

Wavelet series representation for multifractional multistable Riemann-Liouville process

Antoine Ayache¹ and Julien Hamonier²

¹ Univ. Lille, CNRS, UMR 8524 – Laboratoire Paul Painlevé, F-59000 Lille, France
Antoine.Ayache@univ-lille.fr

² Univ. Lille, CHU Lille, ULR 2694 – METRICS: Évaluation des technologies de santé et des pratiques médicales, F-59000 Lille, France
Julien.Hamonier@univ-lille.fr

Abstract. The main goal of this paper is to construct a wavelet-type random series representation for a random field X , defined by a multistable stochastic integral, which generates a multifractional multistable Riemann-Liouville (mmRL) process Y . Such a representation provides, among other things, an efficient method of simulation of paths of Y . In order to obtain it, we expand in the Haar basis the integrand associated with X and we use some fundamental properties of multistable stochastic integrals. Then, thanks to the Abel's summation rule and the Doob's maximal inequality for discrete submartingales, we show that this wavelet-type random series representation of X is convergent in a strong sense: almost surely in some spaces of continuous functions. Also, we determine an estimate of its almost sure rate of convergence in these spaces.

Keywords. Stable and multistable distributions, fractional and multifractional processes, varying Hurst parameter, Haar basis, random series of functions.

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J.E.L. classification. C19, C46.

1 Introduction

The main idea behind multifractional processes is that Hurst parameter which governs path roughness is no longer a constant but a function whose values can change from point to point (see e.g. [1]). Thus, such processes provide more flexible models than the classical fractional Brownian motion whose path roughness

remains everywhere the same. In the same spirit, the articles [4–6] have proposed three different (non-equivalent) approaches allowing to generalize stable stochastic processes (see for instance [9]) in such a way that the parameter α governing the heavy tail behaviour of their distributions becomes a function. Such generalizations are called multistable processes. The approach introduced in [6] relies on the construction of multistable stochastic integrals. Such an integral \mathcal{I} depends on a functional parameter $\alpha(\cdot)$; this deterministic Lebesgue measurable function $\alpha(\cdot)$ is defined on the real line \mathbb{R} and with values in some compact interval $[\underline{\alpha}, \bar{\alpha}]$ included in $(0, 2]$. Throughout this article, we assume that $\alpha(\cdot)$ belongs to the Hölder space $\mathcal{C}^{1+\rho_\alpha}([0, 1])$, for some $\rho_\alpha \in (0, 1)$; in other words $\alpha(\cdot)$ is continuously differentiable on $[0, 1]$ and its derivative $\alpha'(\cdot)$ satisfies a uniform Hölder condition on $[0, 1]$ of order ρ_α , that is one has $|\alpha'(s_1) - \alpha'(s_2)| \leq c|s_1 - s_2|^{\rho_\alpha}$, for some constant $c > 0$ and for all $(s_1, s_2) \in [0, 1]^2$. Moreover, we assume that

$$1 < \underline{\alpha} \leq \alpha(s) \leq \bar{\alpha} < 2, \quad \text{for all } s \in \mathbb{R}. \quad (1)$$

The integrands associated with the multistable stochastic integral \mathcal{I} are the deterministic functions from \mathbb{R} to \mathbb{R} belonging to \mathcal{F}_α , the Lebesgue space of variable order defined as:

$$\mathcal{F}_\alpha := \left\{ f \text{ s.t. } f \text{ is a Lebesgue measurable function from } \mathbb{R} \text{ to } \mathbb{R} \text{ and} \right. \\ \left. \int_{\mathbb{R}} |f(s)|^{\alpha(s)} ds < +\infty \right\}. \quad (2)$$

Notice that, for any fixed $f \in \mathcal{F}_\alpha^* := \mathcal{F}_\alpha \setminus \{0\}$, the function from $(0, +\infty)$ into itself $\lambda \mapsto \int_{\mathbb{R}} \lambda^{-\alpha(s)} |f(s)|^{\alpha(s)} ds$ is continuous and strictly decreasing, and one has $\lim_{\lambda \rightarrow 0^+} \int_{\mathbb{R}} \lambda^{-\alpha(s)} |f(s)|^{\alpha(s)} ds = +\infty$ and $\lim_{\lambda \rightarrow +\infty} \int_{\mathbb{R}} \lambda^{-\alpha(s)} |f(s)|^{\alpha(s)} ds = 0$. Therefore, there exists a unique positive real number denoted by $\|f\|_\alpha$ such that

$$\int_{\mathbb{R}} \|f\|_\alpha^{-\alpha(s)} |f(s)|^{\alpha(s)} ds = 1. \quad (3)$$

The map $\|\cdot\|_\alpha$ defined on \mathcal{F}_α in this way and by using the convention that $\|0\|_\alpha = 0$ is a quasi-norm on \mathcal{F}_α ; notice that the only difference between a norm and a quasi-norm is that in the latter case the triangular inequality holds up to a multiplicative constant, namely there exists $c' \in [1, +\infty)$, such that $\|f + g\|_\alpha \leq c'(\|f\|_\alpha + \|g\|_\alpha)$, for all $f, g \in \mathcal{F}_\alpha$. Also notice that one can derive from (2) and the inequality

$$\int_{\mathbb{R}} |f(s)|^{\alpha(s)} ds \leq \int_{\mathbb{R}} |f(s)|^\alpha ds + \int_{\mathbb{R}} |f(s)|^{\bar{\alpha}} ds,$$

which is satisfied by any Lebesgue measurable function f from \mathbb{R} to \mathbb{R} , that

$$L^\alpha(\mathbb{R}) \cap L^{\bar{\alpha}}(\mathbb{R}) \subseteq \mathcal{F}_\alpha, \quad (4)$$

where, for all $p \in (0, +\infty]$, one denotes $L^p(\mathbb{R})$ the classical Lebesgue space of order p of real-valued functions over \mathbb{R} . Moreover, there is a finite constant κ_1

only depending on $\underline{\alpha}$ and $\bar{\alpha}$, such that, for all $f \in L^{\underline{\alpha}}(\mathbb{R}) \cap L^{\bar{\alpha}}(\mathbb{R})$, one has

$$\|f\|_{\alpha} \leq \kappa_1 (\|f\|_{\underline{\alpha}} + \|f\|_{\bar{\alpha}}) = \kappa_1 \left(\int_{\mathbb{R}} |f(s)|^{\underline{\alpha}} ds \right)^{1/\underline{\alpha}} + \kappa_1 \left(\int_{\mathbb{R}} |f(s)|^{\bar{\alpha}} ds \right)^{1/\bar{\alpha}}. \quad (5)$$

The latter inequality simply results from the fact that

$$\begin{aligned} & \int_{\mathbb{R}} \left(2^{1/\underline{\alpha}} \|f\|_{\underline{\alpha}} + 2^{1/\bar{\alpha}} \|f\|_{\bar{\alpha}} \right)^{-\alpha(s)} |f(s)|^{\alpha(s)} ds \\ & \leq \int_{\mathbb{R}} \left(2^{1/\underline{\alpha}} \|f\|_{\underline{\alpha}} + 2^{1/\bar{\alpha}} \|f\|_{\bar{\alpha}} \right)^{-\underline{\alpha}} |f(s)|^{\underline{\alpha}} ds \\ & \quad + \int_{\mathbb{R}} \left(2^{1/\underline{\alpha}} \|f\|_{\underline{\alpha}} + 2^{1/\bar{\alpha}} \|f\|_{\bar{\alpha}} \right)^{-\bar{\alpha}} |f(s)|^{\bar{\alpha}} ds \\ & \leq 2^{-1} \int_{\mathbb{R}} \|f\|_{\underline{\alpha}}^{-\underline{\alpha}} |f(s)|^{\underline{\alpha}} ds + 2^{-1} \int_{\mathbb{R}} \|f\|_{\bar{\alpha}}^{-\bar{\alpha}} |f(s)|^{\bar{\alpha}} ds = 1. \end{aligned}$$

