On the scaling function of multifractal processes

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Abstract. Benoît Mandelbrot, the father of Fractal Geometry, developed a multifractal model for describing price changes. Despite the commonly used models such as the Brownian motion, the Mutifractal Model of Asset Return (MMAR) takes into account scale-consistency, long-range dependence and heavy tails, thus having a great flexibility in depicting the real-market peculiarities. In section 2 a review of the mathematics involved into multifractals is presented; Section 3 shows how to extend multifractality to stochastic processes. Contributions in Section 4 are new in the literature and extend Mandelbrot's results to canonical multifractal measures. *Proof* of Theorem 5 is unpublished and highlights which are the drivers of heavy tails in the process, thus possibly creating a bridge between multifractal formalism and jump processes.

Keywords. Multifractal processes, multifractal measures, multiplicative cascades, scaling function, MMAR, heavy tails, jump processes.

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1 Introduction

Benoît Mandelbrot, the inventor of Fractal Geometry, introduced in [7] the concept of Multifracta Measures. More than ten years later, he extended the concept of measure towards stochastic processes: With Laurent Calvet and Adlai Fisher, they developed a solid theory on multifractal processes, thoroughly discussing its implication if applied to financial log-returns. They built a self-similar stochastic process, based on a distortion of physical time, exhibiting long-range dependence and heavy tails as, thus able to depict effectively price changes. Despite Browian motion (which can be considered as a particular subcase), it is able to capture many peculiarities of real-market data.

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Subsequently, an article of Mandelbrot appeared in [14] discussing mathematical properties of a particular multifractal process (the binomial cascade) and of the related scaling function. My contribution to this subject is extending Mandelbrot's analysis to general multifractal processes. As a matter of fact, the properties of the so-called *scaling function* $\tau(q)$ of a generic multifractal measure appear to be much more interesting than the ones of a binomial cascade. It will be shown that the asymptotic behaviour of $\tau(q)$ is related to the *maximum* value of the random generators. Since the explanation given for the presence of, and the subsequent capability to generate, heavy tails in the process is quite vague, a possible connection with processes with jumps seems to exist. A very brief justification on where the Multifractal Model of Asset Returns (MMAR) departs form the most commonly used models in finance is presented in the following.[1]

The MMAR appeared for the first time in the three paper series [1], [2], and [3], introducing the concept of multifractality to economics. This model attempts to describe price changes, accounting for several features of financial data: Long memory, fat tails and scale invariance. The authors especially criticized the GARCH-type representations, the latter assuming that the conditional distribution of the return (in respect to the information available until today) has a finite, time-varying second moment.

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and the stochastic process representing log-returns $(X(t))_{t\geq 0}$ adapted to the filtration, defining $h_t := \mathbb{V}(X_t \,|\, \mathcal{F}_{t-1})$, it follows $h_t = f(h_{t-j}, \varepsilon_{t-j})$, where $\varepsilon_t \,|\, \mathcal{F}_{t-1} \overset{d}{\sim} \mathcal{N}(0,1)$ and $t \geq j$ for all j. Moreover, f is assumed to be a mere affine function of its arguments. Note that such a model directly addresses volatility clustering in the data, creating heavy tails. However, neither long-range dependence nor scale invariance can be described with it. Furthermore, being scale invariance the equivalence between representations of the model at different time-scales, the absence of an invariance under scaling in GARCH models implies that, in empirical works, the researcher adds an *additional restriction* to the model when choosing the time-scale of the data. Moreover, as well-summarised in [13], (Brownian) diffusion processes with *nonlinear* dependence of the local volatility, as in the MMAR, can generate heavy tails and skewness.

As a matter of fact, the MMAR generate heavy tails and a divergent variance directly in the directing process of log-returns. Persistence in volatility is given by the use of a *random trading time*, generated as the cumulative distribution function of a random multifractal measure. In Section 3 an in-depth formulation of multifractal process is provided.

Hereunder, a short review on multifractal measures is presented (for a more extensive in-depth analysis, see [5] and [7]).

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[1] Note that in the in this work, the following symbols are used: $\overset{d}{\sim}$ when two random variables have the same distribution, $\overset{d}{\to}$ for convergence in distribution, and, given two functions f and g, $f \sim g$ for the equivalence class of all functions g which are equal to f in the limit (asymptotic).

2 A review on multifractal measures

In the following, I start with introducing the most trivial multifractal measure, namely the (*Bernoulli*) *binomial measure* on the compact interval $[0,1] \subset \mathbb{R}$. Then, the measure will be extended as in [7] in order to construct a *canonical multifractal measures.*

The recursive construction of the binomial measure involves an initiator and a generator. The initiator is the interval $[0,1]$ itself on which a unit of (probability) mass is uniformly spread. This interval will recursively split into halves, leading to, at the k-th stage, dyadic intervals of length 2^{-k}. The generator consists in a single parameter $0 < u_0 < 1$ and $u_0 \neq \frac{1}{2}$, named *multiplier*, which at each stage is spread over the halves of every dyadic interval, with unequal deterministic proportions.

Let u_0 be a multiplier and be u_1 its ones' complement. At stage $k = 0$, a uniform probability measure on $[0,1]$ is defined as

$$f_0(t) := \mu_0([0,1]) = 1.$$

At the step $k = 1$, the measure μ_1 uniformly spread mass equal to u_0 on the first-half subinterval and mass equal to u_1 on the other one, that is

$$f_1(t) := \begin{cases} \mu_1\big(\big[0,\frac{1}{2}\big]\big) = u_0 & \text{if } t \in \big[0,\frac{1}{2}\big) \\ \mu_1\big(\big[\frac{1}{2},1\big]\big) = u_1 & \text{if } t \in \big[\frac{1}{2},1\big]. \end{cases}$$

Here, is trivial to see that the mass is preserved. In fact

$$\mu_1\left(\left[0,\frac{1}{2}\right]\right) + \mu_1\left(\left[\frac{1}{2},1\right]\right) = u_0 + u_1 = 1.$$

