An approach to stochastic input-output modeling

Marta Kozicka

Bioversity International,
Via dei Tre Denari 472/A, 00054 Maccarese (RM), Italy
marta.kozicka@gmail.com

Abstract. An approach to stochastic input-output modeling is proposed in which the external demand and the economy can be random. The economy is allowed to be nonproductive with a certain probability $\varkappa \geq 0$. In this approach, the production plan is set to be feasible if the probability of satisfying the external demand is at least $1 - \alpha$ for some $\alpha \in (0, 1)$. Then the modeling is reduced to minimization of cost functions on the set of feasible plans. In the framework of this approach, the version of the Leontief model which includes both production and import of commodities is proposed and analyzed. Finally, a simple example of this model is solved.

Keywords. Leontief model, convex optimization, stochastic modeling.

M.S.C. classification. 91B70, 91B66.
J.E.L. classification. C67, C69.

1 The Approach

Traditional input-output models [9, 11] describe an economy – a production system consisting of $N \geq 2$ branches, each of which produces a single homogeneous commodity. All the commodities are measured in the same unit, and a part of them is used inside the economy for the production purposes. That is, a part of the output of one branch can be used as an input of the other branches. The related problem consists in finding a *production plan* $x_1, \ldots, x_N \geq 0$ such that the following conditions are satisfied

$$x_i - y_i \geq d_i, \qquad i = 1, \ldots, N, \tag{1}$$

Mathematical Methods in Economics and Finance – m²ef
Vol. 9/10, No. 1, 2014/2015

where y_i is the inner consumption of i-th commodity, whereas $d_i \geq 0$ is the outer demand of this commodity which has to be satisfied. In the simplest model of this kind introduced by Wassily Leontief (Nobel Prize in Economy in 1973), each y_i is a linear function of $x_1, \ldots, x_N$, that is,

$$y_i = \sum_{j=1}^{N} c_{ij} x_j, \qquad i = 1, \ldots, N, \tag{2}$$

where $c_{ij} \geq 0$ is the amount of i-th commodity consumed for producing one unit of j-th commodity. In column vectors X, Y, and D with entries x_i, y_i, d_i, respectively, (1) and (2) can be combined into the following condition

$$X - Y = X - CX = (I - C)X \geq D, \tag{3}$$

where $C = (c_{ij})_{N \times N}$ is the inner consumption matrix and I is the $N \times N$ unit matrix. The inequality in (3) is understood component-wise. The economy is said to be *productive* if the *Leontiev matrix*

$$B = (b_{ij})_{N \times N} = (I - C)^{-1}$$

exists and has all entries nonnegative. In this case, the problem in (3) is solved by setting $X = BD$, which in components has the form

$$x_i = \sum_{j=1}^{N} b_{ij} d_j, \qquad i = 1, \ldots, N.$$

If the series

$$\sum_{n=0}^{\infty} C^n = I + C + C^2 + \cdots \tag{4}$$

converges (component-wise), then it converges to $B = (I - C)^{-1}$ which has all entries nonnegative by this fact. The mathematical (necessary and sufficient) condition for the convergence in (4) is that the spectral radius of C satisfies $r(C) < 1$. For more on the mathematical theory of such models see [1, Chapter 9].

In real life, the entries of the matrix C are subject to various random effects. Thus, more realistic input-output models should consider C as a random matrix. This had been understood quite a long time ago, see a review and historical remarks in [4]. It turns out that the most mathematically advanced works in this direction are concentrated on deriving information on the probability distribution of the Leontief matrix, see [5, 12]. A more modest task in this context is to estimate the coefficients b_{ij}, see [6–8, 10] and [11, Chapter 14]. However, for the very existence of the random Leontief matrix B it is necessary that the economy be productive with probability one, which one cannot expect in general.

In this article, we propose another approach to analyzing stochastic input-output models consisting in the following:

(a) the economy is allowed to be nonproductive with certain probability $\varkappa \in [0,1]$;

(b) a (feasible) production plan is defined as the one for which the outer demand is satisfied with probability $\geq 1-\alpha$ for a prescribed $\alpha \in (0,1)$;

(c) optimal production plans are chosen among feasible plans under the condition of minimizing cost functions given in the model.