Let us now recall some fundamental properties of the multistable stochastic integral \mathcal{I} which was introduced in [6]. Denote by $L^{\gamma}(\Omega, \mathcal{A}, \mathbb{P})$ the space of the real-valued random variables on a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$ whose absolute moment of order γ is finite, where $\gamma \in (0, \underline{\alpha})$ is arbitrary and fixed. The integral \mathcal{I} is a linear map from \mathcal{F}_{α} into $L^{\gamma}(\Omega, \mathcal{A}, \mathbb{P})$ such that, for all $f \in \mathcal{F}_{\alpha}$, the characteristic function $\Phi_{\mathcal{I}(f)}$ of the random variable $\mathcal{I}(f)$ satisfies

$$\Phi_{\mathcal{I}(f)}(\xi) := \mathbb{E}(e^{i\xi\mathcal{I}(f)}) = \exp\left(-\int_{\mathbb{R}} |\xi f(s)|^{\alpha(s)} ds\right), \quad \text{for every } \xi \in \mathbb{R}. \quad (6)$$

Observe that (6) implies that $\mathcal{I}(f)$ has a symmetric distribution. Similarly to stable stochastic integrals (see for instance [9]), one can in a natural way associate to the multistable stochastic integral \mathcal{I} an independently scattered random measure denoted by \mathcal{M}_{α} (see [6]). Thus, $\mathcal{I}(f)$ is frequently denoted by $\int_{\mathbb{R}} f(s)\mathcal{M}_{\alpha}(ds)$. It is worth mentioning that an upper bound of the asymptotic behavior at $+\infty$ of the tail of the distribution of the random variable $\int_{\mathbb{R}} f(s)\mathcal{M}_{\alpha}(ds)$ is provided by Proposition 2.3 of [6]:

$$\mathbb{P}\left(\left|\int_{\mathbb{R}} f(s)\mathcal{M}_{\alpha}(ds)\right| \geq \lambda\right) \leq \kappa_2 \int_{\mathbb{R}} \lambda^{-\alpha(s)} |f(s)|^{\alpha(s)} ds, \quad \text{for all } \lambda \in (0, +\infty), \quad (7)$$

where κ_2 is a constant only depending on $\underline{\alpha}$ and $\bar{\alpha}$. The same proposition also provides, thanks to (7), an estimate for the absolute moment of any order $\gamma \in (0, \underline{\alpha})$ of $\int_{\mathbb{R}} f(s)\mathcal{M}_{\alpha}(ds)$:

$$\mathbb{E}\left(\left|\int_{\mathbb{R}} f(s)\mathcal{M}_{\alpha}(ds)\right|^{\gamma}\right) \leq \kappa_3(\gamma) \|f\|_{\alpha}^{\gamma}, \quad \text{for each fixed } \gamma \in (0, \underline{\alpha}), \quad (8)$$

where $\kappa_3(\gamma)$ is a finite constant only depending on γ , $\underline{\alpha}$ and $\bar{\alpha}$. We mention in passing that the paper [2] has shown that the inequality (7) is sharp: the reverse inequality also holds.

2 Main result and simulations

Let us now give the main motivation behind our present article. The paper [8] has introduced via Haar basis an almost surely uniformly convergent wavelet-type random series representation for the stable stochastic field which generates linear multifractional stable motions [10, 11]. In our present article, we intend to generalize this result to the framework of the multistable stochastic field which generates linear multifractional multistable motions of Riemann-Liouville type. The latter field is denoted by $\{X(u, v) : (u, v) \in [0, 1] \times (1/\underline{\alpha}, 1)\}$, and defined, for all $(u, v) \in [0, 1] \times (1/\underline{\alpha}, 1)$, as:

$$X(u, v) := \int_{\mathbb{R}} K_{u,v}(s) \mathcal{M}_{\alpha}(ds), \quad (9)$$

where, for every $(u, v, s) \in [0, 1] \times (1/\underline{\alpha}, 1) \times \mathbb{R}$,

$$K_{u,v}(s) := (u-s)_+^{v-\frac{1}{\alpha(s)}} \mathbb{1}_{[0,1]}(s) = \begin{cases} 0 & \text{if } s \notin [0, u], \\ (u-s)^{v-\frac{1}{\alpha(s)}} & \text{otherwise.} \end{cases} \quad (10)$$

It can easily be seen that, for each fixed $(u, v) \in [0, 1] \times (1/\underline{\alpha}, 1)$, one has

$$0 \leq K_{u,v}(s) \leq \mathbb{1}_{[0,1]}(s), \quad \text{for all } s \in \mathbb{R}. \quad (11)$$

Thus, the function $K_{u,v}$ belongs to the space \mathcal{F}_{α} (see (2)) which guarantees the existence of the multistable stochastic integral in (9). Also, one can derive from (11) that the function $K_{u,v}$ belongs to all the Lebesgue spaces $L^p([0, 1])$, $p \in (0, +\infty]$, and in particular to the Hilbert space $L^2([0, 1])$. A well-known orthonormal basis of the latter space was introduced by Haar in [7]; it consists in the following collection of functions:

$$\begin{cases} \mathbb{1}_{[0,1]}(\bullet), \\ 2^{j/2} h(2^j \bullet - k) = 2^{j/2} \left(\mathbb{1}_{[2^{-j}k, 2^{-j}(k+\frac{1}{2})]}(\bullet) - \mathbb{1}_{[2^{-j}(k+\frac{1}{2}), 2^{-j}(k+1)]}(\bullet) \right), \\ j \in \mathbb{Z}_+ \text{ and } k \in \{0, \dots, 2^j - 1\}, \end{cases} \quad (12)$$

where $h := \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}$. By expanding, for each fixed $(u, v) \in [0, 1] \times (1/\underline{\alpha}, 1)$, the function $K_{u,v}$ on the latter basis, one gets that

$$K_{u,v}(\bullet) = \|K_{u,v}\|_1 \mathbb{1}_{[0,1]} + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} w_{j,k}(u, v) h(2^j \bullet - k), \quad (13)$$

where $\|K_{u,v}\|_1 := \int_0^1 K_{u,v}(s) ds$ and

$$w_{j,k}(u, v) := 2^j \int_0^1 K_{u,v}(s) h(2^j s - k) ds, \quad \text{for all } j \in \mathbb{Z}_+ \text{ and } k \in \{0, \dots, 2^j - 1\}. \quad (14)$$

A priori, the series in (13) is convergent for the $L^2([0, 1])$ norm; yet (1), (5) and the Hölder inequality imply that this series is also convergent for the quasi-norm $\|\cdot\|_\alpha$. Thus, using (9) and (8) one gets that

$$X(u, v) = \|K_{u,v}\|_1 \eta + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} w_{j,k}(u, v) \varepsilon_{j,k}, \quad (15)$$

where $\eta := \int_{\mathbb{R}} \mathbb{1}_{[0,1)}(s) \mathcal{M}_\alpha(ds) = \mathcal{M}_\alpha([0, 1])$ and

$$\varepsilon_{j,k} := \int_{\mathbb{R}} h(2^j s - k) \mathcal{M}_\alpha(ds), \quad \text{for all } j \in \mathbb{Z}_+ \text{ and } k \in \{0, \dots, 2^j - 1\}. \quad (16)$$

