Asymmetric information in a market with $n + 1$ Brownian motions

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Abstract. This paper covers asymmetric information in financial markets from a micro perspective. Particularly, we aim to extend the asset pricing framework introduced by Guasoni [2], which models price dynamics with both a martingale component, described by permanent shocks, and a stationary component, given by temporary shocks. First, we derive a generalization of this asset pricing model using n Brownian Motions, including an Ornstein-Uhlenbeck process as the $(n + 1)th$ element. We find non-Markovian dynamics for the partially informed agents, which questions the validity of the efficient market hypothesis. Moreover, we compare the positions of informed and partially informed agents. Thereby, the filtration for informed agents is larger and initially specified, whereas the filtration for partially informed agents is smaller and obtained from the Hitsuda representation [3]. Our study examines the logarithmic utility maximization problem.

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1 Introduction: the model

We consider a financial market with one riskless asset D and one risky asset S. The market interest rate is considered deterministic. In order to describe the dynamics of the risky asset, we consider a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ on which are defined $n + 1$, with $n \in N$, independent Brownian Motions:

$$\left(B_t^1\right)_{t\in[0,+\infty[}, \left(B_t^2\right)_{t\in[0,+\infty[}, ..., \left(B_t^n\right)_{t\in[0,+\infty[}, \left(B_t^{n+1}\right)_{t\in[0,+\infty[}.$$

If we set the real parameter $\lambda_{n+1} > 0$, we consider the Ornstein-Uhlenbeck process $\left(U_t^{n+1}\right)_{t\in[0,+\infty)}$ defined by the following equation

$$U_t^{n+1} + \lambda_{n+1} \int_0^t U_s^{n+1} ds = B_t^{n+1}, \qquad t \in [0,+\infty[$$

which, as known, is given by:

$$U_t^{n+1} = \int_0^t e^{-\lambda_{n+1}(t-s)} dB_s^{n+1} \ .$$

Then, if we set the real numbers p_j, with $j = 1, 2, ..., n, n+1$, $p_{n+1} > 0$, and the first n numbers not all zero, let us consider the process $(Y_t)_{t\in[0,+\infty)}$ defined by:

$$Y_t = \sum_{j=1}^{n} p_j B_t^j + p_{n+1} U_t^{n+1} \ .$$

Now, let us introduce two deterministic Lebesgue measurable functions

$$\mu, \sigma : [0,+\infty[\longrightarrow [0,+\infty[$$

such that

$$\forall\, T > 0 \qquad \mu \in L^1\left([0,T]\right), \quad \sigma \in L^2\left([0,T]\right).$$

The price dynamics of S of the risky asset evolve according to:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dY_t$$

and its solution is

$$S_t = S_0 e^{\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dY_s} .$$

Now we can describe the previous situation in the following way: we have "informed agents" who have all the information provided from the Brownian Motions, and a "partially informed agents" who have all the information provided from the process Y_t. The informed agents refer to the filtration $\left(\mathcal{F}_t^1\right)_{t\in[0,+\infty[}$ obtained by completing the natural filtration generated by $n+1$ Brownian Motions $B_t^1, B_t^2, ..., B_t^n, B_t^{n+1}$, which therefore satisfies the usual conditions of completeness and right continuity. The partially informed agents, instead, refer to the filtration $\left(\mathcal{F}_t^0\right)_{t\in[0,+\infty[}$ generated by the process Y_t. Of course, we have that $\mathcal{F}_t^0 \subset \mathcal{F}_t^1, \ \forall\, t$. We might state that the informed agents' risky asset value evolves according to the assigned Brownian Motions; therefore its value is determined as:

$$\frac{dS_t}{S_t} = \left(\mu_t - p_{n+1}\lambda_{n+1}\sigma_t U_t^{n+1}\right) dt + \sigma_t \sum_{j=1}^{n+1} p_j dB_t^j$$

which refers to the Brownian Motion

$$W_t = \left[\sum_{j=1}^{n+1} p_j^2\right]^{-\frac{1}{2}} \sum_{j=1}^{n+1} p_j B_t^j \tag{1}$$

The solution of S_t, if $S_0 > 0$ is the initial wealth, is given, as known, by:

$$S_0 e^{\int_0^t \left[\left(\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}\right) - \frac{1}{2}\left(\sum_{j=1}^{n+1} p_j^2\right)\sigma_s^2\right] ds + \left[\sum_{j=1}^{n+1} p_j^2\right]^{\frac{1}{2}} \int_0^t \sigma_s dW_s}.$$

In the next section, we want to derive, for the partially informed agents, an analogous equation which represents Y_t, and therefore S, in terms of filtration $\mathcal{F}^0$ and of an opportune Brownian Motion B^0.

2 Decomposition of the Process Y_t with respect to $\mathcal{F}^0$

In this section, we shall examine the Markov property of Y_t and will determine, with respect to the filtration $\mathcal{F}^0$, the relative Brownian Motion which represents it.

Theorem 1. *Let Y_t be a Gaussian process, and moreover:*

1. $E(Y_t) = 0 \quad \forall t \in [0, +\infty[$
2. $\Gamma(s,t) := cov(Y_s, Y_t) = \left(\sum_{j=1}^{n} p_j^2\right) t \wedge s + p_{n+1}^2 \frac{e^{-\lambda_{n+1}|t-s|} - e^{-\lambda_{n+1}(t+s)}}{2\lambda_{n+1}}$.

