Fractal analysis of the multifractality of foreign exchange rates

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Abstract. The multifractional model with random exponent (MPRE) is one of the most recent fractional models which extend the fractional Brownian motion (fBm). This paper is an empirical contribution to the justification of the MPRE. Working with several FX rates between 2006 and 2016, sampled every minute, we show the statistical significance of various fractional models applied to log-prices, from the fBm to the MPRE. We propose a method to extract realized Hurst exponents from log-prices. This provides us with a series of Hurst exponents on which we can estimate different models of dynamics. In the MPRE framework, the data justify using a fractional model for the dynamic of the Hurst exponent. We estimate and interpret the value of the key parameter of this model of nested fractality, which is the Hurst exponent of the Hurst exponents.

Keywords. Fractional Brownian motion, Hurst exponent, foreign exchange rate, multifractional Brownian motion, stable process, multifractional process with random exponent.

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J.E.L. classification. C12, C58.

1 Introduction

Since the seminal work of Mandelbrot [45], many articles empirically highlight the fractal nature of financial time series. In the same time, different types of fractal models are developing. Among them, the fractional Brownian motion (fBm) is one of the most simple. Most practitioners however disregard the fBm for modelling log-prices because it contravenes the efficient market hypothesis (EMH), as it assumes correlated increments. But this model is widely used for

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modelling volatility for example [37], as this specific application of the fBm is consistent with the EMH. Besides, the EMH is nowadays seriously challenged. For instance, the adaptive market hypothesis (AMH) does not declare, as does the EMH, that statistical arbitrage is absurd. On the contrary, it considers that models may produce relevant forecasts, at least for a limited period of time [44]. Modelling log-prices with a multifractional Brownian motion (mBm), in which the correlation varies with time, is thus consistent with the AMH, and we can apply the literature on fractal models not only to volatility but also to log-prices.

Once a model is established, it is customary to see if another model cannot do better, by conducting new empirical studies and by proposing alternative models that reproduce the new stylized facts. This is why the fBm is the first of many stages in a journey inside fractal models in financial time series. Indeed, in the fBm, the Hurst exponent describes the fractal feature of the series. It assumes that this feature will not change with time. Empirical evidence suggests adopting a multifractal approach, for example with a time-varying Hurst exponent as in the mBm. All these models are based on the brilliant idea of Mandelbrot of modelling the fractal features of financial time series with either the fractional derivative or the fractional integral of a standard Brownian motion (Bm). Some fractal models go in a different direction, exploiting for example the fractal properties of stable processes instead of the Brownian dynamics. In the vein of fractional Brownian models, one of the most recent and advanced models assumes that the time-varying Hurst exponent follows a stochastic process: it is the multifractional process with random exponent (MPRE) [4]. This model is a progress compared to the mBm as it is in general more parsimonious and thus less subject to overfitting.

While it is a promising multifractal model, the MPRE is only a generic model as it does not state which model should describe the dynamic of H, the Hurst exponent. As we write this article, only a few articles propose a model for H, and even less, if any, justify it empirically. The empirical justification of the MPRE itself, whatever the model used for the Hurst exponent, is limited to a few articles.

The purpose of this paper is thus to explore empirically the relevance of the MPRE and to propose a data-consistent model for the dynamic of the Hurst exponent. We will show that the fBm is an appropriate model for the Hurst exponent. This means that the fractal properties of the log-prices are described by parameters which have also fractal properties. We also explore other fractional models and their statistical significance, going stage by stage from fBm to MPRE, with an additional focus on stable processes. We will see that each extension of the Bm which is studied in our paper is fully justified empirically.

The empirical part of our work is about FX rates. In particular, we are considering ten FX pairs, between December 2006 and January 2016, using prices sampled every minute. These pairs are the following ones: AUD/JPY, AUD/USD, EUR/AUD, EUR/CAD, EUR/GBP, EUR/JPY, EUR/USD, GBP/JPY, GBP/USD and USD/JPY. When working with high-frequency data, the FX market is interesting, more than the equity market, because it is rarely closed, so that we do not need complicated techniques to handle the closing periods. For each pair, we then work with a series of more than 3,350,000 log-prices.

This paper is organized following the history of fractional models. In Section 2, we explore the monofractal properties of log-prices, we see how we can model them with fractional processes, we present an estimation technique, and we apply it to FX rates to study the statistical significance of this approach compared to the standard Bm. We conduct a similar study on multifractal properties in Section 3 and on their random counterpart in Section 4. In Section 5, we discuss hypothetical future extensions of these models, where we replace the nested fractality of the MPRE by a multilayer fractality.