A priori, the series in (15) is convergent in the sense of the $L^\gamma(\Omega, \mathcal{A}, \mathbb{P})$ (quasi)-norm, for each fixed $(u, v) \in [0, 1] \times (1/\underline{\alpha}, 1)$ and $\gamma \in (0, \underline{\alpha})$. The main goal of our article is to show that it is also convergent in a much stronger sense, namely:

Theorem 21 *For all integer $J \geq 1$ and $(u, v) \in [0, 1] \times (1/\underline{\alpha}, 1)$, let $X^J(u, v)$ be the partial sum of the series in (15) defined as:*

$$X^J(u, v) = \|K_{u,v}\|_1 \eta + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} w_{j,k}(u, v) \varepsilon_{j,k}. \quad (17)$$

Then, there exists an event Ω^ of probability 1, such that, for all $\omega \in \Omega^*$ and for every real numbers a and b satisfying $1/\underline{\alpha} < a < b < 1$, $(X^J(\cdot, \cdot, \omega))_{J \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0, 1] \times [a, b])$ the Banach space of the real-valued continuous functions over the rectangle $[0, 1] \times [a, b]$ equipped with the uniform norm denoted by $\|\cdot\|_{\mathcal{C}}$. Thus, it is convergent in this space. Moreover, the multistable stochastic field $\{\tilde{X}(u, v) : (u, v) \in [0, 1] \times [a, b]\}$ with continuous paths, defined as:*

$$\tilde{X}(\cdot, \cdot, \omega) := \lim_{J \rightarrow +\infty} X^J(\cdot, \cdot, \omega), \text{ if } \omega \in \Omega^*, \quad \text{and} \quad \tilde{X}(\cdot, \cdot, \omega) := 0, \text{ else,} \quad (18)$$

is a modification of $\{X(u, v) : (u, v) \in [0, 1] \times [a, b]\}$, and one has, for any fixed $\zeta > 1/\underline{\alpha}$ and $\omega \in \Omega^$,*

$$\sup \left\{ J^{-\zeta} 2^{J \min\{\rho_\alpha, a-1/\underline{\alpha}\}} |\tilde{X}(u, v, \omega) - X^J(u, v, \omega)| : \right. \\ \left. (J, u, v) \in \mathbb{N} \times [0, 1] \times [a, b] \right\} < +\infty. \quad (19)$$

Remark 21 *In view of (17) and of the fact that \tilde{X} is a modification of X , the inequality (19) provides an almost sure estimate of the rate of convergence for the uniform norm $\|\cdot\|_{\mathcal{C}}$ of the random series of functions in (15). Notice that in the particular case where X is an α -stable field (that is $\alpha(s) = \alpha$, for all $s \in [0, 1]$, where $\alpha \in (1, 2)$ is a constant parameter), this estimate of the rate of convergence becomes $\mathcal{O}(2^{-J(a-1/\alpha)} J^{1/\alpha+\eta})$, where η is an arbitrarily small fixed positive real number. Thus, it improves the estimate $\mathcal{O}(2^{-J(a-1/\alpha)} J^{2/\alpha+\eta})$ which was previously obtained in [8, Theorem 2.1].*

Definition 1. Let $H(\cdot)$ be a deterministic function from $[0, 1]$ into $[a, b] \subset (1/\underline{\alpha}, 1)$. The multifractional multistable Riemann-Liouville (mmRL) process of parameter $H(\cdot)$, generated by the field $\{\tilde{X}(u, v) : (u, v) \in [0, 1] \times [a, b]\}$, is the multistable process denoted by $\{Y(t) : t \in [0, 1]\}$ and defined as:

$$Y(t) := \tilde{X}(t, H(t)), \quad \text{for all } t \in [0, 1]. \quad (20)$$

Notice that when the function $H(\cdot)$ is a constant $\{Y(t) : t \in [0, 1]\}$ is called fractional multistable Riemann-Liouville (fmRL) process.

Remark 22 It easily follows from Theorem 21 and Definition 1 that $\{Y(t) : t \in [0, 1]\}$ has almost surely continuous paths as soon as $H(\cdot)$ is a continuous function.

Remark 23 Using (12), (14), and (16), it can be shown by induction on J that, for all $J \in \mathbb{N}$ and for each $(u, v) \in [0, 1] \times (1/\underline{\alpha}, 1)$, the random variable $X^J(u, v)$, defined in (17), can be expressed as:

$$X^J(u, v) = \sum_{l=0}^{2^J-1} \bar{K}_{u,v}^{J,l} \mathcal{M}_\alpha \left([2^{-J}l, 2^{-J}(l+1)) \right), \quad (21)$$

where, for all $J \in \mathbb{N}$ and $l \in \{0, \dots, 2^J - 1\}$, $\bar{K}_{u,v}^{J,l}$ is the average value of the function $K_{u,v}$ on the dyadic interval $[2^{-J}l, 2^{-J}(l+1))$, that is

$$\bar{K}_{u,v}^{J,l} := 2^J \int_{2^{-J}l}^{2^{-J}(l+1)} K_{u,v}(s) ds. \quad (22)$$

The equalities (21), (18) and (20) provide an efficient method for simulating paths of the mmRL process Y . To this end, when J is large enough, one uses the approximation:

$$\mathcal{M}_\alpha \left([2^{-J}l, 2^{-J}(l+1)) \right) \approx \mathcal{Z}_{\alpha(2^{-J}l)} \left([2^{-J}l, 2^{-J}(l+1)) \right), \quad (23)$$

where the $\mathcal{Z}_{\alpha(2^{-J}l)}$, $l = 0, \dots, 2^J - 1$, are independent usual symmetric stable random measures with stability parameters $\alpha(2^{-J}l)$, $l = 0, \dots, 2^J - 1$. Notice that the approximation (23) is justified by [6, Theorem 2.6].

In Figure 1 there are some simulations. In Figure 2, for the same function α as above.

3 Proof of the main result

The main two ideas of the proof of Theorem 21 are:

1. The use of the Abel's summation rule in order to express $\mathcal{L}_j(u, v)$ in a convenient way (see Remark 31).

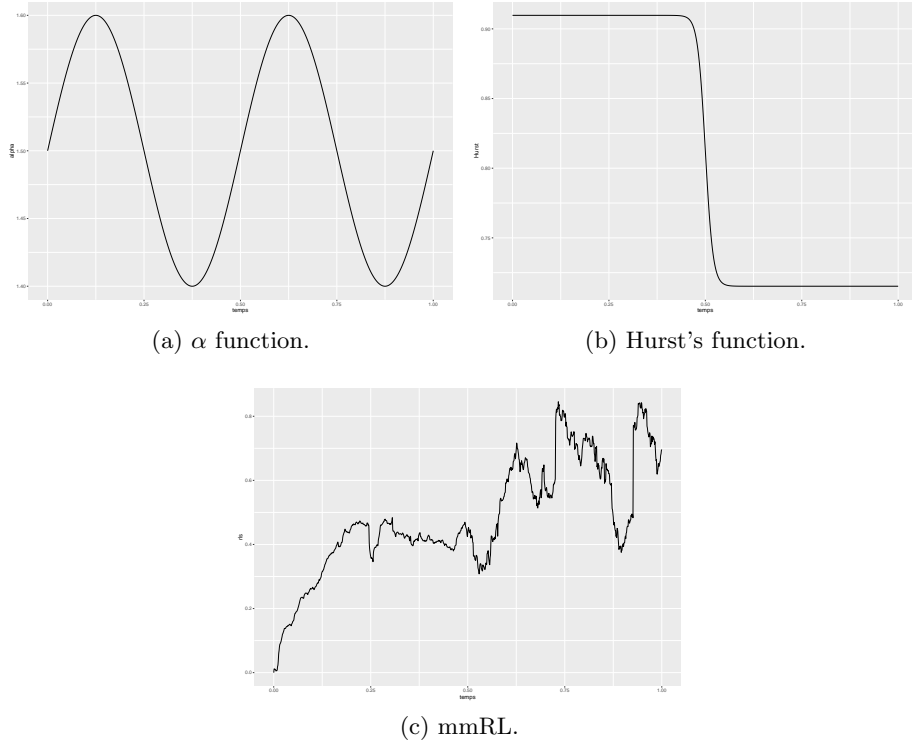


Fig. 1: Multifractional multistable Riemann-Liouville process.