In step $k = 2$, the set $\big[0,\frac{1}{2}\big)$ is split into two subintervals, $\big[0,\frac{1}{4}\big)$ and $\big[\frac{1}{4},\frac{1}{2}\big)$, which respectively receive a percentage u_0 and u_1 of the total mass $\mu_1\big(\big[0,\frac{1}{2}\big]\big)$. Applying the same procedure to the dyadic set $\big[\frac{1}{2},1\big]$, it follows

$$f_2(t) := \begin{cases} \mu_2\big(\big[0,\frac{1}{4}\big]\big) = u_0 \cdot u_0 = u_0^2 & \text{if } t \in \big[0,\frac{1}{4}\big) \\ \mu_2\big(\big[\frac{1}{4},\frac{1}{2}\big]\big) = u_0 \cdot u_1 = u_0 u_1 & \text{if } t \in \big[\frac{1}{4},\frac{1}{2}\big) \\ \mu_2\big(\big[\frac{1}{2},\frac{3}{4}\big]\big) = u_1 \cdot u_0 = u_0 u_1 & \text{if } t \in \big[\frac{1}{2},\frac{3}{4}\big) \\ \mu_2\big(\big[\frac{3}{4},1\big]\big) = u_1 \cdot u_1 = u_1^2 & \text{if } t \in \big[\frac{3}{4},1\big], \end{cases}$$

and the total mass is preserved since

$$\sum_{i=0}^{2^2-1} \mu_2\left(\left[\frac{i}{2^2},\frac{i+1}{2^2}\right]\right) = u_0^2 + 2u_0u_1 + u_1^2 = (u_0+u_1)^2 = 1.$$

It is trivial to prove that at stage k the mass still preserve, and hence the procedure can generate an infinite sequence of conservative measures.

At step $k+1$, assume that the measure μ_k has been defined. To construct μ_{k+1}, it is convenient to consider the general dyadic interval $\left[t, t+2^{-k}\right]$, and express t as *dyadic number* of the form, that is $t := (0.\eta_1\eta_2 \dots \eta_k)_2 = \left(\sum_{i=1}^{k} \eta_i \cdot 2^{-k}\right)_{10}$, for a finite k and $\eta_1, \eta_2, \dots, \eta_k \in (0,1)$.

Then, uniformly spreading a fraction u_0 and u_1 of the mass $\mu_k\left(\left[t, t+2^{-k}\right]\right)$ on the subintervals $\left[t, t+2^{-(k+1)}\right)$ and $\left[t+2^{-(k+1)}, t+2^{-k}\right]$, this procedure repetition (repeated on all the subintervals) define the measure μ_{k+1}.

Let φ_0 and φ_1 denote the relative frequencies of 0's and 1's in the finite binary development $t = (0.\eta_1\eta_2 \dots \eta_k)_2$. The so-called *pre-binomial measure* in the dyadic interval $\left[t, t+2^{-k}\right]$, takes the value

$$\mu_k\left(\left[t, t+2^{-k}\right]\right) = u_0^{k\cdot\varphi_0} \cdot u_1^{k\cdot\varphi_1}. \tag{1}$$

Because of the conservation of the mass at each stage, it follows[2]

$$\sum_{i=0}^{2^k-1} \mu_k\left(\left[\frac{i}{2^k}, \frac{i+1}{2^k}\right]\right) = (u_0+u_1)^k = 1.$$

This construction can receive several extensions. At each stage the interval can be split in more than two ($b > 2$) intervals of equal size. Subintervals, indexed from left to right by j ($0 \le j \le b-1$), receive fraction of the total mass equal to $u_0, u_1, \dots, u_{b-1}$ such that they sum up to one. Under this assumption, the *multinomial measure* on the b-adic interval $\left[t, t+b^{-k}\right]$ follows the conservation rule

$$\sum_{i=0}^{b^k-1} \mu_k\left(\left[\frac{i}{b^k}, \frac{i+1}{b^k}\right]\right) = \left(\sum_{j=0}^{b-1} u_j\right)^k = 1$$

and, the measure is computed as

$$\mu_k(\Delta t) = \prod_{j=0}^{b-1} u_j^{k\cdot\varphi_j}. \tag{2}$$

where $\Delta t = b^{-k}$ and t is the b-adic number $t := (0.\eta_1\eta_2 \dots \eta_k)_b = \left(\sum_{i=1}^{k} \eta_i \cdot b^{-k}\right)_{10}$, for a finite k and $\eta_1, \eta_2, \dots, \eta_k \in (0, 1, \dots, b-1)$, and φ_j are the relative frequencies of the digits of the representation in base b.

The next extension is obtained by making the allocation of the mass random. Thus, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, assume that the multipliers of

[2] The iteration of the procedure generates an infinite sequence of random measure $(\mu_k)_{k\in\mathbb{N}}$ that weakly converges to the *binomial measure* μ, that is $\mu_k \xrightarrow{d} \mu$. Note that the binomial measure has important features common to many multifractal (measures): It is continuous but also a singular probability measure; it thus has no density, being $\lim_{k\to\infty} \mu_k = 0$ and $\sum_{i=0}^{\infty} \mu_k\left(\left[\frac{i}{2^k}, \frac{i+1}{2^k}\right]\right) = 1$.

each subinterval are extracted by a sequence of iid (positive) random variables $(U_j)_{j\in\{0,1,\ldots,b-1\}}$. As for the previous cases, assume that the mass is preserved *at each stage* of the construction, that is $\sum_{j=0}^{b-1} U_j = 1$. Hence, $0 \leq U_j \leq 1$ and $\mathbb{E}(U_j) = 1/b$ must hold for all j. The resulting measure is called *microcanonical measure*.