Proof. 1. Gaussian and mean zero properties are obvious.
2.

$$\Gamma(s,t) := cov(Y_s, Y_t) = E(Y_s Y_t) =$$

$$= E\left(\left[\sum_{j=1}^{n} p_j B_s^j + p_{n+1} U_s^{n+1}\right]\left[\sum_{j=1}^{n} p_j B_t^j + p_{n+1} U_t^{n+1}\right]\right) =$$

applying the Brownian Motion's independence property, we get:

$$= E\left(\sum_{j=1}^{n} p_j^2 B_s^j B_t^j\right) + E\left(p_{n+1}^2 U_s^{n+1} U_t^{n+1}\right) =$$

$$= \sum_{j=1}^{n} p_j^2 E\left(B_s^j B_t^j\right) + p_{n+1}^2 E\left(\int_0^s e^{-\lambda_{n+1}(s-u)} dB_u^{n+1} \int_0^t e^{-\lambda_{n+1}(t-u)} dB_u^{n+1}\right) =$$

$$= \left(\sum_{j=1}^{n} p_j^2\right) s \wedge t + p_{n+1}^2 \int_0^{s\wedge t} e^{-\lambda_{n+1}(t-u) - \lambda_{n+1}(s-u)} du =$$

$$= \left(\sum_{j=1}^{n} p_j^2\right) s \wedge t + p_{n+1}^2 e^{-\lambda_{n+1}(t+s)} \frac{e^{2\lambda_{n+1}(s\wedge t)} - 1}{2\lambda_{n+1}} =$$

$$= \left(\sum_{j=1}^{n} p_j^2\right) s \wedge t + p_{n+1}^2 \frac{e^{-\lambda_{n+1}|t-s|} - e^{-\lambda_{n+1}(t+s)}}{2\lambda_{n+1}} \quad \square$$

To verify the Markov property of the process Y_t, we recall the following result [4] (III.1.13)

Theorem 2. *Let Y_t be a Markov process if, and only if:*

$$\Gamma(s,t)\Gamma(t,u) = \Gamma(t,t)\Gamma(s,u), \quad \forall\ s \leq t \leq u. \tag{2}$$

Theorem 3. *1. Let $\lambda_{n+1} = 0$, then Y_t is a Markov process.*
2. Let $\lambda_{n+1} > 0$ then Y_t is a Markov process if, and only if: $p_j = 0 \quad \forall j = 1, 2, ..., n$ or $p_{n+1} = 0$.

Proof. Property 1 is obvious. Besides it is obvious that Y_t is a Markov process if $p_j = 0 \quad \forall j = 1, 2, ..., n$ or if $p_{n+1} = 0$. We suppose that $\sum_{j=1}^{n} p_j^2 > 0$ and consider (2).

Considering

$$\lim_{u\to+\infty} \Gamma(s,t)\Gamma(t,u)$$

and

$$\lim_{u\to+\infty} \Gamma(t,t)\Gamma(s,u)$$

we get:

$$tp_{n+1}^2 \frac{e^{-\lambda_{n+1}(t-s)} - e^{-\lambda_{n+1}(t+s)}}{2\lambda_{n+1}} = sp_{n+1}^2 \frac{1 - e^{-2\lambda_{n+1}t}}{2\lambda_{n+1}}$$

which can be written also as follows

$$p_{n+1}^2 \left[\frac{e^{\lambda_{n+1}s} - e^{-\lambda_{n+1}s}}{s} - \frac{e^{\lambda_{n+1}t} - e^{-\lambda_{n+1}t}}{t}\right] = 0, \qquad \forall s \leq t$$

and considering the limit for $t \to +\infty$, we get the thesis: $p_{n+1} = 0$. $\square$

Now let us consider the process Z_t defined by:

$$Z_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{-\frac{1}{2}} \sum_{j=1}^{n} p_j \left(B_t^j + \lambda_{n+1}\int_0^t B_u^j du\right) + \left(\sum_{j=1}^{n+1} p_j^2\right)^{-\frac{1}{2}} p_{n+1} B_t^{n+1}.$$

It verifies the following result:

Theorem 4. *1. Let Z_t be a Gaussian process.*
2. $E(Z_t) = 0 \quad \forall t \in [0, +\infty[$.
3. $cov(Z_t, Z_s) = t \wedge s +$

$$\left(\sum_{j=1}^{n+1} p_j^2\right)^{-1} \left(\sum_{j=1}^{n} p_j^2\right) \int_0^t \int_0^s \left(\lambda_{n+1} + \lambda_{n+1}^2 u \wedge v\right) du dv$$

Proof. We note that the Z_t process can be rewritten in the form:

$$Z_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{-\frac{1}{2}} \sum_{j=1}^{n} p_j \left(B_t^j + \lambda_{n+1} \int_0^t (t-u)\, dB_u^j\right) + \left(\sum_{j=1}^{n+1} p_j^2\right)^{-\frac{1}{2}} p_{n+1} B_t^{n+1} =$$

$$= \left(\sum_{j=1}^{n+1} p_j^2\right)^{-\frac{1}{2}} \sum_{j=1}^{n} p_j \int_0^t \left[1 + \lambda_{n+1}(t-u)\right] dB_u^j + \left(\sum_{j=1}^{n+1} p_j^2\right)^{-\frac{1}{2}} p_{n+1} B_t^{n+1}.$$