2. The use of the Doob's maximal inequality for discrete submartingales in order to derive, for each $j \in \mathbb{N}$, a suitable upper for the supremum of the absolute values of the partial sums $\tau_{j,k}$, $k \in \{0, \dots, 2^j - 1\}$, of the multistable random variables $\epsilon_{j,0}, \dots, \epsilon_{j,2^j-1}$ (see Remark 32, Lemma 1 and its proof).

The first idea is borrowed from [8] while the second one is completely new.

Remark 31 For all $j \in \mathbb{Z}_+$ and $(u, v) \in [0, 1] \times (1/\underline{\alpha}, 1)$, one sets

$$\mathcal{L}_j(u, v) := \sum_{k=0}^{2^j-1} w_{j,k}(u, v) \epsilon_{j,k} = \sum_{k=0}^{[2^j u]} w_{j,k}(u, v) \epsilon_{j,k}. \quad (24)$$

One mentions in passing that the last equality in (24), in which $[2^j u]$ denotes the integer part of $2^j u$, follows from (12) and (14). Using the Abel's summation rule one has that:

$$\mathcal{L}_j(u, v) = \tau_{j, [2^j u]} w_{j, [2^j u]}(u, v) + \sum_{k=0}^{[2^j u]-1} \tau_{j,k} (w_{j,k}(u, v) - w_{j,k+1}(u, v)), \quad (25)$$

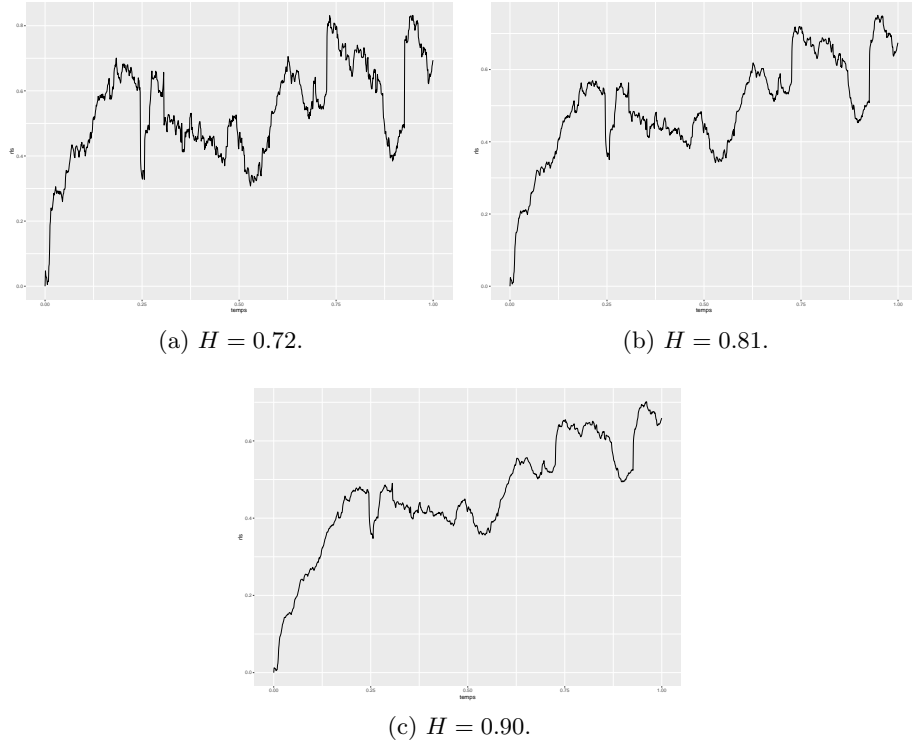


Fig. 2: Fractional multistable Riemann-Liouville processes.

where $\{\tau_{j,k} : j \in \mathbb{Z}_+ \text{ and } k \in \{0, \dots, 2^j - 1\}\}$ is the sequence of the multistable random variables defined, for all $j \in \mathbb{Z}_+$ and $k \in \{0, \dots, 2^j - 1\}$, as:

$$\tau_{j,k} := \sum_{m=0}^k \varepsilon_{j,m} = \int_{\mathbb{R}} \left(\sum_{m=0}^k h(2^j s - m) \right) \mathcal{M}_\alpha(ds). \quad (26)$$

Notice that the last equality in (26) follows from (16).

Remark 32 One knows from (16) and (8) that the multistable random variables $\varepsilon_{j,k}$, $j \in \mathbb{N}$ and $k \in \{0, \dots, 2^j - 1\}$, belong to $L^\gamma(\Omega, \mathcal{A}, \mathbb{P})$, for all $\gamma \in (0, \underline{\alpha})$; which in particular means that they are in $L^1(\Omega, \mathcal{A}, \mathbb{P})$ since $\underline{\alpha} > 1$; notice that they are centered since their distributions are symmetric. Moreover, for each fixed $j \in \mathbb{N}$, the random variables $\varepsilon_{j,k}$, $k \in \{0, \dots, 2^j - 1\}$, are independent since the random measure \mathcal{M}_α is independently scattered and the supports of the functions $h(2^j \bullet - k)$, $k \in \{0, \dots, 2^j - 1\}$, are pairwise disjoint (up to Lebesgue negligible sets). Thus, in view of the first equality in (26), it turns out that, for each fixed $j \in \mathbb{N}$, the sequence of random variables $\{\tau_{j,k}\}_{0 \leq k < 2^j}$ is a discrete martingale

with respect to the filtration $(\mathcal{A}_{j,k})_{0 \leq k < 2^j - 1}$ such that, for any $k \in \{0, \dots, 2^j - 2\}$, $\mathcal{A}_{j,k}$ denotes the smallest σ -algebra for which the random variables $\varepsilon_{j,0}, \dots, \varepsilon_{j,k}$ are measurable.