Given a date $t = (0.\eta_1\eta_2\ldots\eta_k)_b$ and a length $\Delta t = b^{-k}$, the measure of the b-adic cell $\left[t, t+b^{-k}\right]$ satisfies

$$\mu_k(\Delta t) = U_{\eta_1} \cdot U_{\eta_1\eta_2} \cdot \ldots \cdot U_{\eta_1\eta_2\ldots\eta_k}$$

where $\eta_1\eta_2\ldots\eta_k$ is one of the element (it is a b-adic number) of the ordered selection with repetition made with b digits. Because of its properties (iid random variables), it follows

$$\mathbb{E}\left[\mu_k(\Delta t)^q\right] = \mathbb{E}\left(U^q_{\eta_1}\right) \cdot \mathbb{E}\left(U^q_{\eta_1\eta_2}\right) \cdot \ldots \cdot \mathbb{E}\left(U^q_{\eta_1\eta_2\ldots\eta_k}\right) = \mathbb{E}\left(U^q\right)^k$$

for all $q \geq 0$.

Setting $\tau(q) := -\log_b \mathbb{E}(U^q) - 1$ (which is named *scaling function*), the expression can be written as

$$\mathbb{E}\left[\mu_k(\Delta t)^q\right] = \Delta t^{\tau(q)+1}, \tag{3}$$

which is the typical behaviour of a multifractal measure (see [5]).

The very last extension define the multifractal *canonical measure*. If, given a sequence of iid (positive) random variables, each iteration conserves probability mass only "on average" in the sense that

$$\mathbb{E}\left(\sum_{j=0}^{b-1} U_j\right) = 1,$$

a less restrictive class on multipliers is achieved, just allowing for the positivity of the random variables, that is $U_j \geq 0$ for all j. Hence, the total mass of the canonical measure, denoted as Υ, is generally random.

Thus, given a time $t = 0.\eta_1\eta_2\ldots\eta_k$, at the k-th stage, since the conservation of the mass holds only "on average", the canonical measure does *not* has an extra term than the microcanonical measure, that is

$$\mu_k(\Delta t) = \Upsilon_{\eta_1\eta_2\ldots\eta_k} \cdot \left(U_{\eta_1} \cdot U_{\eta_1\eta_2} \cdot \ldots \cdot U_{\eta_1\eta_2\ldots\eta_k}\right)$$

Since it can be proven that $\Upsilon_{\eta_1\eta_2\ldots\eta_k}$ are iid random variables[3] and, being independent from U_j for all j, it follows

$$\mathbb{E}\left[\mu_k(\Delta t)^q\right] = \mathbb{E}\left(\Upsilon^q\right) \cdot \mathbb{E}\left(U^q\right)^k.$$

[3] The random variable Υ is the fixed point of the operation of randomly weighted averaging using as weights the random quantities Υ_j, that is $\sum_{j=0}^{b-1} \Upsilon_j \cdot U_j \overset{d}{\sim} \Upsilon$.

Setting $c(q) := \mathbb{E}(\Upsilon^q)$ (usually called *prefactor*), we finally get the sought expression for the canonical measure

$$\mathbb{E}\big[\mu_k(\Delta t)^q\big] = c(q) \cdot \Delta t^{\tau(q)+1}. \tag{4}$$

To sum up, four different measures have been introduced with different levels of sophistication: (1) (*pre-binomial*), (2) (*multinomial*), (3) (*microcanonical*), and (4) (*canonical*). The latter is the cornerstone for constructing multifractal stochastic processes.

3 Multifractal processes and the application to the MMAR

Now the concept of multifractality can be extended to stochastic process. Because of the way multifractal random measures have been constructed, it is convenient defining *multifractal processes* in terms of their moments. Nevertheless, we have to remark that dealing with measures rather than stochastic process might be similar; however it is not the same. In the following, we will discuss about some discrepancies.

Definition 1. *Given a filtered probability space* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$, *a* $\mathcal{F}_t$-*adapted real-valued stochastic process* $(X(t))_{t\in[0,T]}$ *is called a* multifractal process, *if it has stationary increments and satisfies the following properties*[4]*:*

(a) $X(0) = 0$ *a.s.;*
(b) *The expectation of its absolute increments raised to the q-th power are such that*
$$\mathbb{E}\big[|X(t) - X(s)|^q\big] = c(q) \cdot |t - s|^{\tau(q)+1}$$
where $q \in Q \subseteq \mathbb{R}$ *and* $c, \tau : Q \to \mathbb{R}$.

Thank to the definition, setting $s = 0$, it follows

$$\mathbb{E}\big[|X(t)|^q\big] = c(q) \cdot t^{\tau(q)+1}.$$

Such a definition of multifractal process extend the one of self-similar process (whose standard and fractional Browninan motions belong to). As a matter of fact, since a self-similar process is a process such that, given $a \in \mathbb{R}^+$

$$(X(a \cdot t))_{t\in[0,T]} \overset{d}{\sim} \left(a^H \cdot X(t)\right)_{t\in[0,T]}$$

with $0 < H < 1$, it also satisfies

$$(|X(t)|^q)_{t\in[0,T]} \overset{d}{\sim} \left(t^{Hq} \cdot |X(1)|^q\right)_{t\in[0,T]}.$$

[4] Such process need to be defined on closed intervals $[0, T]$ in order to preserve multifractality. See [2].

Taking their expectation, it follows

$$\mathbb{E}\big[|X(t)|^q\big] = t^{Hq} \cdot \mathbb{E}\big[|X(1)|^q\big]$$

that is a multifractal process with $\tau(q) = Hq - 1$ and $c(q) = \mathbb{E}\big[|X(1)|^q\big]$. In the special case of self-similar processes, the scaling function $\tau(q)$ is thus linear and fully determined by the unique Hurst exponent H. For this reason, multifractal processes with linear $\tau(q)$ are called *unifractal*. In this work, however, the main analysis will be devote to multifractal processes with *non-linear* functions $\tau(q)$. Moreover, the fact that in the uniscaling case the crucial exponent is (the only one) Hurst exponent, it should be expected that the (continuum of) exponent of the multiscaling case might be related to the first derivative $\tau'(q)$ of the scaling function. This point can be analyzed though the multifractal spectrum and the Legendre transform (see Section 5).

As already sketched, trading time plays a notable role in transmitting multifractality to financial records. A preliminary definition regarding a particular class of stochastic processes is needed for understanding this transmission mechanism.