2 Fractal properties

The standard Bm has a fractal property, that is to say it is self-similar: the scaled process has indeed the same probability distribution at two distinct times. The scaling rule consists in dividing the process at time $t > 0$ by $\sqrt{t}$. One often applies this simple model in finance to describe log-prices. But empirical studies stress the fact that this scaling rule is often not realistic [24, 45, 48, 60]. One should instead use a scaling rule in t^H, where H is a Hurst exponent [40]. Once we know this stylized fact, we have to introduce a stochastic process consistent with this scaling rule. The fBm is then a natural extension of the standard Bm that follows Hurst's scaling rule. We now introduce this model as well as other useful fractional processes and show an empirical application to FX rates.

2.1 Fractional processes

In finance, following the old seminal work of Bachelier, the standard Bm has long been the (largely) predominating model to describe the dynamic of the log-prices of a stock [5]. Some extensions, such as the Ornstein-Uhlenbeck model, allow to take into account the stationarity of a series. This makes it possible to describe the dynamic of rates, for example. Finally, with few adjustments, practitioners and researchers have used the standard Bm to describe almost all the asset classes.

Mandelbrot was one of the first to look for an alternative to using the standard Bm in finance. For this purpose, he applied the fBm he had introduced with van Ness in the 1960s. After some decades, during which the corresponding literature slowly increased, the fBm is becoming a serious challenger to the standard Bm for financial applications. Several financial crisis underlined the limitation of the Bm and have favored the rise of the fBm concept in the financial literature [2, 41, 51]. Among the many applications of the fBm in finance, we can cite predictions [27, 31, 33, 50] and pricing [10, 29, 59]. These publications often interest in priority the econophysics community. But the emergence of the rough volatility concept, which is also based on the fBm, tends to settle this model as a mainstream approach for modelling volatility [37, 43].

The simplicity of the definition of the fBm partly explains the success of this concept in finance. Following Mandelbrot and van Ness, the fBm is a weighted integral of a standard Bm [47]:

$$B_t^H = \frac{\sigma}{\Gamma\left(H+\frac{1}{2}\right)} \int_{-\infty}^{\infty} \left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\right) dW_s, \tag{1}$$

where W is a standard Bm and B^H is a fBm of Hurst exponent H and volatility σ. This definition is simple in the sense that it only relies on one more parameter than the Bm itself. In particular, the standard Bm is a particular case of the fBm, for $H = 1/2$. Besides its simplicity, this model also properly describes the fractal property of the observed log-prices, thanks to an appropriate weight kernel. We already explained that the standard Bm also follows a fractal property, but the corresponding scaling rule is then axiomatic, contrary to the fBm case, for which it is estimated on the data. Taking into account the fractal property contained in the log-prices is important, because decisions can be made for example with the help of risk measures scaled to the risk horizon, and, for this purpose, the square-root rule is often too simplistic [63].

It is worth noting that the fractal property of the fBm is obtained thanks to a fractional technique. Indeed, the weighting kernel in the integral definition of the fBm corresponds to the fractional derivative (respectively the fractional integral) of order $|H - 1/2|$ of a standard Bm if $H < 1/2$ (resp. $H > 1/2$). Therefore, the fBm is a natural generalization of the Bm and it is rougher or smoother than the Bm itself, depending on the value of the Hurst exponent. This provides a simple interpretation of the value of H:

- If $H = 1/2$, non-overlapping increments of the fBm are independent and the process is even a standard Bm.
- If $H > 1/2$, the process is persistent, which means that non-overlapping increments of the fBm are positively correlated, what is in contradiction with the EMH if the fBm is applied directly to log-prices [12, 16, 33, 55].
- If $H < 1/2$, the process is anti-persistent, which means that non-overlapping increments of the fBm are negatively correlated. This makes the prediction more complicated than in the persistent case [31, 33], but this model is still in contradiction with the EMH.

We can see the fBm as a Gaussian process with increments made dependent thanks to a single parameter, H, and totally defined by the following covariance structure, which is a consequence of the integral definition:

$$\mathbb{E}\{B_t^H B_s^H\} = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

But making the increments of the Bm correlated is not the only way to modify the fractal property of the Bm. One can indeed alternatively play on another shared property of the Bm and the fBm, which is the Gaussian feature. One can then define a process with symmetric alpha-stable increments. This wide parametric family of random variables is parametrized by $\alpha \in (0, 2]$. It contains the particular case of Gaussian variables, for $\alpha = 2$. It has been used for example in signal processing [35, 36] or in medicine [56]. It is also possible to combine dependence of the increments and alpha-stable distribution in the fractional Lévy stable motion (fLsm) [56, 62]. An integral definition exists for this process:

$$W_t^{\alpha,m} = \sigma \int_{-\infty}^{\infty} \left((t-s)_+^m - (-s)_+^m\right) dW_s^{\alpha,0},$$

where $W_s^{\alpha,0}$ is a symmetric Lévy α-stable motion, that is to say a stochastic process with i.i.d. increments of symmetric Lévy α-stable distribution.[1] The fLsm of parameters m and α has the same fractal property as the fBm of Hurst exponent H such that:

$$H = \frac{1}{\alpha} + m.$$

The parameter m is a memory parameter. It is positive (respectively negative) if non-overlapping increments of the fLsm are positively (resp. negatively) correlated. The parameter α is related to the law of the increments. The larger α is, the closer the law is to the Gaussian law. Therefore, estimating the Hurst exponent is sometimes not enough for financial applications, because knowing the structure of the dependence of the increments is often overriding. For example, in the fLsm framework, a negative correlation is possible even with an estimated Hurst exponent above 1/2. This is in contradiction with the interpretation made of the fBm in econophysics, but it is not only a theoretical particular case as we will see in the following application to FX rates.