Let us now explain this in more detail. According to (a) the economy is productive with probability $1-\varkappa \geq 0$, and hence, for some nonnegative $x_1, \dots, x_N$, the event

$$\mathcal{A}(x_1, \dots, x_N): \qquad x_i - \sum_{j=1}^{N} c_{ij} x_j \geq d_i, \qquad i = 1, \dots, N \tag{5}$$

can occur with a prescribed positive probability. Note that this makes sense also for random outer demands $d_1, \dots, d_N$. In (b), we propose to define the collection of feasible production plans as the set of nonnegative $x_1, \dots, x_N$ such that

$$\mathbb{P}\left[\mathcal{A}(x_1, \dots, x_N)\right] := \mathrm{Prob}\left\{\mathcal{A}(x_1, \dots, x_N)\right\} \geq 1-\alpha, \tag{6}$$

for some fixed $\alpha \in (0,1)$, e.g. $\alpha = 0.1$. Then (c) means that optimal plans are chosen to minimize a cost function $\varphi(x_1, \dots, x_N)$ on the set of feasible plans defined in (6). The important advantage of the proposed approach is that one deals with the matrix C only, and hence avoids complex and tiresome procedure of getting information on the distribution of the Leontief matrix B. Moreover, our approach admits the following extensions:

(i) Along with random c_{ij} one can also allow d_i in (5) be random.

(ii) One can also include import of some commodities, especially in the case where the set of feasible plans defined in (6) is empty for a given α.

Note that considering probabilities as in (6) was suggested already in paper [3]. However, the discussion therein was restricted to constructing feasible production plans. In Section 2 below, we use both possibilities (i) and (ii) and propose an open stochastic input-output model which realizes the approach proposed above. The presentation is then illustrated by a simple example analyzed in the concluding part of the paper.

2 The Open Stochastic Leontief Model

2.1 The model

We consider an economy with $N \geq 2$ branches, as described in the Introduction, in which both matrices C and D in (3) can be random. We also assume that the commodities produced by the economy are available at outer market, and hence the economy is a part of an *open* economic system. Let $z_i \geq 0$, $i = 1, \dots, N$

be the amount of i-th imported commodity. Then the balance condition, cf. (3), takes the form

$$X - CX + Z \geq D, \tag{7}$$

where Z is the corresponding column vector. The fact that (7) is satisfied is a random event, which can be presented in the form

$$\mathcal{A}(X, Z) = \bigcap_{i=1}^{N} \mathcal{A}_i(X, Z), \tag{8}$$

where

$$\mathcal{A}_i(X, Z) = \{\omega : x_i - \sum_{j=1}^{N} c_{ij}(\omega)x_j + z_i - d_i(\omega) \geq 0\}, \tag{9}$$

with ω being an elementary event, and $i = 1, \ldots, N$.

Definition 1. *For a given $\alpha \in (0, 1)$, the set M_α of nonnegative vectors X and Z defined by the condition*

$$\mathbb{P}\left[\mathcal{A}(X, Z)\right] \geq 1 - \alpha \tag{10}$$

is called the set of feasible plans.

Like in statistical inference, standard values of α can be 0.1, 0.05, 0.01. For the open model described above, feasible plans always exist. For instance, $X = 0$ and $Z = D$ is one of such plans. If the probability of nonproductivity satisfies $\varkappa < 1$, i.e., if the economy is productive with positive probability, then there exist feasible plans with $x_i > 0$, for at least some of $i = 1, \ldots, N$. Moreover, M_α is a convex set. That is, if $(X', Z') \in M_\alpha$ and $(X'', Z'') \in M_\alpha$, and also

$$X = \delta X' + (1 - \delta)X'', \qquad Z = \delta Z' + (1 - \delta)Z'',$$

for some $\delta \in [0, 1]$, then $(X, Z) \in M_\alpha$. This follows from the fact that the conditions wich define $\mathcal{A}_i$, see (9), are linear with respect to the components of X and Z.

For the production vector X, by $\varphi(X)$ we denote the cost of its realization. Let also $\psi(Z)$ denote the cost of import of commodities involved in (7) – (10).