Definition 2. *Given a filtered probability space* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$, *let* $(Z(t))_{t\,\in\,[0,T]}$ *be a* $\mathcal{F}_t$*-adapted real-valued stochastic process and* $\theta(t)$ *an increasing function of the time* t*. The process*

$$X(t) := Z\left[\theta(t)\right]$$

is called a compound process.

Since t denotes the clock physical time, the function $\theta(t)$ reproduces the so-called trading time. We are now able to state the theoretical assumption of the MMAR.

Definition 3. *Given a filtered probability space* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$, *let* $(X(t))_{t\,\in\,[0,T]}$ *the* $\mathcal{F}_t$*-adapted real-valued stochastic process describing the log-return on the stock price, that is*

$$X(t) := \ln S(t) - \ln S(0) \tag{5}$$

where $(S(t))_{t\,\in\,[0,T]}$ *is the* $\mathcal{F}_t$*-adapted real-valued stochastic process of the price of the stock. The MMAR bases on the following three hypotheses:*

1. $(X(t))_{t\,\in\,[0,T]}$ *is a compound process*

$$X(t) = B_H[\theta(t)]$$

where $B_H(t)$ *is a fractional Brownian motion under* $\mathbb{P}$ *with Hurst exponent* H*, and* $\theta(t)$ *is a stochastic trading time;*

2. *The trading time $\theta(t)$ is the cumulative distribution function of a multifractal measure defined on $[0,T]$. Thus, $\theta(t)$ is a multifractal process with continuous, non-decreasing paths, and stationary increments;*
3. $(B_H(t))_{t\,\in\,[0,T]}$ *and* $(\theta(t))_{t\,\in\,[0,T]}$ *are independent.*

We first note that $\theta(0)$ must be null due to the definition of $X(t)$, which imposes $X(0) = 0$. Moreover the assumptions on $\theta(t)$ impose that the latter is the cumulative distribution function of a self-similar random measure, such as the microcanonical or the canonical.

It is quite natural to expect that trading time $\theta(t)$ "transfers" multifractality to $X(t)$, and that scaling functions $\tau_\theta(q)$ and $\tau_X(q)$ may be related. The following theorem expresses this intuition.

Theorem 1. *The process $X(t)$ is multifractal, with stationary increments and scaling function such that*

$$\tau_X(q) = \tau_\theta(Hq).$$

Proof. : Since

$$X(t) = B_H[\theta(t)],$$

using the law of total expectation, $\mathbb{E}\big[|X(t)|^q\big]$ can be written as

$$\mathbb{E}\big[|X(t)|^q\big] = \mathbb{E}\Big[\mathbb{E}\big[|X(t)|^q \,|\, \theta(t) = \theta\big]\Big].$$

Due to independence between trading time $(\theta(t))_{t\,\in\,[0,T]}$ and the fractional Brownian motion $(B_H(t))_{t\,\in\,[0,T]}$, applying the properties of fractional Brownian motions, it follows

$$\mathbb{E}\big[|X(t)|^q \,|\, \theta(t) = \theta\big] = \mathbb{E}\big[|B_H(\theta)|^q \,|\, \theta(t) = \theta\big] = \mathbb{E}\big[|B_H(1)|^q\big] \cdot \theta(t)^{Hq}.$$

Thus,

$$\mathbb{E}\big[|X(t)|^q\big] = \mathbb{E}\Big[\mathbb{E}\big[|B_H(1)|^q\big] \cdot \theta(t)^{Hq}\Big] = \mathbb{E}\big[|B_H(1)|^q\big] \cdot \mathbb{E}\left[\theta(t)^{Hq}\right].$$

Since $\theta(t)$ satisfies the scaling relation

$$\mathbb{E}\big[|\theta(t)|^q\big] = c_\theta(q) \cdot t^{\tau_\theta(q)+1},$$

it implies

$$\mathbb{E}\big[|X(t)|^q\big] = \mathbb{E}\big[|B_H(1)|^q\big] \cdot c_\theta(Hq) \cdot t^{\tau_\theta(Hq)+1}.$$

Setting $c_X(q) := \mathbb{E}\big[|B_H(1)|^q\big] \cdot c_\theta(Hq)$ and $\tau_X(q) := \tau_\theta(Hq)$, the thesis follows

$$\mathbb{E}\big[|X(t)|^q\big] = c_X(q) \cdot t^{\tau_X(q)+1}. \tag{6}$$

□

Hence, choosing a fractional Brownian motion as compounder and a multifractal measure to deform time, we are able to "spread" multiscaling to the process of the asset return $X(t)$. This is one of the most important property of the MMAR which allows to make further statements about the finiteness of the moments of the process.

Firstly, (6) shows that, if $\mathbb{E}\left[|X(t)|^q\right]$ is finite for some t, then it is finite for all t. Moreover, since $\mathbb{E}\left[|X(t)|^q\right]$ depends on $\mathbb{E}\left[\theta(t)^{Hq}\right]$, it is finite if and if only the process $\theta(t)$ has finite moment of order Hq. Hence, the trading time *controls* the moment of the return $X(t)$.

Since the trading time is generated by a multifractal random measure, the choice of using either a microcanonical or a canonical multifractal measure influences the process features. As a matter of fact, microcanonical measures have a fixed mass on the interval $[0, T]$ on which are defined. So $\theta(t)$ is bounded, and the compound process $X(t)$ has finite moments of all order. As Mandelbrot pointed out in [1] and [11], it generates *mild* randomness, with relatively thin tails.

On the other hand, being canonical measure depending on the random variable Υ, it permits the model to have divergent moments. This eventuality is analyzed in 4. The corresponding process $X(t)$ will be then *wild*. Overall, the MMAR has enough flexibility to allow for a wide variety of tail behaviour, both thin and fat.