Therefore the covariance, because of the independence of the Brownian Motions, is given by:

$$cov(Z_t, Z_s) = \left(\sum_{j=1}^{n+1} p_j^2\right)^{-1} \sum_{j=1}^{n} p_j^2 \int_0^{t\wedge s} \left[1 + \lambda_{n+1}(t-u)\right]\left[1 + \lambda_{n+1}(s-u)\right] du +$$

$$+ \left(\sum_{j=1}^{n+1} p_j^2\right)^{-1} p_{n+1}^2 \; t \wedge s$$

with standard calculations, we obtain the thesis. □

Now let us consider the following function:

$$\tilde{f}(t,s) = -\left(\sum_{j=1}^{n+1} p_j^2\right)^{-1} \left(\sum_{j=1}^{n} p_j^2\right) \left(\lambda_{n+1} + \lambda_{n+1}^2 t \wedge s\right) \tag{3}$$

which is part of the covariance of the process Z_t. For our further aims, if $0 \le s \le t$, then we can rewrite the (3):

$$\tilde{f}(t,s) = -\left(\sum_{j=1}^{n+1} p_j^2\right)^{-1} \left(\sum_{j=1}^{n} p_j^2\right) \left(\lambda_{n+1} + \lambda_{n+1}^2 s\right) \quad \forall\, 0 \le s \le t. \tag{4}$$

To simplify, if $A^2 = \left(\sum_{j=1}^{n+1} p_j^2\right)^{-1} \left(\sum_{j=1}^{n} p_j^2\right)$, we can consider the following result:

Theorem 5. *The function*

$$\tilde{g}\left(t,s\right)=\begin{cases}\lambda_{n+1}\eta(s) & \textit{for } 0\leq s\leq t\\ \\ 0 & \textit{otherwise}\end{cases}\tag{5}$$

verifies the following integral equation

$$\tilde{f}\left(t,s\right)=\tilde{g}\left(t,s\right)-\int_{0}^{s}\tilde{g}\left(t,u\right)\tilde{g}\left(s,u\right)du\qquad\forall\ 0\leq s\leq t\tag{6}$$

and $\eta\left(s\right)$ verifies the following Cauchy problem

$$\begin{cases}\eta'\left(s\right)=\lambda_{n+1}\left(\eta^{2}(s)-A^{2}\right)\\ \eta\left(0\right)=-A^{2}.\end{cases}$$

Proof. It is easy to verify this, considering the following integral equation:

$$-A^{2}\left(\lambda_{n+1}+\lambda_{n+1}^{2}s\right)=\lambda_{n+1}\eta\left(s\right)-\lambda_{n+1}^{2}\int_{0}^{s}\eta^{2}\left(u\right)du\qquad\forall\ 0\leq s\leq t$$

from which we easily obtain the Cauchy problem. Its solution, as already verified, is given by the function:

$$\eta\left(s\right)=A\frac{1-A-\left(1+A\right)e^{2A\lambda_{n+1}s}}{1-A+\left(1+A\right)e^{2A\lambda_{n+1}s}}.\ \square$$

Now we are able to enunciate the following theorem:

Theorem 6. *Let $g(t,s)$ be the negative resolvent of $\tilde{g}\left(t,s\right)$ defined by*

$$g\left(t,s\right)=\begin{cases}-\lambda_{n+1}\eta(s)e^{\lambda_{n+1}\int_{s}^{t}\eta(u)du} & \textit{for } 0\leq s\leq t\\ \\ 0 & \textit{otherwise}\end{cases}$$

then the process

$$B_{t}^{0}=Z_{t}-\int_{0}^{t}\left(\int_{0}^{s}g\left(s,u\right)dZ_{u}\right)ds\tag{7}$$

is a Brownian Motion with respect to the filtration $\left(\mathcal{F}_{t}^{0}\right)_{t\in[0,+\infty[}$.

Also we have that

$$Z_{t}=B_{t}^{0}-\int_{0}^{t}\left(\int_{0}^{s}\tilde{g}\left(s,u\right)dB_{u}^{0}\right)ds=B_{t}^{0}-\lambda_{n+1}\int_{0}^{t}\left(\int_{0}^{s}\eta\left(u\right)dB_{u}^{0}\right)ds.$$

Proof. From Proposition 2 in [3], it follows the existence of the Brownian Motion B_{t}^{0}. $\square$

Theorem 7. *Let the processes Y_t and Z_t verify the following equation:*

$$Y_t + \lambda_{n+1} \int_0^t Y_u du = \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} Z_t$$

so we have

$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} \int_0^t e^{-\lambda_{n+1}(t-u)} dZ_u. \tag{8}$$

Proof. Since:

$$Y_t - \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} Z_t =$$

$$= \sum_{j=1}^{n} p_j B_t^j + p_{n+1} U_t^{n+1} - \sum_{j=1}^{n} p_j \left(B_t^j + \lambda_{n+1} \int_0^t B_u^j du \right) - p_{n+1} B_t^{n+1} =$$

$$= -\lambda_{n+1} \int_0^t \left[\sum_{j=1}^{n} p_j B_u^j + p_{n+1} U_u^{n+1} \right] du = -\lambda_{n+1} \int_0^t Y_u du.$$

then by integration we easily obtain (8). □

Now it is possible to establish the link between the process Y_t and the Brownian Motion B_t^0. Namely, we have the following (fundamental) result:

Theorem 8. *Let $Y_t = \sum_{j=1}^{n} p_j B_t^j + p_{n+1} U_t^{n+1}$ and $\left(\mathcal{F}_t^0\right)_{t\in[0,+\infty[}$ be the completed natural filtration of Y_t. Let B_t^0 be the Brownian Motion with respect to the filtration $\left(\mathcal{F}_t^0\right)_{t\in[0,+\infty[}$. Besides, we have the following results:*

1.

$$B_t^0 = \left(\sum_{j=1}^{n+1} p_j^2 \right)^{-\frac{1}{2}} \left[Y_t + \lambda_{n+1} \int_0^t \left(\int_0^s [1+\eta(u)] \, e^{\lambda_{n+1} \int_u^s \eta(l) dl} dY_u \right) ds \right].$$

2.