Definition 2. *A feasible plan (X_*, Z_*) is called optimal if, for each $(X, Z) \in M_\alpha$, the following holds*

$$\varphi(X_*) + \psi(Z_*) \leq \varphi(X) + \psi(Z). \tag{11}$$

Then the aim of the description of the model which we propose is to find optimal plans. Its realization consists in finding the probability of the event (8), (9), which yields the (convex) set M_α, and then in solving the optimization problem (11). If both φ and ψ are convex functions, then one can apply here powerful tools of convex optimization, see, e.g., [2]. Clearly, this realization crucially depends on the concrete model and may be quite complex. We illustrate this in a simple example below.

2.2 Example

Setup For the sake of simplicity, we take $N = 2$ and assume that only c_{12} and c_{21} are random, whereas d_1 and d_2 are deterministic. Next, we suppose that c_{12} and c_{21} are independent and uniformly distributed on the intervals $[0, a]$ and $[0, b]$, respectively. Here a and b are positive parameters. If $ab < 1$, then the spectral radius of C satisfies $r(C) < 1$ and hence the economy is productive with probability one. Thus, in the sequel we assume that

$$ab > 1, \tag{12}$$

which means that the economy in question can be nonproductive with probability

$$\varkappa = [ab - 1 - \ln(ab)] / ab > 0.$$

Then the events as in (9) are

$$\begin{aligned} \mathcal{A}_1(x_1, x_2, z_1, z_2) &= \{\omega : c_{12}(\omega) \in [0, (x_1 + z_1 - d_1)/x_2]\}, \\ \mathcal{A}_2(x_1, x_2, z_1, z_2) &= \{\omega : c_{21}(\omega) \in [0, (x_2 + z_2 - d_2)/x_1]\}. \end{aligned} \tag{13}$$

Set $\pi_i(x_1, x_2, z_1, z_2) = \mathbb{P}[\mathcal{A}_i(x_1, x_2, z_1, z_2)]$, $i = 1, 2$. Since c_{12} and c_{21} are independent, condition (10) can be written

$$\pi_1(x_1, x_2, z_1, z_2)\pi_2(x_1, x_2, z_1, z_2) \geq 1 - \alpha. \tag{14}$$

By the very definition in (13)

$$x_1 + z_1 \geq d_1, \qquad x_2 + z_2 \geq d_2. \tag{15}$$

Under these conditions we have

$$\pi_1(x_1, x_2, z_1, z_2) = \begin{cases} (x_1 + z_1 - d_1)/ax_2, & \text{if } \ x_1 + z_1 - d_1 \leq ax_2; \\ 1, & \text{if } \ x_1 z_1 - d_1 \geq ax_2. \end{cases} \tag{16}$$

$$\pi_2(x_1, x_2, z_1, z_2) = \begin{cases} (x_2 + z_2 - d_2)/bx_1, & \text{if } \ x_2 + z_2 - d_2 \leq bx_1; \\ 1, & \text{if } \ x_2 + z_2 - d_2 \geq bx_1. \end{cases}$$

Along with (15) x_1, x_2, z_1, and z_2 satisfy one of the following pairs of conditions:

$$\begin{aligned} &x_1 + z_1 - d_1 \leq ax_2 \quad \text{and} \quad x_2 + z_2 - d_2 \leq bx_1 \ \ (a) \\ &x_1 + z_1 - d_1 \leq ax_2 \quad \text{and} \quad x_2 + z_2 - d_2 > bx_1 \ \ (b) \\ &x_1 + z_1 - d_1 > ax_2 \quad \text{and} \quad x_2 + z_2 - d_2 \leq bx_1 \ \ (c) \end{aligned} \tag{17}$$

Now we choose the cost functions, which we take also in the simplest form

$$\varphi(x_1, x_2) = \theta_1 x_1 + \theta_2 x_2, \qquad \psi(z_1, z_2) = \tau_1 z_1 + \tau_2 z_2, \tag{18}$$

with nonnegative parameters θ_i and τ_i, $i = 1, 2$. Instead of z_1 and z_2 it is convenient to introduce new variables

$$u_1 = \frac{x_1 + z_1 - d_1}{ax_2}, \qquad u_2 = \frac{x_2 + z_2 - d_2}{bx_1}. \tag{19}$$