Lemma 1. *There exists an event Ω^* of probability 1 such that on Ω^* , one has, for all fixed $\zeta > 1/\underline{\alpha}$,*

$$\sup_{j \in \mathbb{Z}_+} \left\{ (1+j)^{-\zeta} \sup_{0 \leq k < 2^j} |\tau_{j,k}| \right\} < +\infty. \quad (27)$$

Proof (Proof of Lemma 1). Let $\zeta > 1/\underline{\alpha}$ and $\gamma \in [1, \underline{\alpha}]$ be fixed and such that

$$\zeta > 1/\gamma > 1/\underline{\alpha}. \quad (28)$$

Observe $z \mapsto |z|^\gamma$ is a convex function from \mathbb{R} to \mathbb{R}_+ , and one has $\mathbb{E}(|\tau_{j,k}|^\gamma) < +\infty$, for all j and k (see the last equality in (26), (8) and (12)). Thus, it follows Remark 32 and from [3, Theorem 10.3.3 on page 354], that, for each fixed $j \in \mathbb{N}$, the sequence of random variables $\{|\tau_{j,k}|^\gamma\}_{0 \leq k < 2^j}$ is a discrete submartingale with respect to the filtration $(\mathcal{A}_{j,k})_{0 \leq k < 2^j - 1}$. Hence, using the Doob's maximal inequality (see [3, Theorem 10.4.2 on page 360]) one has, for all positive real number M ,

$$\mathbb{P}\left(\sup_{0 \leq k < 2^j} |\tau_{j,k}|^\gamma > M\right) \leq M^{-1} \mathbb{E}(|\tau_{j,2^j-1}|^\gamma). \quad (29)$$

Observe that it follows from the last equality in (26), (8) and the fact that

$$\int_{\mathbb{R}} \left| \sum_{m=0}^{2^j-1} h(2^j s - m) \right|^{\alpha(s)} ds = 1,$$

that

$$\mathbb{E}(|\tau_{j,2^j-1}|^\gamma) \leq \kappa_3(\gamma), \quad \text{for all } j \in \mathbb{Z}_+, \quad (30)$$

where $\kappa_3(\gamma)$ is the same finite constant as in (8). Next, taking in (29) $M = (1+j)^{\gamma\zeta}$ and using (30) and (28), one obtains that

$$\sum_{j=1}^{+\infty} \mathbb{P}\left(\sup_{0 \leq k < 2^j} |\tau_{j,k}| > (1+j)^\zeta\right) \leq \kappa_3(\gamma) \sum_{j=1}^{+\infty} (1+j)^{-\gamma\zeta} < +\infty.$$

Thus, it follows from the Borel-Cantelli's Lemma that the probability of the event

$$\Omega_\zeta^* := \left\{ \omega \in \Omega : \sup_{j \in \mathbb{Z}_+} \left\{ (1+j)^{-\zeta} \sup_{0 \leq k < 2^j} |\tau_{j,k}(\omega)| \right\} < +\infty \right\}$$

is equal to 1. For finishing the proof, one sets

$$\Omega^* := \bigcap_{\zeta \in \mathbb{Q} \cap (1/\underline{\alpha}, +\infty)} \Omega_\zeta^*$$

where \mathbb{Q} denotes the countable set of the rational numbers.

In order to derive Theorem 21, one also needs the following five lemmas whose proofs are given in the Appendix A. From now on, for the sake of simplicity one denotes by I the interval $[0, 1]$.

Lemma 2. *There exists a positive and finite constant c_1 such that, for any $j \in \mathbb{N}$, for each $(u, v) \in I \times [a, b]$ satisfying $u \geq 4 \cdot 2^{-j-1}$, and for all $s \in [0, u - 4 \cdot 2^{-j-1}]$, the following inequality holds*

$$\begin{aligned} & |K_{u,v}(s) - K_{u,v}(s + 2^{-j-1}) - K_{u,v}(s + 2 \cdot 2^{-j-1}) + K_{u,v}(s + 3 \cdot 2^{-j-1})| \\ & \leq c_1 2^{-j} (2^{-j\rho_\alpha} + 2^{-j} |u - s - 3 \cdot 2^{-j-1}|^{a-1/\alpha-2}). \end{aligned} \quad (31)$$

Lemma 3. *There exists a positive and finite constant c_2 such that, for any $j \in \mathbb{Z}_+$ and $(u, v) \in I \times [a, b]$, the following inequality, in which $[2^j u]$ denotes the integer part of $2^j u$, is satisfied*

$$|w_{j, [2^j u]}(u, v)| \leq c_2 2^{-j(a-\frac{1}{\alpha})}. \quad (32)$$

Lemma 4. *There exists a positive and finite constant c_3 such that, for any $j \in \mathbb{Z}_+$ and $(u, v) \in I \times [a, b]$, one has*

$$\begin{aligned} I_j^1(u, v) & := 2^j \int_{u-2 \cdot 2^{-(j+1)}}^{u-2^{-(j+1)}} \left| (u-s)^{v-\frac{1}{\alpha(s)}} - (u-s-2^{-j-1})^{v-\frac{1}{\alpha(s+2^{-j-1})}} \right| ds \\ & \leq c_3 2^{-j(a-\frac{1}{\alpha})}, \end{aligned} \quad (33)$$

with the convention that $I_j^1(u, v) := 0$ when $u \leq 2^{-(j+1)}$.

Lemma 5. *There exists a positive and finite constant c_4 such that, for any $j \in \mathbb{Z}_+$ and $(u, v) \in I \times [a, b]$, one has*

$$\begin{aligned} I_j^2(u, v) & := 2^j \int_{u-3 \cdot 2^{-(j+1)}}^{u-2 \cdot 2^{-(j+1)}} \left| (u-s)^{v-\frac{1}{\alpha(s)}} - (u-s-2^{-(j+1)})^{v-\frac{1}{\alpha(s+2^{-j-1})}} \right. \\ & \quad \left. - (u-s-2 \cdot 2^{-(j+1)})^{v-\frac{1}{\alpha(s+2 \cdot 2^{-j-1})}} \right| ds \\ & \leq c_4 2^{-j(a-\frac{1}{\alpha})}, \end{aligned} \quad (34)$$

with the convention that $I_j^2(u, v) := 0$ when $u \leq 2 \cdot 2^{-(j+1)}$.

Lemma 6. *There exists a positive and finite constant c_5 such that, for any $j \in \mathbb{Z}_+$ and $(u, v) \in I \times [a, b]$, one has*

$$\begin{aligned} I_j^3(u, v) & := 2^j \int_{u-4 \cdot 2^{-(j+1)}}^{u-3 \cdot 2^{-(j+1)}} \left| (u-s)^{v-\frac{1}{\alpha(s)}} - (u-s-2^{-(j+1)})^{v-\frac{1}{\alpha(s+2^{-j-1})}} \right. \\ & \quad \left. - (u-s-2 \cdot 2^{-(j+1)})^{v-\frac{1}{\alpha(s+2 \cdot 2^{-j-1})}} \right. \\ & \quad \left. + (u-s-3 \cdot 2^{-(j+1)})^{v-\frac{1}{\alpha(s+3 \cdot 2^{-j-1})}} \right| ds \\ & \leq c_5 2^{-j(a-\frac{1}{\alpha})}, \end{aligned} \quad (35)$$

with the convention that $I_j^3(u, v) := 0$ when $u \leq 3 \cdot 2^{-(j+1)}$.

We are now in position to prove Theorem 21.