$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} \int_0^t \left[e^{-\lambda_{n+1}(t-u)} [1+\eta(u)] - \eta(u) \right] dB_u^0.$$

3.

$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} \left[B_t^0 - \lambda_{n+1} \int_0^t \left[e^{-\lambda_{n+1}} (t-u) \left(B_u^0 + \int_0^u \eta(v) dB_v^0 \right) \right] du \right].$$

4.

$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \left[B_t^0 - \lambda_{n+1} \int_0^t \left(\int_0^s [1+\eta(u)]\, e^{-\lambda_{n+1}(s-u)} dB_u^0\right) ds\right].$$

Proof. 1. Substituting

$$Z_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{-\frac{1}{2}} \left[Y_t + \lambda_{n+1} \int_0^t Y_u du\right]$$

in (7):

$$B_t^0 = \left(\sum_{j=1}^{n+1} p_j^2\right)^{-\frac{1}{2}} \left[Y_t + \int_0^t \left(\lambda_{n+1} Y_s - \int_0^s g(s,u) dY_u - \int_0^s g(s,u) Y_u du\right) ds\right]. \tag{9}$$

Integrating by parts

$$\int_0^s g(s,u) Y_u du$$

and if we suppose $G(s,u) = \displaystyle\int_0^u g(s,v) dv$, it is easy to have:

$$\int_0^s g(s,u) Y_u du = \lambda_{n+1} Y_s - \int_0^s G(s,u) dY_u \tag{10}$$

To obtain the result 1, we substitute (10) in (9).

2. In

$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \int_0^t e^{-\lambda_{n+1}(t-s)} dZ_s$$

we substitute Z_s with

$$Z_s = B_s^0 - \lambda_{n+1} \int_0^s \left(\int_0^u \eta\,(l)\, dB_l^0\right) du$$

so that we obtain

$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \left[\int_0^t e^{-\lambda_{n+1}(t-s)} dB_s^0 - \lambda_{n+1} \int_0^t \left(\int_0^s \eta(u) e^{-\lambda_{n+1}(t-s)} dB_u^0\right) ds\right] \tag{11}$$

which, applying Fubini Tonelli theorem, can be written as:

$$Y_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \left[\int_0^t e^{-\lambda_{n+1}(t-s)} dB_s^0 - \lambda_{n+1} \int_0^t \left(\int_u^t \eta(u) e^{-\lambda_{n+1}(t-s)} ds\right) dB_u^0\right]$$

which, simplified, gives the result 2.

3. Integrating by parts the first integral of (11) we obtain:

$$\int_0^t e^{-\lambda_{n+1}(t-s)} dB_s^0 = B_t^0 - \lambda_{n+1} \int_0^t e^{-\lambda_{n+1}(t-s)} B_s^0 ds.$$

Substituting and simplifying we get the result 3.

4. Substituting $W_t = \left(\sum_{j=1}^{n+1} p_j^2\right)^{-\frac{1}{2}} Y_t$ in result 1, we obtain that

$$B_t^0 = W_t + \lambda_{n+1} \int_0^t \left(\int_0^s [1+\eta(u)]\, e^{\lambda_{n+1} \int_u^s \eta(l) dl} dW_u\right) ds$$

which, written in standard form

$$B_t^0 = W_t - \int_0^t \left(\int_0^s -\lambda_{n+1} [1+\eta(u)]\, e^{\lambda_{n+1} \int_u^s \eta(l) dl} dW_u\right) ds$$

identifies the following Volterra Kernel

$$k(t,s) = \begin{cases} -\lambda_{n+1} [1+\eta(s)]\, e^{\lambda_{n+1} \int_s^t \eta(l) dl} & \text{for } 0 \le s \le t \\ 0 \quad \text{otherwise.} \end{cases}$$

Utilizing the relation 2, we also identify the negative resolvent $\tilde{k}(t,s)$

$$\tilde{k}(t,s) = \begin{cases} -k(t,s) e^{\int_s^t k(u,u) du} & \text{for } 0 \le s \le t \\ 0 \quad \text{otherwise} \end{cases}$$

from which, by substitution, we obtain:

$$\tilde{k}(t,s) = \begin{cases} \lambda_{n+1} [1+\eta(s)]\, e^{-\lambda_{n+1}(t-s)} & \text{se } 0 \le s \le t \\ 0 \quad \text{otherwise.} \end{cases}$$

As a consequence

$$W_t = B_t^0 - \int_0^t \left(\int_0^s \tilde{k}(s,u) dB_u^0\right) ds$$

from which, we deduce result 4. □

Using result 4, the asset price dynamics for the partially informed agents, can be written as:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} \left[dB_t^0 - \lambda_{n+1} \left(\int_0^t \left[1+\eta(u)\right] e^{-\lambda_{n+1}(t-u)} dB_u^0 \right) dt \right]$$

which can be rewritten as:

$$\frac{dS_t}{S_t} = \left[\mu_t - \lambda_{n+1} \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} \left(\sigma_t \int_0^t \left[1+\eta(u)\right] e^{-\lambda_{n+1}(t-u)} dB_u^0 \right) \right] dt +$$

$$+\sigma_t \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} dB_t^0 .$$

Conversely, for the informed agents, the asset price dynamics can be written as:

$$\frac{dS_t}{S_t} = \left(\mu_t - p_{n+1} \lambda_{n+1} \sigma_t U_t^{n+1} \right) dt + \sigma_t \sum_{j=1}^{n+1} p_j dB_t^j$$

which refers to the Brownian Motion (1).