In these new variables, the probabilities π_i defined in (16) take the form

$$\pi_i = \min\{u_i, 1\}, \qquad i = 1, 2. \tag{20}$$

By (19) we also have

$$z_1 = au_1x_2 - x_1 + d_1, \qquad z_2 = bu_2x_1 - x_2 + d_2, \tag{21}$$

which implies that x_i and u_i, $i = 1, 2$, satisfy the following conditions

$$\begin{cases} au_1x_2 - x_1 + d_1 \geq 0 \\ bu_2x_1 - x_2 + d_2 \geq 0. \end{cases} \tag{22}$$

The conditions (14) and (17) imply, see (20), that u_1 and u_2 ought to satisfy one of the following triple of constraints

$$\begin{aligned} &u_1 \leq 1, \quad u_2 \leq 1, \quad u_1u_2 \geq 1 - \alpha, && (a) \\ &1 - \alpha \leq u_1 \leq 1, \quad u_2 > 1, && (b) \\ &u_1 > 1, \quad 1 - \alpha \leq u_2 \leq 1, && (c) \end{aligned} \tag{23}$$

The values of u_1 and u_2 which satisfy (23) will be called feasible. In the new variables, the total cost function takes the form

$$\begin{aligned} f(x_1, x_2, u_1, u_2) &= \varphi(x_1, x_2) + \psi(au_1x_2 - x_1 + d_1, bu_2 - x_2 + d_2) \qquad (24) \\ &= g(x_1, x_2, u_1, u_2) + \tau_1 d_1 + \tau_2 d_2. \\ g(x_1, x_2, u_1, u_2) &:= -(\tau_1 - \theta_1 - b\tau_2u_2)x_1 - (\tau_2 - \theta_2 - a\tau_1u_1)x_2 \end{aligned}$$

Then the problem consists in minimizing g on the set of feasible values of x_i and u_i, $i = 1, 2$, defined by the conditions given in (22) and (23).

Import is preferable Suppose now that, for some feasible u_1 and u_2, the prices τ_1 and τ_2 are such that the following holds

$$\begin{cases} \tau_1 - bu_2\tau_2 \leq \theta_1 \\ \tau_2 - au_1\tau_1 \leq \theta_2. \end{cases} \tag{25}$$

In this case, both coefficients at x_i in g are nonnegative, which means that its minimum is attained at the point $x_1 = x_2 = 0$, that by (21) yields $z_1 = d_1$ and

$z_2 = d_2$. That is, both commodities are to be imported and no production is planned. If

$$abu_1u_2 \geq 1, \tag{26}$$

then the set of positive pairs (τ_1, τ_2) which satisfy (25) is unbounded. That is, even for very high import prices it is more reasonable to import rather than produce if the import prices satisfy (25) with some u_1 and u_2 satisfying (23). A sufficient condition for (26) to hold is

$$ab(1-\alpha) \geq 1, \tag{27}$$

which readily follows from (23). In the opposite case where

$$\gamma := 1 - (1-\alpha)ab > 0, \tag{28}$$

the set of pairs (u_1, u_2) that satisfy, see (12),

$$1 - \alpha \leq u_1u_2 < 1/ab \tag{29}$$

is non-empty. For such u_1 and u_2, we set

$$\tau_1^* = \frac{\theta_1 + bu_2\theta_2}{1 - abu_1u_2}, \qquad \tau_2^* = \frac{\theta_2 + au_1\theta_1}{1 - abu_1u_2}. \tag{30}$$

Then, for $\tau_1 > \tau_1^*$ and $\tau_2 > \tau_2^*$, both coefficients in the expression for g in (24) are negative. Hence, the minimum of g is attained at some positive x_1 and x_2. We will consider this case in more detail later.