Proof (Proof of Theorem 21). Let $J \in \mathbb{N}$, $Q \in \mathbb{N}$, $(u, v) \in I \times [a, b]$ and $\omega \in \Omega^*$ be arbitrary and fixed. Using (17), (24), (25), the triangular inequality, (14) and (12), one gets that

$$\begin{aligned}
|X^{J+Q}(u, v, \omega) - X^J(u, v, \omega)| &= \left| \sum_{j=J}^{J+Q-1} \mathcal{L}_j(u, v, \omega) \right| \\
&\leq \sum_{j=J}^{J+Q-1} \left(\sup_{0 \leq k < 2^j} |\tau_{j,k}(\omega)| \right) \left(|w_{j,[2^j u]}(u, v)| + \sum_{k=0}^{[2^j u]-1} |w_{j,k}(u, v) - w_{j,k+1}(u, v)| \right) \\
&= \sum_{j=J}^{J+Q-1} \left(\sup_{0 \leq k < 2^j} |\tau_{j,k}(\omega)| \right) \left(|w_{j,[2^j u]}(u, v)| \right. \\
&\quad \left. + \sum_{k=0}^{[2^j u]-1} \left| 2^j \int_{k2^{-j}}^{(k+1/2)2^{-j}} \left(K_{u,v}(s) - K_{u,v}(s + 2^{-j-1}) - K_{u,v}(s + 2 \cdot 2^{-j-1}) \right. \right. \right. \\
&\quad \left. \left. + K_{u,v}(s + 3 \cdot 2^{-j-1}) \right) ds \right| \right) \\
&\leq \sum_{j=J}^{J+Q-1} \left(\sup_{0 \leq k < 2^j} |\tau_{j,k}(\omega)| \right) \left(|w_{j,[2^j u]}(u, v)| \right. \\
&\quad \left. + 2^j \int_0^{u-2^{-j-1}} \left| K_{u,v}(s) - K_{u,v}(s + 2^{-j-1}) - K_{u,v}(s + 2 \cdot 2^{-j-1}) \right. \right. \\
&\quad \left. \left. + K_{u,v}(s + 3 \cdot 2^{-j-1}) \right| ds \right). \tag{36}
\end{aligned}$$

Next, putting together (36), (10) and Lemmas 1 to 6, one obtains, for any fixed $\zeta > 1/\underline{\alpha}$, that:

$$\begin{aligned}
|X^{J+Q}(u, v, \omega) - X^J(u, v, \omega)| &\leq C'(\omega) \sum_{j=J}^{J+Q-1} (1+j)^\zeta \left(2^{-j \min\{a-\frac{1}{\underline{\alpha}}, \rho_\alpha\}} + 2^{-j} \right. \\
&\quad \left. \cdot \int_0^{u-4 \cdot 2^{-j-1}} (u-s-3 \cdot 2^{-j-1})^{a-1/\underline{\alpha}-2} ds \right) \\
&\leq C''(\omega) \sum_{j=J}^{J+Q-1} (1+j)^\zeta 2^{-j \min\{a-\frac{1}{\underline{\alpha}}, \rho_\alpha\}} \leq C'''(\omega) \sum_{j=J}^{+\infty} (1+j)^\zeta 2^{-j \min\{a-\frac{1}{\underline{\alpha}}, \rho_\alpha\}}, \tag{37}
\end{aligned}$$

where C' and C''' are two positive and finite random variables not depending on J , Q and (u, v) . Thus, one can derive from (37) that $(X^J(\cdot, \cdot, \omega))_{J \in \mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{C}(I \times [a, b])$; its limit in this space is denoted by $\tilde{X}(\cdot, \cdot, \omega)$.

Let us now prove that (19) is satisfied. When Q goes to $+\infty$, it follows from (37) that

$$\begin{aligned} |\tilde{X}(u, v, \omega) - X^J(u, v, \omega)| &\leq C''(\omega) \sum_{j=J}^{+\infty} (1+j)^\zeta 2^{-j \min\{a - \frac{1}{\alpha}, \rho_\alpha\}} \\ &\leq C''(\omega) (1+J)^\zeta 2^{-J \min\{a - \frac{1}{\alpha}, \rho_\alpha\}} \sum_{j=0}^{+\infty} (1+j)^\zeta 2^{-j \min\{a - \frac{1}{\alpha}, \rho_\alpha\}} \\ &\leq C'''(\omega) J^\zeta 2^{-J \min\{a - \frac{1}{\alpha}, \rho_\alpha\}}, \end{aligned} \quad (38)$$

where C''' is a positive and finite random variable not depending on J and (u, v) . Thus, (38) implies that (19) holds.

A Appendix

Proof (Proof of Lemma 2). One assumes that $j \in \mathbb{N}$ and $(u, v) \in I \times [a, b]$ are arbitrary and such that $u \geq 4 \cdot 2^{-j-1}$. Then, one denotes by $L_{u,v}$ the infinitely differentiable function on the open set $(-\infty, u) \times (1/v, +\infty) \subset \mathbb{R}^2$ defined as:

$$L_{u,v}(x, y) := (u - x)^{v-1/y}, \quad \text{for all } (x, y) \in (-\infty, u) \times (1/v, +\infty). \quad (39)$$

Thus, using (10), one has

$$K_{u,v}(z) = L_{u,v}(z, \alpha(z)) \quad \text{for all } z \in [0, u - 2^{-j-1}]. \quad (40)$$

One can derive from (40) and the triangular inequality that, for every $s \in [0, u - 4 \cdot 2^{-j-1}]$,

$$\begin{aligned} &|K_{u,v}(s) - K_{u,v}(s + 2^{-j-1}) - K_{u,v}(s + 2 \cdot 2^{-j-1}) + K_{u,v}(s + 3 \cdot 2^{-j-1})| \\ &= \left| L_{u,v}(s, \alpha(s)) - L_{u,v}(s + 2^{-j-1}, \alpha(s + 2^{-j-1})) \right. \\ &\quad \left. - L_{u,v}(s + 2 \cdot 2^{-j-1}, \alpha(s + 2 \cdot 2^{-j-1})) + L_{u,v}(s + 3 \cdot 2^{-j-1}, \alpha(s + 3 \cdot 2^{-j-1})) \right| \\ &\leq A_{u,v}^j(s) + B_{u,v}^j(s), \end{aligned} \quad (41)$$

where

$$\begin{aligned} A_{u,v}^j(s) &:= \left| L_{u,v}(s, \alpha(s)) - L_{u,v}(s + 2^{-j-1}, \alpha(s + 2^{-j-1})) \right. \\ &\quad \left. - L_{u,v}(s + 2 \cdot 2^{-j-1}, \alpha(s)) + L_{u,v}(s + 3 \cdot 2^{-j-1}, \alpha(s + 2^{-j-1})) \right| \end{aligned} \quad (42)$$

and

$$\begin{aligned} B_{u,v}^j(s) &:= \left| L_{u,v}(s + 2 \cdot 2^{-j-1}, \alpha(s)) - L_{u,v}(s + 3 \cdot 2^{-j-1}, \alpha(s + 2^{-j-1})) \right. \\ &\quad \left. - L_{u,v}(s + 2 \cdot 2^{-j-1}, \alpha(s + 2 \cdot 2^{-j-1})) \right. \\ &\quad \left. + L_{u,v}(s + 3 \cdot 2^{-j-1}, \alpha(s + 3 \cdot 2^{-j-1})) \right|. \end{aligned} \quad (43)$$

First step: The goal of this step is to provide a suitable upper bound for the quantity $A_{u,v}^j(s)$.