3 The value functions for two agents

As already said in the previous section, the informed agents consider the underlying value starting from an initial wealth $x > 0$, and investing H_t units of S_t. They obtain the self-financed value of wealth X_t, at time t, through all the assigned Brownian Motion. Conversely the partially informed agents, invest the same monetary item $x > 0$, and they utilize the Brownian Motion B_t^0 in order to assess the dynamics of the wealth obtained. In this section, we want to examine these two situations, and we also want to evaluate the utility functions the two type of agents use.

3.1 The value function for the informed agents

Let $x > 0$ be the initial monetary item that the partially informed agents invest in asset S_t. To do this, they utilize an opportune stochastic process H_t which, at time t, represents the asset shares used. The value (self-financed) of wealth X_t, at time t, is given by:

$$X_t = x + \int_0^t H_s dS_s .$$

As already said, the process H_t, which will be said admissible, must be predictable with respect to filtration $\left(\mathcal{F}_t^1\right)_{t\in[0,+\infty[}$, integrable with respect to process S_t and such that almost certainly we also have $X_t > 0$, $\forall t \in [0,T]$.

Finally, if $\mathcal{U}$ is the utility function, the agents maximize the mean utility of the wealth obtained in the final instant T. Thereby it solves the following problem:

$$\sup\left\{E\left(\mathcal{U}\left(x+\int_0^T H_t dS_t\right)\right) : \quad H_t \quad admissible \right\}.$$

In order to guarantee the positivity of the wealth produced at every instant t, we can consider the process π_t defined by $H_t = \pi_t \frac{X_t}{S_t}$.

Therefore we obtain:

$$X_t = x + \int_0^t \pi_s \frac{X_s}{S_s} dS_s$$

it allows

$$\frac{dX_t}{X_t} = \pi_t \frac{dS_t}{S_t}$$

and

$$\frac{dX_t}{X_t} = \pi_t\left(\mu_t - p_{n+1}\lambda_{n+1}\sigma_t U_t^{n+1}\right)dt + \pi_t\sigma_t\sum_{j=1}^{n+1} p_j dB_t^j.$$

If we isolate the Brownian Motion (1):

$$\frac{dX_t}{X_t} = \pi_t\left(\mu_t - p_{n+1}\lambda_{n+1}\sigma_t U_t^{n+1}\right)dt + \left[\sum_{j=1}^{n+1} p_j^2\right]^{\frac{1}{2}} \pi_t\sigma_t dW_t.$$

Therefore the value of wealth X_t, at time t, is given by:

$$X_t = xe^{\int_0^t \left[\pi_s\left(\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}\right) - \frac{1}{2}\left(\sum_{j=1}^{n+1} p_j^2\right)\pi_s^2\sigma_s^2\right]ds}$$

$$e^{\left[\sum_{j=1}^{n+1} p_j^2\right]^{\frac{1}{2}} \int_0^t \pi_s\sigma_s dW_s}$$

and, as a consequence, X_t at final instant T, is given by:

$$X_T = xe^{\int_0^T \left[\pi_s\left(\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}\right) - \frac{1}{2}\left(\sum_{j=1}^{n+1} p_j^2\right)\pi_s^2\sigma_s^2\right]ds}$$

$$e^{\left[\sum_{j=1}^{n+1} p_j^2\right]^{\frac{1}{2}} \int_0^T \pi_s \sigma_s dW_s}.$$

Now, considering the logarithmic utility function, we have the following result:

Theorem 9. *Let $\mathcal{U}(y) = \log y$. The process*

$$\pi_s = \frac{\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}}{\left(\sum_{j=1}^{n+1} p_j^2\right)\sigma_s^2}$$

provides the optimal investment share and the value function is given by:

$$u(x) = \log x + \frac{1}{2\left(\sum_{j=1}^{n+1} p_j^2\right)} E\left[\int_0^T \frac{\left(\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}\right)^2}{\sigma_s^2} ds\right].$$

Proof. $\forall\ \pi_s$ admissible, it results

$$\mathcal{U}(X_T) = \log(X_T) =$$

$$= \log x + \int_0^T \left[\pi_s \left(\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}\right) - \frac{1}{2}\left(\sum_{j=1}^{n+1} p_j^2\right)\pi_s^2\sigma_s^2\right] ds +$$

$$+ \left[\sum_{j=1}^{n+1} p_j^2\right]^{\frac{1}{2}} \int_0^T \pi_s \sigma_s dW_s$$

as a consequence, considering the mean value, we obtain that:

$$E\left(\log(X_T)\right) =$$

$$= \log x + E\left[\int_0^T \left[\pi_s \left(\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}\right) - \frac{1}{2}\left(\sum_{j=1}^{n+1} p_j^2\right)\pi_s^2\sigma_s^2\right] ds\right].$$

The maximum value is obtained from

$$\pi_s = \frac{\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}}{\left(\sum_{j=1}^{n+1} p_j^2\right)\sigma_s^2}.$$

It is given by:

$$u(x) = \log x + \frac{1}{2} E\left[\int_0^T \frac{\left(\mu_s - p_{n+1}\lambda_{n+1}\sigma_s U_s^{n+1}\right)^2}{\left(\sum_{j=1}^{n+1} p_j^2\right)\sigma_s^2} ds\right]$$

from which we deduce the relation looked for. □

We can rewrite the value function, in the following way:

Theorem 10. *Let*

$$u(x) = \log x + \frac{1}{2\sum_{j=1}^{n+1} p_j^2} \int_0^T \frac{\mu_s^2}{\sigma_s^2} ds + \frac{p_{n+1}^2}{\sum_{j=1}^{n+1} p_j^2}\left[\frac{T\lambda_{n+1}}{4} - \frac{1-e^{-2T\lambda_{n+1}}}{8}\right].$$