Intermediate case Now let only one of the inequalities in (25), say the first one, hold for some feasible u_1 and u_2. Then the minimum of g is attained at $x_2 = 0$ and the biggest possible $x_1 > 0$. By the first line in (23) we have $x_1 \leq d_1$; hence, to minimize g we set $x_1 = d_1$. In this case, we have

$$g = -(\tau_1 - \theta_1)d_1 + b\tau_2u_2d_1 \geq -(\tau_1 - \theta_1)d_1 + b\tau_2(1-\alpha)d_1,$$

where we take the least possible value of u_2 which satisfies (23), i.e., we set $u_2 = 1 - \alpha$. Note that $\tau_1 - (1-\alpha)b\tau_2 > \theta_1$ whenever $\tau_1 - u_2b\tau_2 > \theta_1$ holds for some feasible u_2. Then the solution for this case is:

$$x_1 = d_1, \quad z_1 = 0, \quad x_2 = 0, \quad z_2 = (1-\alpha)bd_1 + d_2. \tag{31}$$

It exists regardless which of the inequalities (27) or (28) holds. The case where only the second inequality in (25) holds can be considered in the same way.

Production is preferable Let us turn now to the case where (28) holds, and hence the maximum import prices given in (30) exist. We then assume that $\tau_i \geq \tau_i^*$, $i = 1, 2$. The set of feasible values of x_1 and x_2 is defined by the system of linear constraints given in (22). For fixed values of u_1 and u_2, g is a linear function of x_1 and x_2; hence, its least value is attained at the corner point

$$x_1 = x_1^* = \frac{d_1 + au_1 d_2}{1 - abu_1 u_2}, \qquad x_2 = x_2^* = \frac{d_2 + bu_2 d_1}{1 - abu_1 u_2}, \tag{32}$$

which corresponds to the equalities in (22). By (21) this yields

$$z_1 = z_2 = 0, \tag{33}$$

that is, no import is planned. In this case, it is more convenient to minimize the total cost function f, which now depends only on x_1 and x_2, see (18) and (24),

$$f(x_1, x_2, 0, 0) = \theta_1 x_1 + \theta_2 x_2,$$

with x_1 and x_2 given in (32) with u_1 and u_2 in the set defined in (29). That is, on the set just mentioned we have to minimize the function

$$h(u_1, u_2) = \theta_1 x_1^* + \theta_2 x_2^* = \frac{\theta_1 d_1 + \theta_2 d_2 + a\theta_1 d_2 u_1 + bd_1 \theta_2 u_2}{1 - abu_1 u_2}. \tag{34}$$

By direct calculations we get that

$$\frac{\partial h(u_1, u_2)}{\partial u_1} > 0, \qquad \frac{\partial h(u_1, u_2)}{\partial u_2} > 0,$$

which means that the minimum of h is attained on the boundary $u_1 u_2 = 1 - \alpha$, $u_1 \leq 1$, $u_2 \leq 1$. Then we set

$$u_1 = \xi, \qquad u_2 = (1 - \alpha)/\xi,$$

and consider, cf. (34) and (28),

$$\begin{aligned} w(\xi) &= h(\xi, (1 - \alpha)/\xi) \\ &= \frac{1}{\gamma}(\theta_1 d_1 + \theta_2 d_2) + \frac{a\theta_1 d_2}{\gamma}\xi + \frac{(1 - \alpha) bd_1 \theta_2}{\gamma \xi}, \end{aligned} \tag{35}$$

with ξ taking values in $[1 - \alpha, 1]$. Thus, we have to minimize w on the latter interval. By (35) we have

$$w'(\xi) = \frac{bd_1 \theta_2}{\gamma} \left(\frac{\theta_1}{\theta_2} \cdot \frac{ad_2}{bd_1} - \frac{1 - \alpha}{\xi^2} \right).$$

Suppose that the following holds

$$\frac{(1 - \alpha) bd_1}{ad_2} < \frac{\theta_1}{\theta_2} < \frac{bd_1}{(1 - \alpha) ad_2}.$$