4 Analytical properties of the scaling function $\tau(q)$

In the this section all the relevant properties of the scaling function $\tau(q)$ are analysed: These are its zeros and concavity/convexity. Mandelbrot in [14] gave a proof of the concavity of the scaling function only for the canonical measure with $b = 2$ (that is the dyadic case). Hereinafter, the proof for every $b \in \mathbb{N} : b \geq 2$ is given. Furthermore, special attention is addressed to the asymptotic behavior of the function, and its connections within the MMAR.

Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where canonical measure is defined, starting from the definition of the scaling function

$$\tau(q) := -\log_b \mathbb{E}\left(U^q\right) - 1,$$

where U is a discrete random variable extracted form the sequence $(U_j)_{j=\{0,1,\ldots,b-1\}}$ (which are iid). Thus, the q-th moment can be written as

$$\tau(q) = -\log_b \left(\sum_{j=0}^{b-1} \mathbb{P}\left(U = u_j\right) \cdot u_j^q \right) - 1$$

where u_j are the values taken by the random variables.

Theorem 2. *: The points* $(0, -1)$ *and* $(1, 0)$ *are respectively the intercept and a zero for the scaling function* $\tau(q)$.

Proof. Setting $q = 0$, it follows

$$\tau(0) = -\log_b \left(\sum_{j=0}^{b-1} \mathbb{P}\left(U = u_j\right) \cdot u_j^0 \right) - 1 = -\log_b \left(\sum_{j=0}^{b-1} \mathbb{P}\left(U = u_j\right) \right) - 1 =$$
$$= -\log_b\left(1\right) - 1 = -1,$$

hence the point $(0, -1)$ is the intercept of the scaling function. Studying the first moment of the random variable, which corresponds to the value $q = 1$, the trivial zero of τ is given. In fact

$$\tau(1) = -\log_b \left(\sum_{j=0}^{b-1} \mathbb{P}\left(U = u_j\right) \cdot u_j \right) - 1 = -\log_b \left(\frac{1}{b} \right) - 1 =$$
$$= -(-1) - 1 = 0,$$

since $\mathbb{E}(U) = \frac{1}{b}$. Thus the point $(1, 0)$ is one of the zeros of the function $\tau(q)$. □

On the existence of other zeros, it will be discussed later. The following theorems (Theorem 3, 4, and 5), with the related proofs, are new to the literature. General conditions about the increasing/decreasing paths and about the concavity/convexity of the scaling function are derived. These conditions provide a possible explanations about the fact that the canonical measure is able to produce fat tails in the distribution of the measure, thus connecting multifractal formalism with jump processes.

Theorem 3. *: The scaling function $\tau(q)$ is* non-decreasing *if the measure is microcanical; if the measure is canonical the function might exhibit* both increasing and decreasing *regions.*

Proof. The first derivative of $\tau(q)$ is such that

$$\tau'(q) = \frac{\mathrm{d}\left[-\log_b \left(\sum_{j=0}^{b-1} \mathbb{P}\left(U = u_j\right) \cdot u_j^q \right) - 1 \right]}{\mathrm{d}q} =$$
$$= -\log_b(e) \cdot \frac{\sum_{j=0}^{b-1} \mathbb{P}\left(U = u_j\right) \cdot u_j^q \cdot \ln(u_j)}{\sum_{j=0}^{b-1} \mathbb{P}\left(U = u_j\right) \cdot u_j^q}.$$

Since $b \in \mathbb{N} \setminus \{0, 1\}$, $-\log_b(e)$ is always negative. Moreover, since $U_j \geq 0$ for all j, all the moments of the random variable are positive[5]. Hence the denominator is positive. The only entity whose sign may vary is the numerator of the fraction, whose positivity/negativity is due to the quantities $\ln(u_j)$. If we consider the microcanical measure, that is $0 \leq U_j \leq 1$, the logarithms are all negative, making the numerator negative as well. In this case, we find $\tau'(q) \geq 0$, being so *non-decreasing*. However, if the measure is canonical, hence allowing $U_j \geq 0$, it is not possible to state a priori whether the scaling function is increasing or decreasing. □

[5] They would be exactly equal to zero only in the degenerate case, that is if the random variables were all null.

Theorem 4. *: Regardless the measure is microcanical or canonical, the scaling function $\tau(q)$ is* convcave*.*

Proof. : Studying the concavity/convexity of $\tau(q)$ through the second derivative

$$\tau''(q) = \frac{\mathrm{d}^2\left[-\log_b\left(\sum_{j=0}^{b-1}\mathbb{P}\left(U=u_j\right)\cdot u_j^q\right)-1\right]}{\mathrm{d}q^2} =$$
$$= -\log_b(e)\cdot\frac{\sum_{j=0}^{b-1}\sum_{i=j+1}^{b-1}\mathbb{P}\left(U=u_j\right)\cdot\mathbb{P}\left(U=u_i\right)\cdot u_j^q\cdot u_i^q\cdot\left[\ln\left(u_j\right)-\ln\left(u_i\right)\right]^2}{\left(\sum_{j=0}^{b-1}\mathbb{P}\left(U=u_j\right)\cdot u_j^q\right)^2},$$

shows that the fraction is always positive. Thus, the second derivative is always negative. Hence $\tau''(q) < 0$ and the scaling function is always *concave*, despite the fact that the measure is microcanonical or canonical. □

The last features we are interested in investigating is the existence of asymptotes for the function $\tau(q)$.