Proof. From theorem 9 we obtain that:

$$u(x) = \log x + \frac{1}{2\sum_{j=1}^{n+1} p_j^2} E\left[\int_0^T \frac{\mu_s^2}{\sigma_s^2} ds\right] - \frac{p_{n+1}\lambda_{n+1}}{\sum_{j=1}^{n+1} p_j^2} E\left[\int_0^T \frac{\mu_s}{\sigma_s} U_s^{n+1} ds\right] +$$

$$+\frac{p_{n+1}^2\lambda_{n+1}^2}{2\sum_{j=1}^{n+1} p_j^2} E\left[\int_0^T \left[U_s^{n+1}\right]^2 ds\right].$$

Let the functions μ_s and σ_s be deterministic, then:

$$E\left[\int_0^T \frac{\mu_s^2}{\sigma_s^2} ds\right] = \int_0^T \frac{\mu_s^2}{\sigma_s^2} ds.$$

Moreover it results:

$$E\left[\int_0^T \frac{\mu_s}{\sigma_s} U_s^{n+1} ds\right] = E\left[\int_0^T \left(\frac{\mu_s}{\sigma_s}\int_0^s e^{-\lambda_{n+1}(s-u)} dB_u^{n+1}\right) ds\right] =$$

$$= \int_0^T E\left(\frac{\mu_s}{\sigma_s}\int_0^s e^{-\lambda_{n+1}(s-u)} dB_u^{n+1}\right) ds =$$

$$= \int_0^T \frac{\mu_s}{\sigma_s} E\left(\int_0^s e^{-\lambda_{n+1}(s-u)} dB_u^{n+1}\right) ds = \int_0^T \frac{\mu_s}{\sigma_s} 0 ds = 0$$

and

$$E\left[\int_0^T \left[U_s^{n+1}\right]^2 ds\right] = \int_0^T E\left[\left[\int_0^s e^{-\lambda_{n+1}(s-u)} dB_u^{n+1}\right]^2\right] ds =$$
$$= \int_0^T \left[\int_0^s e^{-2\lambda_{n+1}(s-u)} du\right] ds = \frac{T}{2\lambda_{n+1}} - \frac{1-e^{-2\lambda_{n+1}T}}{4\lambda_{n+1}^2}.$$

In this way we get the thesis. □

3.2 The value function for the partially informed agents

Similarly, the partially informed agents consider the investment shares provided through processes K_t admissible: they are predictable with respect to the filtration $\left(\mathcal{F}_t^0\right)_{t\in[0,+\infty[}$, integrable with respect to the process S_t and such that $X_t > 0$ almost certainly and $\forall t \in [0,T]$. If $\mathcal{V}$ is their utility function, then the agents maximize their expected utility of wealth at time T. Therefore it solves the following problem:

$$\max\left\{E\left(\mathcal{V}\left(x+\int_0^T K_t dS_t\right)\right) : \quad K_t \quad admissible \quad\right\}.$$

Also in this case the agents consider the process κ_t defined by:

$$K_t = \kappa_t \frac{X_t}{S_t}.$$

Therefore we have:

$$X_t = x + \int_0^t \kappa_s \frac{X_s}{S_s} dS_s$$

from which we obtain

$$\frac{dX_t}{X_t} = \kappa_t \frac{dS_t}{S_t}$$

and if

$$\nu_t = -\lambda_{n+1}\int_0^t \left[1+\eta(u)\right] e^{-\lambda_{n+1}(t-u)} dB_u^0$$

we have:

$$\frac{dX_t}{X_t} = \kappa_t\left[\mu_t + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_t\sigma_t\right] dt + \kappa_t\sigma_t\left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} dB_t^0.$$

The wealth at time t, if $x > 0$ is the initial one, is therefore given by

$$X_t = xe^{\int_0^t \left(\kappa_s\left[\mu_s + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_s\sigma_s\right] - \frac{1}{2}\kappa_s^2\sigma_s^2\left(\sum_{j=1}^{n+1} p_j^2\right)\right) ds}$$

$$e^{\int_0^t \kappa_s \sigma_s \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} dB_s^0}$$

and, as a consequence, at final time T, is given by:

$$X_T = xe^{\int_0^T \left(\kappa_s \left[\mu_s + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_s \sigma_s\right] - \frac{1}{2}\kappa_s^2 \sigma_s^2 \left(\sum_{j=1}^{n+1} p_j^2\right)\right) ds}$$

$$e^{\int_0^T \kappa_s \sigma_s \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} dB_s^0}$$

Considering the logarithmic utility function, we have the following result:

Theorem 11. *Let $\mathcal{V}(y) = \log y$. The process*

$$\kappa_s = \frac{\mu_s + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_s \sigma_s}{\left(\sum_{j=1}^{n+1} p_j^2\right) \sigma_s^2}$$

where

$$\nu_s = -\lambda_{n+1} \int_0^s \left[1 + \eta(u)\right] e^{-\lambda_{n+1}(s-u)} dB_u^0$$

provides the optimal investment share. The value function is given by:

$$v(x) = \log x + \frac{1}{2\left(\sum_{j=1}^{n+1} p_j^2\right)} E\left[\int_0^T \frac{1}{\sigma_s^2} \left(\mu_s + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_s \sigma_s\right)^2 ds\right].$$