Then w' changes its sign, and hence w attains its minimum, at some $\xi_* \in (1 - \alpha, 1)$. For

$$\frac{\theta_1}{\theta_2} \leq \frac{(1 - \alpha) bd_1}{ad_2},$$

$w'(\xi) \leq 0$ for all $\xi \in [1-\alpha, 1]$; hence, the minimum of w is attained at $\xi_* = 1$. Finally, for

$$\frac{\theta_1}{\theta_2} \geq \frac{bd_1}{(1-\alpha)ad_2},$$

$w'(\xi) \geq 0$ for all $\xi \in [1-\alpha, 1]$, and the minimum of w is attained at $\xi_* = 1-\alpha$. In all these situations, the optimal solution is obtained from (32)

$$x_1^* = \frac{d_1}{\gamma} + \frac{ad_2\xi_*}{\gamma}, \qquad x_2^* = \frac{d_2}{\gamma} + \frac{(1-\alpha)bd_1}{\gamma\xi_*}, \tag{36}$$

with the corresponding value of ξ_*.

Concluding remarks The intervals characterizing the probability distributions of c_{12} and c_{21} were chosen for simplicity of calculations to be $[0, a]$ and $[0, b]$, respectively. A more realistic version would be $[a_-, a_+]$ and $[b_-, b_+]$ for some $0 < a_- < a_+$ and $0 < b_- < b_+$. In that case, one allows the coefficients to randomly oscillate in the mentioned intervals. Another choices were discussed in papers [3] and [5]. The choice of the objective functions in (18) was also dictated by our wish to make the calculations simple and transparent. Note that this choice is pretty reasonable and that the functions are convex.

The parametr γ introduced in (28) reflects the very essence of our approach. Namely, typically ab exceeds 1, cf. (12), but not too much. Then the 'numerical effect' of passing to the condition in (10) is just multiplying ab by $1-\alpha$, and making thereby the economy productive with probability $1-\alpha$ if (28) holds. If (27) holds, then the economy is still nonproductive, and the solution $x_1 = x_2 = 0$ is the only feasible one in this case. The conditions on the import prices in (25) mean that the costs of production, i.e., the left-hand sides of the inequalities, would be higher, and hence the import is preferable. In the intermediate case, the imported amount z_2 is used to satisfy the demand and as an input of the first branch, see (31). The fact that we have no solutions where both x_i and z_i are positive for some $i = 1, 2$, cf. (31) and (33), is related to the linearity of the objective functions chosen in (18). The case where the production is preferable can be realized only if γ is positive. If, however, it is small, the economy is 'nearly nonproductive', and hence the optimal plans given in (36) are much bigger than those corresponding to $\gamma = 1$.

3 Acknowledgement

The author acknowledges the referee's suggestions which helped to improve the presentation of the paper.

References

1. Berman, A., Plemmons, R. J.: Nonnegative matrices in the mathematical sciences. Revised reprint of the 1979 original. Classics in Applied Mathematics, 9. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (1994)

2. Boyd, S., Vandenberghe, L.: Convex optimization, Cambridge University Press, Cambridge (2004)
3. Goicoechea, A., Hansen, D. R.: An input-output model with stochastic parameters for economic analysis. A IIE Transactions **10** (1978), 285–291
4. Gurgul, H.: Stochastic input-output modeling. Ekonomia Menedzerska **2** (2007) 57–70
5. Kogelschatz, H.: On the solution of stochastic input-output models. University of Heidelberg. Discussion Paper Series No. 447 (2007)
6. Lahiri, S.: A note on the underestimation and overestimation in stochastic input-output models. Econom. Lett. **13** (1983) 361–366
7. Lahiri, S., Satchel, S.: Underestimation and overestimation of the Leontief inverse revisited. Econom. Lett. **18** (1985) 181–186
8. Lahiri, S., Satchel, S.: Properties of the expected value of the Leontief inverse: some further results. Math. Soc. Sciences **11** (1986) 69–82
9. Leontief, W.: Input-output economics. Second edition. Oxford University Press, New York Oxford (1986)
10. Simonovits, A.: A note on the underestimations and overestimation of the Leontief matrix. Econometrica **43** (1975), 493–498
11. ten Raa, T.: Linear analysis of competitive economies. LSE Handbooks in Economics. Harvester Wheatsheaf. New York London Toronto Sydney Tokyo Songapore (1995)
12. West, G. R. A stochastic analysis of an input-output model. Econometrica **54** (1986) 363–374