For any fixed $s \in [0, u - 4 \cdot 2^{-j-1}]$, one denotes by $g_{1,s}$ the infinitely differentiable function defined as:

$$g_{1,s} : \begin{cases} [0, 2^{-j}] \rightarrow \mathbb{R} \\ x \mapsto L_{u,v}(s+x, \alpha(s)) - L_{u,v}(s+2^{-j-1}+x, \alpha(s+2^{-j-1})). \end{cases} \quad (44)$$

Thus, it follows from (42) that $A_{u,v}^j(s) = |g_{1,s}(2^{-j}) - g_{1,s}(0)|$. Then using the mean value theorem, one obtains that $A_{u,v}^j(s) = 2^{-j}|g'_{1,s}(x_*)|$, for some $x_* \in (0, 2^{-j-1})$. Therefore, one can derive from (44), (39), the triangular inequality and the inequalities $|v - 1/\alpha(s+2^{-j-1})| < 1$ and $|v - 1/\alpha(s)| < 1$ that

$$\begin{aligned} A_{u,v}^j(s) &\leq 2^{-j} \left(\left| (u-s-2^{-j-1}-x_*)^{v-1-1/\alpha(s+2^{-j-1})} \right. \right. \\ &\quad \left. \left. - (u-s-2^{-j-1}-x_*)^{v-1-1/\alpha(s)} \right| \right. \\ &\quad \left. + \left| \frac{1}{\alpha(s+2^{-j-1})} - \frac{1}{\alpha(s)} \right| (u-s-2^{-j-1}-x_*)^{v-1-1/\alpha(s)} \right. \\ &\quad \left. + \left| (u-s-2^{-j-1}-x_*)^{v-1-1/\alpha(s)} - (u-s-x_*)^{v-1-1/\alpha(s)} \right| \right). \end{aligned} \quad (45)$$

Next, notice that it follows from the assumption $\alpha \in \mathcal{C}^{1+\rho_\alpha}(I)$, (1), $x_* \in (0, 2^{-j-1})$ and $v \in [a, b]$, that

$$\begin{aligned} &\left| \frac{1}{\alpha(s+2^{-j-1})} - \frac{1}{\alpha(s)} \right| (u-s-2^{-j-1}-x_*)^{v-1-1/\alpha(s)} \\ &\leq c_1 2^{-j-1} (u-s-2^{-j-1}-x_*)^{v-1-1/\alpha(s)} \\ &\leq c_1 2^{-j-1} (u-s-3 \cdot 2^{-j-1})^{a-1-1/\alpha}, \end{aligned} \quad (46)$$

where c_1 is a constant not depending on j, u, s, v . Thus, using (45) and (46), and applying the mean value theorem to the functions:

$$\begin{aligned} g_{2,s,x_*} &: \begin{cases} [0, 2^{-j-1}] \rightarrow \mathbb{R} \\ w \mapsto (u-s-x_*-w)^{v-1-\frac{1}{\alpha(s)}} \end{cases} \\ g_{3,s,x_*} &: \begin{cases} [\alpha(s) \wedge \alpha(s+2^{-j-1}), \alpha(s) \vee \alpha(s+2^{-j-1})] \rightarrow \mathbb{R} \\ z \mapsto (u-s-2^{-j-1}-x_*)^{v-1-\frac{1}{z}} \end{cases}, \end{aligned}$$

one obtains that

$$\begin{aligned} A_{u,v}^j(s) &\leq c_2 2^{-2j} \left((u-s-3 \cdot 2^{-j-1})^{a-1-1/\alpha} + (u-s-3 \cdot 2^{-j-1})^{a-2-1/\alpha} \right. \\ &\quad \left. + |\log(u-s-2^{-j-1})| (u-s-3 \cdot 2^{-j-1})^{a-1-1/\alpha} \right) \\ &\leq c_2 2^{-2j} \left(2(u-s-3 \cdot 2^{-j-1})^{a-2-1/\alpha} + |\log(u-s-2^{-j-1})| \right. \\ &\quad \left. \cdot (u-s-3 \cdot 2^{-j-1})^{a-1-1/\alpha} \right), \end{aligned} \quad (47)$$

where c_2 is a constant not depending on j, u, s, v . Finally, combining (47) and the inequality $|\log(x)| \leq x^{-1}$, for all $x \in (0, 1]$, one gets that

$$A_{u,v}^j(s) \leq c_3 2^{-2j} (u - s - 3 \cdot 2^{-j-1})^{\alpha-2-1/\alpha}, \quad (48)$$

where c_3 is a constant not depending on j, u, s, v .

Second step: The goal of this step is to provide a suitable upper bound for the quantity $B_{u,v}^j(s)$.

In view of (43), the quantity $B_{u,v}^j(s)$ can be rewritten as:

$$\begin{aligned} B_{u,v}^j(s) = & \left| (L_{u,v}(s + 2 \cdot 2^{-j-1}, \alpha(s)) - L_{u,v}(s + 2 \cdot 2^{-j-1}, \alpha(s + 2 \cdot 2^{-j-1}))) \right. \\ & - (L_{u,v}(s + 3 \cdot 2^{-j-1}, \alpha(s + 2^{-j-1})) \\ & \left. - L_{u,v}(s + 3 \cdot 2^{-j-1}, \alpha(s + 3 \cdot 2^{-j-1}))) \right|. \end{aligned}$$

Thus applying the mean value theorem to the functions

$$\begin{aligned} g_{4,s}: & \begin{cases} [\alpha(s) \wedge \alpha(s + 2 \cdot 2^{-j-1}), \alpha(s) \vee \alpha(s + 2 \cdot 2^{-j-1})] \longrightarrow \mathbb{R} \\ y \longmapsto L_{u,v}(s + 2 \cdot 2^{-j-1}, y), \end{cases} \\ g_{5,s}: & \begin{cases} [\alpha(s + 2^{-j-1}) \wedge \alpha(s + 3 \cdot 2^{-j-1}), \alpha(s + 2^{-j-1}) \vee \alpha(s + 3 \cdot 2^{-j-1})] \longrightarrow \mathbb{R} \\ y \longmapsto L_{u,v}(s + 3 \cdot 2^{-j-1}, y). \end{cases} \end{aligned}$$

and putting together the triangular inequality, the assumption $\alpha \in \mathcal{C}^{1+\rho_\alpha}(I)$, (1), $v \in [a, b]$, and the equality

$$df - gh = d(f - h) + h(d - g), \quad \text{for all } d, f, g, h \in \mathbb{R},$$

one obtains, for some

$$y_* \in (\alpha(s) \wedge \alpha(s + 2 \cdot 2^{-j-1}), \alpha(s) \vee \alpha(s + 2 \cdot 2^{-j-1})) \quad (49)$$

and

$$y_{**} \in (\alpha(s + 2^{-j-1}) \wedge \alpha(s + 3 \cdot 2^{-j-1}), \alpha(s + 2^{-j-1}) \vee \alpha(s + 3 \cdot 2^{-j-1})), \quad (50)$$

that

$$\begin{aligned} B_{u,v}^j(s) \leq & c_4 2^{-j(1+\rho_\alpha)} \left| \frac{1}{y_*^2} \log(u - s - 2 \cdot 2^{-j-1}) \right| (u - s - 2 \cdot 2^{-j-1})^{\alpha-1/\alpha} \\ & + c_4 2^{-j} \left| \left(\frac{1}{y_*^2} - \frac{1}{y_{**}^2} \right) \log(u - s - 2 \cdot 2^{-j-1}) \right| (u - s - 2 \cdot 2^{-j-1})^{\alpha-1/\alpha} \\ & + c_4 2^{-j} \left| \frac{1}{y_{**}^2} \log(u - s - 2 \cdot 2^{-j-1}) \right| \left| (u - s - 2 \cdot 2^{-j-1})^{v-1/y_*} \right. \\ & \left. - (u - s - 2 \cdot 2^{-j-1})^{v-1/y_{**}} \right| \\ & + c_4 2^{-j} \left| \frac{1}{y_{**}^2} \left((u - s - 2 \cdot 2^{-j-1})^{v-1/y_{**}} \log(u - s - 2 \cdot 2^{-j-1}) \right. \right. \\ & \left. \left. - (u - s - 3 \cdot 2^{-j-1})^{v-1/y_{**}} \log(u - s - 3 \cdot 2^{-j-1}) \right) \right|, \quad (51) \end{aligned}$$

where c_4 is a constant not depending on j, u, s, v, y_*, y_{**} . Next, notice that using (1), (49), (50) and the assumption $\alpha \in \mathcal{C}^{1+\rho_\alpha}(I)$, one gets that