Proof. $\forall \ \kappa_s$ admissible, it results

$$\mathcal{V}(X_T) = \log(X_T) =$$

$$= \log x + \int_0^T \left[\kappa_s \left(\mu_s + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_s \sigma_s\right) - \frac{1}{2}\left(\sum_{j=1}^{n+1} p_j^2\right) \kappa_s^2 \sigma_s^2\right] ds +$$

$$+ \left[\sum_{j=1}^{n+1} p_j^2\right]^{\frac{1}{2}} \int_0^T \pi_s \sigma_s dW_s$$

as a consequence, considering the mean value, we obtain that:

$$E\left(\log\left(X_T\right)\right) =$$

$$= \log x + E\left[\int_0^T \left[\kappa_s\left(\mu_s + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_s\sigma_s\right) - \frac{1}{2}\left(\sum_{j=1}^{n+1} p_j^2\right)\kappa_s^2\sigma_s^2\right] ds\right].$$

The maximum value is obtained from

$$\kappa_s = \frac{\mu_s + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_s\sigma_s}{\left(\sum_{j=1}^{n+1} p_j^2\right)\sigma_s^2}$$

It is given by:

$$v(x) = \log x + \frac{1}{2}E\left[\int_0^T \frac{\left(\mu_s + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_s\sigma_s\right)^2}{\left(\sum_{j=1}^{n+1} p_j^2\right)\sigma_s^2} ds\right]$$

from which, we obtain the relation looked for. □

We can rewrite the value function, in the following way:

Theorem 12. *Let*

$$v(x) = \log x + \frac{1}{2\sum_{j=1}^{n+1} p_j^2}\int_0^T \frac{\mu_s^2}{\sigma_s^2} ds + \frac{\lambda_{n+1}^2}{2}\int_0^T \left[\int_0^s \left[1+\eta\left(u\right)\right]^2 e^{-2\lambda_{n+1}(s-u)} du\right] ds.$$

Proof. From theorem 11 we have:

$$v(x) = \log x + \frac{1}{2\left(\sum_{j=1}^{n+1} p_j^2\right)} E\left[\int_0^T \frac{1}{\sigma_s^2}\left(\mu_s + \left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} \nu_s\sigma_s\right)^2 ds\right] =$$

$$= \log x + \frac{1}{2\sum_{j=1}^{n+1} p_j^2} E\left[\int_0^T \frac{\mu_s^2}{\sigma_s^2} ds\right] + \frac{1}{\left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}}} E\left[\int_0^T \frac{\mu_s}{\sigma_s} \nu_s ds\right] + \frac{1}{2} E\left[\int_0^T \nu_s^2 ds\right] =$$

$$= \log x + \frac{1}{2\sum_{j=1}^{n+1} p_j^2} \int_0^T \frac{\mu_s^2}{\sigma_s^2} ds + \frac{1}{\left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}}} \int_0^T \frac{\mu_s}{\sigma_s} E\left[\nu_s\right] ds + \frac{1}{2} E\left[\int_0^T \nu_s^2 ds\right] =$$

$$= \log x + \frac{1}{2\sum_{j=1}^{n+1} p_j^2} \int_0^T \frac{\mu_s^2}{\sigma_s^2} ds + \frac{1}{\left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}}} \int_0^T \frac{\mu_s}{\sigma_s} E\left[\nu_s\right] ds + \frac{1}{2} E\left[\int_0^T \nu_s^2 ds\right].$$

Besides it results that:

$$E\left[\nu_s\right] = E\left[-\lambda_{n+1} \int_0^s \left[1 + \eta(u)\right] e^{-\lambda_{n+1}(s-u)} dB_u^0\right] = 0.$$

Finally we have:

$$E\left[\int_0^T \nu_s^2 ds\right] = \int_0^T E\left[\nu_s^2\right] ds =$$

$$= \int_0^T E\left[\left(-\lambda_{n+1} \int_0^s \left[1 + \eta(u)\right] e^{-\lambda_{n+1}(s-u)} dB_u^0\right)^2\right] ds =$$

$$= \lambda_{n+1}^2 \int_0^T \left[\int_0^s \left[1 + \eta(u)\right]^2 e^{-2\lambda_{n+1}(s-u)} du\right] ds.$$

From the results obtained we get the thesis. □

In order to compare the two value functions so far obtained, we consider the following theorem:

Theorem 13. *Let*

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \left[\int_0^s \left[1 + \eta\left(u\right)\right]^2 e^{-2\lambda_{n+1}(s-u)} du\right] ds = \frac{\left(1 - A\right)^2}{2\lambda_{n+1}}$$

where

$$A = \left(\sum_{j=1}^{n+1} p_j^2\right)^{-\frac{1}{2}} \left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}}$$

.

Proof. First, we verify that

$$\lim_{T\to+\infty}\int_0^T [1+\eta(u)]^2 e^{-2\lambda_{n+1}(T-u)}du = \frac{(1-A)^2}{2\lambda_{n+1}}$$

and substituting $v = T - u$, we obtain:

$$\int_0^T [1+\eta(T-v)]^2 e^{-2\lambda_{n+1}(v)}dv.$$

Besides, if we denote the indicator function of the interval $[0, T]$ with $I_{[0,T]}$, we can write

$$\lim_{T\to+\infty}\int_0^{+\infty} [1+\eta(T-v)]^2 e^{-2\lambda_{n+1}(v)} I_{[0,T]}(v)\, dv.$$

Now, note that, since the $\lim_{T\to+\infty}\eta(T-v) = -A$, then the integrand function tends punctually to the function $v \in [0, +\infty[\mapsto (1-A)^2 e^{-2\lambda_{n+1}v}$. Moreover the integrand function is:

$$0 \le [1+\eta(T-v)]^2 e^{-2\lambda_{n+1}(v)} I_{[0,T]}(v)\, dv \le (1+A)\, e^{-2\lambda_{n+1}v} \quad \forall\, v \in [0, +\infty[$$

and the decrease and increase functions are integrable on $[0, +\infty[$.