Theorem 5. *: Regardless the measure is microcanical or canonical, the scaling function $\tau(q)$ is* asymptotic linear *both for $q \to -\infty$ and $q \to +\infty$.*

Proof. : First, the function is unbounded above and below. In fact

$$\lim_{q\to+\infty}\tau(q) = \lim_{q\to+\infty}\left[-\log_b\left(\sum_{j=0}^{b-1}\mathbb{P}\left(U=u_j\right)\cdot u_j^q\right)-1\right] = \begin{cases}+\infty & \text{if } 0\le U_j\le 1\\ -\infty & \text{if } U_j\ge 0\end{cases}$$

and

$$\lim_{q\to-\infty}\tau(q) = \lim_{q\to-\infty}\left[-\log_b\left(\sum_{j=0}^{b-1}\mathbb{P}\left(U=u_j\right)\cdot u_j^q\right)-1\right] = -\infty.$$

Hence, there might be at most two *oblique* asymptotes (one for $q \to -\infty$ and one for $q \to +\infty$). Since the previous limits are necessary conditions but not sufficient, to individuate the asymptotes (if any), the following limit (which calculate the slope of the asymptote) need to be calculated

$$\lim_{q\to+\infty}\frac{\tau(q)}{q} = \lim_{q\to+\infty}\frac{-\log_b\left(\sum_{j=0}^{b-1}\mathbb{P}\left(U=u_j\right)\cdot u_j^q\right)-1}{q} =$$
$$= -\lim_{q\to+\infty}\frac{\log_b\left(\sum_{j=0}^{b-1}\mathbb{P}\left(U=u_j\right)\cdot u_j^q\right)}{q}.$$

Since, as $q \to +\infty$

$$\log_b\left(\sum_{j=0}^{b-1}\mathbb{P}\left(U=u_j\right)\cdot u_j^q\right) \sim \log_b\Big(\mathbb{P}\left(U=u_{max}\right)\cdot u_{max}^q\Big),$$

where $u_{max} = \max_{j\in\{0,1,\ldots,b-1\}}(u_j)$, it follows

$$\begin{aligned}\lim_{q\to+\infty}\frac{\tau(q)}{q} &= -\lim_{q\to+\infty}\frac{\log_b\Big(\mathbb{P}\left(U=u_{max}\right)\cdot u_{max}^q\Big)}{q} = \\ &= -\lim_{q\to+\infty}\frac{\log_b\big(\mathbb{P}\left(U=u_{max}\right)\big)+q\cdot\log_b\left(u_{max}\right)}{q} = \\ &= -\log_b\left(u_{max}\right) < \infty\,.\end{aligned}$$

To ascertain that there is an oblique asymptote actually, the computation of the following limit (which calculate the intercept of the asymptote) is required

$$\begin{aligned}\lim_{q\to+\infty}&\Big[\tau(q)-q\cdot\left[-\log_b\left(u_{max}\right)\right]\Big] = \lim_{q\to+\infty}\Big[\tau(q)+q\cdot\log_b\left(u_{max}\right)\Big] = \\ &= \lim_{q\to+\infty}\left[-\log_b\left(\sum_{j=0}^{b-1}\mathbb{P}\left(U=u_j\right)\cdot u_j^q\right)-1+q\cdot\log_b\left(u_{max}\right)\right] = \\ &= \lim_{q\to+\infty}\Big[-\log_b\Big(\mathbb{P}\left(U=u_{max}\right)\cdot u_{max}^q\Big)-1+q\cdot\log_b\left(u_{max}\right)\Big] = \\ &= \lim_{q\to+\infty}\big[-\log_b\big(\mathbb{P}\left(U=u_{max}\right)\big)-q\cdot\log_b\left(u_{max}\right)-1+q\cdot\log_b\left(u_{max}\right)\big] = \\ &= -\log_b\big(\mathbb{P}\left(U=u_{max}\right)\big)-1 < \infty\,.\end{aligned}$$

Thus, for $q\to+\infty$, the scaling function $\tau(q)$ is asymptotic to the straight line with equation

$$a_1(q) = -\log_b\left(u_{max}\right)\cdot q - \log_b\big(\mathbb{P}\left(U=u_{max}\right)\big)-1\,. \tag{7}$$

The same procedure has to be repeated for $q\to-\infty$.

$$\begin{aligned}\lim_{q\to-\infty}\frac{\tau(q)}{q} &= \lim_{q\to-\infty}\frac{-\log_b\left(\sum_{j=0}^{b-1}\mathbb{P}\left(U=u_j\right)\cdot u_j^q\right)-1}{q} = \\ &= -\lim_{q\to-\infty}\frac{\log_b\left(\sum_{j=0}^{b-1}\mathbb{P}\left(U=u_j\right)\cdot u_j^q\right)}{q}\,.\end{aligned}$$

Because, as $q\to-\infty$

$$\log_b\left(\sum_{j=0}^{b-1}\mathbb{P}\left(U=u_j\right)\cdot u_j^q\right) \sim \log_b\Big(\mathbb{P}\left(U=u_{min}\right)\cdot u_{min}^q\Big)\,,$$

where $u_{min} = \min_{j\in\{0,1,\ldots,b-1\}}(u_j)$, the limit simplifies as

$$\begin{aligned}\lim_{q\to-\infty}\frac{\tau(q)}{q} &= -\lim_{q\to-\infty}\frac{\log_b\Big(\mathbb{P}\left(U=u_{min}\right)\cdot u_{min}^q\Big)}{q} = \\ &= -\lim_{q\to-\infty}\frac{\log_b\big(\mathbb{P}\left(U=u_{min}\right)\big)+q\cdot\log_b\left(u_{min}\right)}{q} = \\ &= -\log_b\left(u_{min}\right) < \infty\,.\end{aligned}$$

The limit for the existence of the intercept is

$$\begin{aligned}
&\lim_{q\to-\infty}\Big[\tau(q)-q\cdot[-\log_b(u_{min})]\Big]=\lim_{q\to-\infty}\Big[\tau(q)+q\cdot\log_b(u_{min})\Big]=\\
&\quad=\lim_{q\to-\infty}\left[-\log_b\left(\sum_{j=0}^{b-1}\mathbb{P}(U=u_j)\cdot u_j^q\right)-1+q\cdot\log_b(u_{min})\right]=\\
&\quad=\lim_{q\to-\infty}\Big[-\log_b\Big(\mathbb{P}(U=u_{min})\cdot u_{min}^q\Big)-1+q\cdot\log_b(u_{min})\Big]=\\
&\quad=\lim_{q\to-\infty}\Big[-\log_b\big(\mathbb{P}(U=u_{min})\big)-q\cdot\log_b(u_{min})-1+q\cdot\log_b(u_{min})\Big]=\\
&\quad=-\log_b\big(\mathbb{P}(U=u_{min})\big)-1<\infty\,.
\end{aligned}$$

Hence, even for $q\to-\infty$, the scaling function $\tau(q)$ has another slant asymptote, that is

$$a_2(q)=-\log_b(u_{min})\cdot q-\log_b\big(\mathbb{P}(U=u_{min})\big)-1\,. \tag{8}$$

□

The slope and the intercept of the asymptotes allow to make further statements about the behaviour of the scaling function. Consider (7) and (8). The intercept of the left (right) asymptote is positive if and only if $\mathbb{P}(U=u_{min})<1/b$ ($\mathbb{P}(U=u_{max})<1/b$). The implications for the right asymptote are however more interesting (see *infra*, about the existence of other zeros).