$$\max \left\{ \frac{1}{y_*^2}, \frac{1}{y_{**}^2} \right\} < 1 \quad (52)$$

and

$$\begin{aligned} \left| \frac{1}{y_*^2} - \frac{1}{y_{**}^2} \right| &= \\ \left| \frac{y_{**}^2 - y_*^2}{y_{**}^2 y_*^2} \right| &\leq 4 |y_{**} - y_*| \leq 4 \left| \max_{0 \leq i \leq 3} \alpha(s + i2^{-j-1}) - \min_{0 \leq i \leq 3} \alpha(s + i2^{-j-1}) \right| \\ &\leq c_5 2^{-j}, \end{aligned} \quad (53)$$

where c_5 is a constant not depending on j, u, s, v, y_*, y_{**} . Also, notice that applying the mean value theorem to the function

$$g_{6,s} : \begin{cases} [y_* \wedge y_{**}, y_* \vee y_{**}] \longrightarrow \mathbb{R} \\ x \longmapsto (u - s - 2 \cdot 2^{-j-1})^{v-1/x} \end{cases}$$

and using $v \in [a, b]$, $y_*, y_{**} \in (\underline{\alpha}, \bar{\alpha})$, the second and the third inequality in (53), one obtains that

$$\begin{aligned} &\left| (u - s - 2 \cdot 2^{-j-1})^{v-1/y_*} - (u - s - 2 \cdot 2^{-j-1})^{v-1/y_{**}} \right| \\ &\leq c_6 2^{-j} (u - s - 2 \cdot 2^{-j-1})^{a-1/\alpha} \left| \log(u - s - 2 \cdot 2^{-j-1}) \right|, \end{aligned} \quad (54)$$

where c_6 is a constant not depending on j, u, s, v, y_*, y_{**} . Moreover, notice that applying the mean value theorem to the function

$$g_{7,s,y_{**}} : \begin{cases} [0, 2^{-j-1}] \longrightarrow \mathbb{R} \\ x \longmapsto (u - s - 2 \cdot 2^{-j-1} - x)^{v-1/y_{**}} \log(u - s - 2 \cdot 2^{-j-1} - x) \end{cases}$$

and making use of $v \in [a, b]$ and $y_*, y_{**} \in (\underline{\alpha}, \bar{\alpha})$, it follows that

$$\begin{aligned} &\left| (u - s - 2 \cdot 2^{-j-1})^{v-1/y_{**}} \log(u - s - 2 \cdot 2^{-j-1}) \right. \\ &\quad \left. - (u - s - 3 \cdot 2^{-j-1})^{v-1/y_{**}} \log(u - s - 3 \cdot 2^{-j-1}) \right| \\ &\leq c_7 2^{-j} (u - s - 3 \cdot 2^{-j-1})^{a-1/\alpha-1} \left(1 + \log(u - s - 3 \cdot 2^{-j-1}) \right), \end{aligned} \quad (55)$$

where c_7 is a constant not depending on j, u, s, v, y_*, y_{**} . Next putting together (51) to (55), one gets that

$$\begin{aligned} B_{u,v}^j(s) &\leq c_4 2^{-j(1+\rho_\alpha)} \left| \log(u - s - 2 \cdot 2^{-j-1}) \right| (u - s - 2 \cdot 2^{-j-1})^{a-1/\alpha} \\ &\quad + c_8 2^{-2j} \left((u - s - 2 \cdot 2^{-j-1})^{a-1/\alpha} \left| \log(u - s - 2 \cdot 2^{-j-1}) \right| \right. \\ &\quad + (u - s - 2 \cdot 2^{-j-1})^{a-1/\alpha} \log^2(u - s - 2 \cdot 2^{-j-1}) \\ &\quad \left. + (u - s - 3 \cdot 2^{-j-1})^{a-1/\alpha-1} \left(1 + \log(u - s - 3 \cdot 2^{-j-1}) \right) \right), \end{aligned} \quad (56)$$

where c_8 is a constant not depending on j, u, s, v, y_*, y_{**} . Then, one can derive from (56) and the inequalities

$$\sup_{z \in (0,1]} |\log(z)| z^{a-1/\underline{\alpha}} < +\infty, \quad \sup_{z \in (0,1]} |\log^2(z)| z^{a-1/\underline{\alpha}} < +\infty \text{ and } |\log(x)| \leq |x|^{-1},$$

for all $x \in (0, 1]$, that

$$B_{u,v}^j(s) \leq c_9 (2^{-j(1+\rho_\alpha)} + 2^{-2j} |u - s - 3 \cdot 2^{-j-1}|^{a-1/\underline{\alpha}-2}), \quad (57)$$

where c_9 is a constant not depending on j, u, s, v . Finally combining (41), (48) and (57), one obtains (31).

Proof (Proof of Lemma 3). It easily follows from (14), (12), (10), (1) and the assumption that $v \in [a, b]$ that

$$\begin{aligned} |w_{j,[2^j u]}(u, v)| &\leq 2^j \int_{2^{-j}[2^j u]}^u (u-s)^{v-\frac{1}{\alpha(s)}} ds \\ &\leq 2^j \int_{2^{-j}[2^j u]}^u 2^{-j(a-1/\underline{\alpha})} ds \leq 2^{-j(a-1/\underline{\alpha})}. \end{aligned}$$

The proofs of Lemmas 4, 5 and 6 are very similar so we only give that of Lemma 6.

Proof (Proof of Lemma 6). Let $j \in \mathbb{Z}_+$ and $(u, v) \in I \times [a, b]$ be arbitrary and such that $u > 3 \cdot 2^{-(j+1)}$. In view of the assumptions on (u, v) and (1), it can easily be seen that, for all $s \in [u - 4 \cdot 2^{-(j+1)}, u - 3 \cdot 2^{-(j+1)}]$ and for any $q \in \{0, 1, 2, 3\}$, one has

$$\begin{aligned} 0 &\leq (u - s - q \cdot 2^{-(j+1)})^{v - \frac{1}{\alpha(s+q \cdot 2^{-j-1})}} \\ &\leq (7 \cdot 2^{-(j+1)})^{v - \frac{1}{\alpha(s+q \cdot 2^{-j-1})}} \leq 7^{b-1/\bar{\alpha}} \cdot 2^{-(j+1)(a-1/\underline{\alpha})}. \end{aligned}$$

Thus, using the triangular inequality one gets that

$$\begin{aligned} I_j^3(u, v) &:= 2^j \int_{u-4 \cdot 2^{-(j+1)}}^{u-3 \cdot 2^{-(j+1)}} \left| (u-s)^{v-\frac{1}{\alpha(s)}} - (u-s-2^{-(j+1)})^{v-\frac{1}{\alpha(s+2^{-j-1})}} \right. \\ &\quad \left. - (u-s-2 \cdot 2^{-(j+1)})^{v-\frac{1}{\alpha(s+2 \cdot 2^{-j-1})}} \right. \\ &\quad \left. + (u-s-3 \cdot 2^{-(j+1)})^{v-\frac{1}{\alpha(s+3 \cdot 2^{-j-1})}} \right| ds \\ &\leq 2^j \int_{u-4 \cdot 2^{-(j+1)}}^{u-3 \cdot 2^{-(j+1)}} 4 \cdot 7^{b-1/\bar{\alpha}} \cdot 2^{-(j+1)(a-1/\underline{\alpha})} ds \\ &= 2^{-1} \cdot 4 \cdot 7^{b-1/\bar{\alpha}} \cdot 2^{-(j+1)(a-1/\underline{\alpha})}, \end{aligned}$$

which shows that (35) is satisfied.

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