_H and λ, the NPV is lower for 65 and 85 years old populations. In terms of the fixed rate K_H which makes the contract par, this translates into the smile effect observed in Fig. 4, i.e. the par rate must be larger for 65 and 85 years old populations;

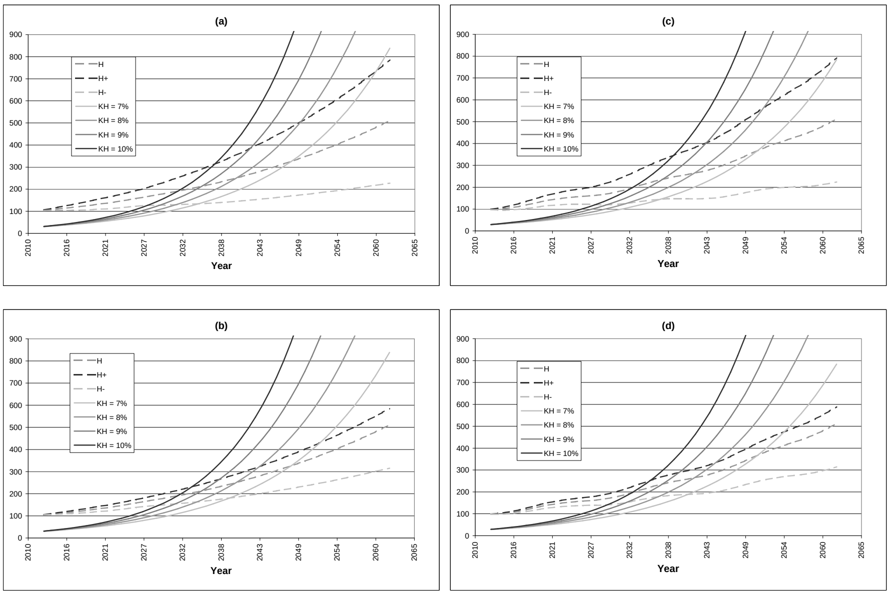

Fig. 6. Housing values *vs* debt (for different fixed rates K_H). (a) Linear regression, σ^{mkt}; (b) Linear regression, $0.5 \times \sigma^{mkt}$; (c) cyclic regression, σ^{mkt}; (d) cyclic regression, $0.5 \times \sigma^{mkt}$.

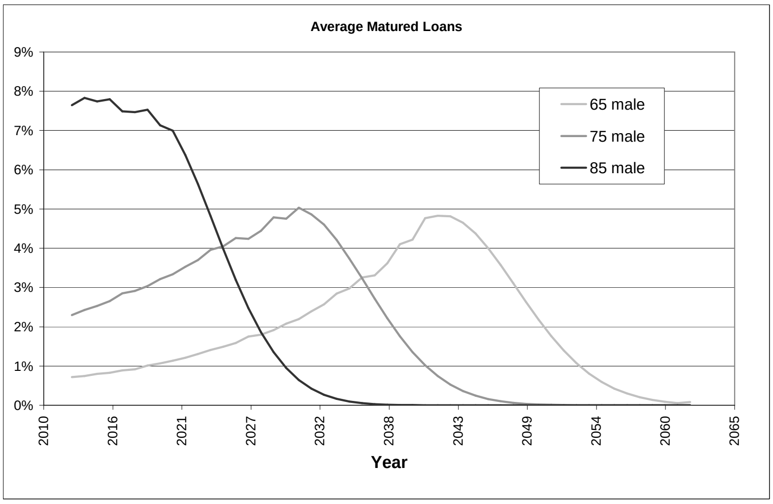

Fig. 7. Average matured loans for cohorts of 65, 75 and 85.

- for 65 and 75 years old populations the NPV increases as the risk premium λ decreases. This implies that for this age range we are short longevity risk. Vice versa, for 85 years old population the NPV is an increasing function of λ and we are long longevity risk.

Table 3. Net present value (NPV) as a function of the fixed rate K_H and the risk premium λ applied to mortality improvements for populations of 65, 75 and 85 years old males.

			65 Male			**75 Male**			**85 Male**	
Cyclic			λ			λ			λ	
Regression		0%	0.5%	1%	0%	0.5%	1%	0%	0.5%	1%
K_H	7%	23.26	22.94	22.53	24.67	24.62	24.55	24.53	24.54	24.55
	8%	26.84	26.43	25.90	27.96	27.95	27.92	26.33	26.40	26.47
	9%	29.72	29.16	28.48	31.25	31.24	31.19	28.26	28.39	28.52

6 Conclusion

In this work we have proposed a simplified, yet comprehensive, framework to value equity release mortgage contracts. These products have a hybrid nature characterized by the interplay of three main sources of risk: longevity risk, real estate risk and inflation/interest rates risk. Longevity has been assumed independent of the other drivers of risk and has been described by adopting the

Lee Carter model [11] corrected with a risk premium in the spirit of LLMA (Longevity Pricing Framework) [14]. Real estate risk, for which no consensus model exists, has been dealt with by using a regression of housing data on inflation, combining this idea with different scenarios of volatility. A detailed numerical analysis has been carried out in order to study the combined effects of these different kinds of randomness.

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