The Lebesgue dominated convergence theorem allows that

$$\lim_{T\to+\infty}\int_0^{+\infty} [1+\eta(T-v)]^2 e^{-2\lambda_{n+1}(v)} I_{[0,T]}(v)\, dv = \int_0^{+\infty} (1-A)^2 e^{-2\lambda_{n+1}v}dv.$$

Integrating the second term, we obtain that

$$\lim_{T\to+\infty}\int_0^T [1+\eta(u)]^2 e^{-2\lambda_{n+1}(T-u)}du = \frac{(1-A)^2}{2\lambda_{n+1}}$$

and with the standard results it results:

$$\lim_{T\to+\infty}\int_0^T \left[\int_0^s [1+\eta(u)]^2 e^{-2\lambda_{n+1}(s-u)}du\right] ds = +\infty.$$

Applying De L'Hopital's rule, we obtain the thesis. □

3.3 A comparison between the two value functions

In the previous sections, we have determined the value functions for the two type of agents: $u(x)$ for the informed one, $v(x)$ for the partially informed one. In this section, we want to focus on the divergence, when $T \to +\infty$,the two utility functions. Besides, when $T \to +\infty$, the difference between the expected utility of two agents, $u(x)$ and $v(x)$, diverges.

We can enunciate the following result:

Theorem 14. *Consider the following properties:*

1.

$$\lim_{T\to+\infty} u(x) =$$

$$= \lim_{T\to+\infty}\left(\log x + \frac{1}{2\sum_{j=1}^{n+1} p_j^2}\int_0^T \frac{\mu_s^2}{\sigma_s^2}ds + \; + \frac{p_{n+1}^2}{\sum_{j=1}^{n+1} p_j^2}\left[\frac{2T\lambda_{n+1} - 1 + e^{-2T\lambda_{n+1}}}{8}\right]\right) =$$

$$= +\infty.$$

2.

$$\lim_{T\to+\infty} v(x) =$$

$$\lim_{T\to+\infty}\left(\log x + \frac{1}{2\sum_{j=1}^{n+1} p_j^2}\int_0^T \frac{\mu_s^2}{\sigma_s^2}ds + \frac{\lambda_{n+1}^2}{2}\int_0^T\left[\int_0^s \left[1+\eta\left(u\right)\right]^2 e^{-2\lambda_{n+1}(s-u)}du\right]ds\right)$$

$$= +\infty$$

3.

$$\lim_{T\to+\infty} \frac{u(x)-v(x)}{T} =$$

$$= \frac{\lambda_{n+1}}{2}\left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n+1} p_j^2\right)^{-1}\left[\left(\sum_{j=1}^{n+1} p_j^2\right)^{\frac{1}{2}} - \left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}}\right]$$

as a consequence we have:

$$\lim_{T\to+\infty}\left[u(x)-v(x)\right] = +\infty.$$

Proof. The first two properties can be easily verified. About property 3, we note that

$$u(x) - v(x) =$$

$$= \frac{p_{n+1}^2}{\sum_{j=1}^{n+1} p_j^2}\left[\frac{2T\lambda_{n+1} - 1 + e^{-2T\lambda_{n+1}}}{8}\right] +$$

$$-\frac{\lambda_{n+1}^2}{2}\int_0^T\left[\int_0^s \left[1+\eta\left(u\right)\right]^2 e^{-2\lambda_{n+1}(s-u)}du\right]ds$$

and therefore

$$\frac{u(x)-v(x)}{T} =$$

$$= \frac{p_{n+1}^2}{\sum_{j=1}^{n+1} p_j^2} \left[\frac{2T\lambda_{n+1} - 1 + e^{-2T\lambda_{n+1}}}{8T} \right] +$$

$$-\frac{\lambda_{n+1}^2}{2T} \int_0^T \left[\int_0^s \left[1 + \eta(u)\right]^2 e^{-2\lambda_{n+1}(s-u)} du \right] ds =$$

$$= \frac{\lambda_{n+1}}{4} \left(\frac{p_{n+1}^2}{\sum_{j=1}^{n+1} p_j^2} - \left[1 - \frac{\left(\sum_{j=1}^{n} p_j^2 \right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}}} \right]^2 \right) =$$

$$= \frac{\lambda_{n+1}}{4} \left(\frac{p_{n+1}^2}{\sum_{j=1}^{n+1} p_j^2} - 1 + 2 \frac{\left(\sum_{j=1}^{n} p_j^2 \right)^{\frac{1}{2}}}{\left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}}} - \frac{\sum_{j=1}^{n} p_j^2}{\sum_{j=1}^{n+1} p_j^2} \right) =$$

$$= \frac{\lambda_{n+1}}{4} \frac{p_{n+1}^2 - \sum_{j=1}^{n+1} p_j^2 + 2 \left(\sum_{j=1}^{n} p_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{n} p_j^2}{\sum_{j=1}^{n+1} p_j^2} =$$

$$= \frac{\lambda_{n+1}}{2} \frac{\left(\sum_{j=1}^{n} p_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{n} p_j^2}{\sum_{j=1}^{n+1} p_j^2} =$$

$$= \frac{\lambda_{n+1}}{2} \frac{\left(\sum_{j=1}^{n} p_j^2 \right)^{\frac{1}{2}}}{\sum_{j=1}^{n+1} p_j^2} \left[\left(\sum_{j=1}^{n+1} p_j^2 \right)^{\frac{1}{2}} - \left(\sum_{j=1}^{n} p_j^2 \right)^{\frac{1}{2}} \right]$$

which, simplified, provides the thesis. $\square$

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