About the slope of the asymptotes, the inclination varies according to the fact the measure is microcanonical or canonical for $q\to+\infty$. In fact, for a_1, the rate of growth $-\log_b(u_{max})$ is *positive* if we are dealing with microcanical measures (since $0\le U_j\le 1$ a.s., that is $u_{max}<1$), but becomes *negative* if the measure is canonical (since $U_j\ge 0$ a.s., thus being $u_{max}>1$). On the contrary, for a_2 the slope of the asymptote is always *positive*, since it must be the case that $u_{min}<1$, both in the microcanonical case and the canonical one.

Hereunder, all the relevant properties of the scaling function are summarised:

1. The point $(1,0)$ is a zero for $\tau(q)$;
2. The point $(0,-1)$ is the intercept of $\tau(q)$;
3. If the measure is microcanonical, the function is non-decreasing;
4. The function is concave;
5. The function is asymptotical linear

$$\begin{aligned}
\tau(q)&\sim-\log_b(u_{max})\cdot q-\log_b\big(\mathbb{P}(U=u_{max})\big)-1\quad\text{for } q\to+\infty\\
\tau(q)&\sim-\log_b(u_{min})\cdot q-\log_b\big(\mathbb{P}(U=u_{min})\big)-1\quad\text{for } q\to-\infty.
\end{aligned}$$

These properties allow us to infer another property of $\tau(q)$ if a canonical measure is involved. Since the function is always concave and also asymptotic, for $q\to+\infty$, to the straight line which has a negative slope, thus it must have

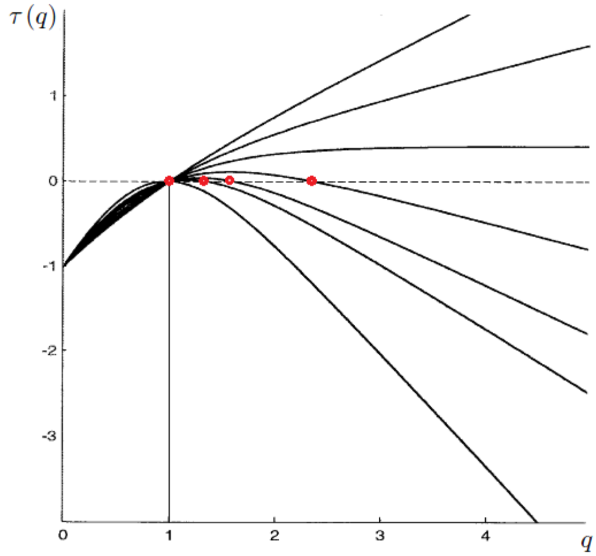

Fig. 1. Possible shapes of $\tau(q)$ for different sequences (U_j). Red dots are different q_{crit}.

a "cap" form, such as $\bigcap$. Moreover, since the function as a zero for $q = 1$, there must exist another one, which is usually addressed as q_{crit}, that is

$$q_{crit} := \{q > 1 : \tau(q) = 0\} \,,$$

remarking that the second zero $(q_{crit}, 0)$ exists *only* if a canonical measures has been chosen. That zero has a great impact on the finiteness of the moments of the measure. As a matter of fact Mandelbrot in [14] showed that, for $q > 1$, the moments of the measure are finite if and only if $\tau(q) > 0$. That eventuality occurs only for $1 < q < q_{crit}$.

In fact, since the q-th moment of a canonical measure is given by

$$\mathbb{E}\big[\mu\,(\Delta t)^q\big] = \mathbb{E}\,(\Upsilon^q) \cdot \Delta t^{\tau(q)+1} \,,$$

its finiteness may depends only on either $\tau(q)$ or $\mathbb{E}\,(\Upsilon^q)$. Since, for finite q, $\tau(q)$ is finite a.s., hence the infiniteness can be achieved only by a particular behaviour of the random variable Υ which is the random mass on the interval $[0, 1]$.

Indeed, Guivarc'h proved in [6] that Υ has Paretian tails and allows infinite moments, for those values $q \geq q_{crit}$. Thus, the random variable Υ follows a Pareto's distribution of exponent q_{crit}.

5 Conclusions and future developments

In order to study further properties of $\tau(q)$ and thus draw the sought conclusion, the definition of *local Hölder exponent* of a function is required. This quantity characterises the smoothness of a function at a given point.

Definition 4. *Let $g : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on a neighborhood of a given point t. The number*

$$\alpha(t) := \sup\left(\beta > 0 : |g(t+h) - g(t)| = O\left(|h|^{\beta}\right)\right)$$

as $h \to 0$, is called the local Hölder exponent *of g at t.*

We note that $\alpha(t_0)$ is non-negative if and only if the function g is bounded around t_0. Local Hölder exponent is a notion that can be well applied to functions and measures, deterministic or stochastic, with some adjustment. In the case of stochastic processes, the local Hölder exponent depends on the particular sample path considered. There are however some famous exceptions: continuous stochastic process such as Brownian motion and fractional Brownian motion are characterized by a *unique* Hölder exponent almost everywhere, for almost all sample paths. Differently, for the MMAR a continuum of Hölder exponent is allowed.

Defining log-returns in the following way

$$X(t, \Delta t) := X(t) - X(t - \Delta t),$$

where $X(t)$ is defined as in (5), it follows

$$\mathbb{E}\big[|X(t, \Delta t)|\big] \sim \Delta t^{\alpha(t)} \tag{9}$$

for all t, as $\Delta t \to 0$. In the standard Brownian motion and in the standard fractional Brownian motion cases, the local absolute variation are always proportional to $\Delta t^{1/2}$ and to Δt^{H} respectively. On the other hand, multifractal processes generate variety in local regularity while filling with a *continuum* of values of $\alpha(t)$. Given that the local Hölder exponent may vary from sample path to sample path, it is not a robust statistical tool for characterising the roughness of a stochastic process. The notion of *multifractal/singularity spectrum* was introduced to give a less detailed but more stable characterisation of the local smoothness of a function in a "statistical" sense.

Definition 5. *Let $g : \mathbb{R} \to \mathbb{R}$ be a real-valued function and for each $\alpha > 0$ define the set of point at which g has Hölder exponent $\alpha(t)$*

$$\Lambda(\alpha) := \{t, \alpha(t) = \alpha\} .$$

The multifractal/singularity spectrum *of g is the function $\tau^{\star} : \mathbb{R}^{+} \to \mathbb{R}$ which associates to each $\alpha > 0$ the Hausdorff-Besicovich dimension of $\Lambda(\alpha)$:*

$$\tau^{\star}(\alpha) := \dim_{HB} \Lambda(\alpha) .$$

Using the above definition, one may associate to each sample path $X(\omega, t)$ of a stochastic process $X(t)$ its singularity spectrum. If it depends on the path ω, then the singularity spectrum is not likely to give much information about the properties of the process $X(t)$. Fortunately, it has been shown that, for large classes of stochastic processes, the singularity spectrum is the same for almost all sample paths. Moreover, for multifractal process it has been proven that the multifractal spectrum is equivalent to the the *Legendre transform* of the scaling function (which is easier to compute than the spectrum). Hence, for such processes, the Legendre transform is the main tool for describing the distribution of local Hölder exponents. In fact

$$\tau^{\star}(\alpha) = \inf_{q \in \mathbb{R}} \left\{ \alpha \cdot q - \tau(q) \right\},$$

where the expression on the right-hand side is the Legendre transform of τ. Given these premises and given the result of Theorem 5, a few statements about the set of local Hölder exponents can be done.

Due to the asymptotic relations

$$\begin{aligned} \tau(q) &\sim -\log_b\left(u_{max}\right) \cdot q - \log_b\big(\mathbb{P}\left(U = u_{max}\right)\big) - 1 \quad \text{for } q \to +\infty \\ \tau(q) &\sim -\log_b\left(u_{min}\right) \cdot q - \log_b\big(\mathbb{P}\left(U = u_{min}\right)\big) - 1 \quad \text{for } q \to -\infty, \end{aligned}$$

the slope of the oblique asymptotes are respectively

$$\alpha_{max} = -\log_b\left(u_{min}\right) > 0 \quad \text{and} \quad \alpha_{min} = -\log_b\left(u_{max}\right) \lesseqgtr 0\,.$$

The relation max/min is inverted since the the slope of the asymptote for $q \to -\infty$ is grater than the one of the asymptote for $q \to +\infty$. Moreover, these two values give the bounds of the support of the multifractal spectrum, that is

$$\tau^{\star} : [\alpha_{min}, \alpha_{max}] \to \mathbb{R}\,,$$

since they are the least and the highest value α can take. Hence, if the scaling function $\tau(q)$ is defined on the entire real line, its asymptotic linear behaviour implies the multifractal spectrum $\tau^{\star}(\alpha)$ to be defined only on a closed set of values. Note that, if the measure involved in the process physical time in trading time is canonical, hence $[\alpha_{min}, \alpha_{max}] \not\subset \mathbb{R}^{+}$ since $\alpha_{min} < 0$ (as the definition of the multifractal spectrum requires). Hence the multifractal spectrum should be considered as the restriction of that set up to zero. However, the negative values of alpha have a deeper financial meaning: When the local local Hölder exponents become negative, it must be the case the process exhibits *jumps*. Thus choosing a canonical measure for deforming time implies the presence of fat tails (the moments diverge for $q > q_{crit}$) and jumps in the sample paths ($\alpha_{min} < 0$).

Since (9) holds, the largest the range of possible $\alpha(t)$, the riskiest is the asset since the local Hölder exponents can take more values. It conveys a more variability of the log-returns, and hence an higher uncertainty in the magnitude of future price variations. Moreover, the smaller the value of α_{min}, the riskiest the asset should be considered.

The asymptotic behavior of the scaling function and the connection with the αs through the Legendre transform should lead to reconsider the information connected with the higher moments of the log-returns' distribution. As a matter of fact, the right asymptote of $\tau(q)$ might have positive or negative slope according to the intrinsic riskiness of the asset. Since this quantity is linked to the higher moments of the distribution $(q \to +\infty)$, more consistent estimation techniques[6] ought to be developed being α_{min} the most ruinous local exponent that can occur. As a matter of fact, on the drawback of these estimation methods is that their finite sample properties are not well known.

Moreover, a more in-depth study of the link between the quantity $\mathbb{P}(U = u_{max})$ and the choice of the distribution for deforming time would be necessary. Since U_j are random, a suitable distribution should be chosen [7] for practical purpose. Subsequently, the distribution of U_j would require special attention since the maximum value of u_{max} defines α_{min}.

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[6] The authors in [3] suggested a procedure for the values of the scaling function based on multiple regressions at different time-scales

$$\mathbb{E}\Big[\ln P(q, \Delta t)\Big] - \ln(T) = \ln c_X(q) + \tau_X(q) \cdot \ln(\Delta t)\,,$$

where $P(q, \Delta t) := \sum_{i=1}^{N} \big|X(i\Delta t, \Delta t)\big|^q$, having divided the interval $[0, T]$ into N subintervals of length Δt.

[7] In [2], the authors analyzed different possibilities for the multipliers' distribution. Mandelbrot proved that, in order to have canonical measure and hence allowing fat tails, the most immediate distribution to be chosen is the log-normal.

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