2.2 Statistics of fractional processes

Before applying fractional processes, namely fBm and fLsm, to FX rates, we have to explain the estimation method for all the parameters introduced above.

First, several estimation techniques are possible for the Hurst exponent of a fBm. In empirical sciences and econophysics, the rescaled range analysis (R/S) [40] and the detrended fluctuation analysis (DFA) [53, 57] are popular methods. But both R/S and DFA rely on a big number of observations, due to multiple regressions. We then put forward a simple method based on the empirical absolute moments of log-price increments. The absolute moment of order k of the increments of a process X, in a time interval $[0, N]$ and for a given scale τ, is defined by:

$$M_{k,\tau,N}(X) = \frac{1}{\lfloor N/\tau \rfloor} \sum_{i=1}^{\lfloor N/\tau \rfloor} |X_{i\tau} - X_{(i-1)\tau}|^k.$$

This statistic is well defined for a fBm because its increments are stationary. But, for other models, the distribution of $M_{k,\tau,N}(X)$ may depend on the times

[1] A Γ function appeared in the definition of the fBm, equation (1), for normalizing the process. The α-stable distribution has a scale parameter which makes it possible to tune the magnitude of $W^{\alpha,m}$ directly in the definition of $W^{\alpha,0}$, or in σ.

of observation. In the fBm case, $\ln(M_{k,\tau,N}(X))$ is proportional to H. We can then define several estimators of Hurst exponents using these absolute moments [8, 31]. The rationale consists in confronting these absolute-moment statistics at several scales, two in the most simple case and a great number of scales in more advanced estimators. The estimator of H is then $1/k$ times the slope of the regression of $\ln(M_{k,\tau,N}(X))$ on $\ln(\tau)$ [22].

When we work in the framework of the fLsm, we can still estimate the key parameter of the scaling rule, that is to say the Hurst exponent, exactly like in the fBm case. We then estimate the α parameter using the characteristic function of a centred symmetric α-stable random variable:

$$u \in \mathbb{R} \mapsto \exp(-\gamma|u|^{\alpha}),$$

where γ is a scale parameter of the α-stable variable and u is a transform variable.[2] Confronting the values of the characteristic function at various transform variables and scales provides us with an estimate of α. We can work with many transform variables and scales or, more easily, with only two properly chosen transform variables and at the finest scale:

$$\hat{\alpha} = \frac{1}{\ln(2)} \ln\left(\frac{\ln\left(\frac{1}{N}\sum_{t=1}^{N}\cos\left(2u(X_t - X_{t-1})\right)\right)}{\ln\left(\frac{1}{N}\sum_{t=1}^{N}\cos\left(u(X_t - X_{t-1})\right)\right)}\right), \qquad (2)$$

where the transform variable u in what follows is $u = \pi/4Q_{90\%}$, where $Q_{90\%}$ is the quantile of the absolute increments $|X_t - X_{t-1}|$ of probability 90% [56].

The estimation of m is a direct consequence of the estimation of H and of α:

$$\hat{m} = \hat{H} - \frac{1}{\hat{\alpha}},$$

where $\hat{H}$ is the Hurst exponent estimated with the method presented above.

2.3 Application to FX rates

Using the absolute-moment estimator presented above, we estimate a Hurst exponent for all the FX pairs. We use an absolute moment of order 2 and consider the 200 lowest scales, between 1 and 200 minutes. For all the FX pairs studied, we obtain an affine regression with a very high R^2, always above 0.999, which thus confirms the fractal property of log-rates. For example, we provide the plot of the logarithm of the empirical absolute moment as a function of the logarithm of the time scale for the EUR/CAD pair, in Figure 1. The Hurst exponent for all the pairs is between 0.476 and 0.506, as exposed in Table 1, indicating only a slight deviation from the standard case $H = 1/2$. In the next section, we will see that the series does not look like a standard Bm because the Hurst exponent varies significantly over time. Here, we can only say that, on the very

[2] In the case where $\alpha = 2$, that is to say for Gaussian variables, γ is half the variance.

long run, the series is not that far from a standard Bm, even though, given the high number of observations, all the estimated Hurst exponents, except for USD/JPY, are significantly different from 1/2, either higher or lower. We have determined the corresponding confidence level, displayed in Table 1, with the help of a bootstrap.

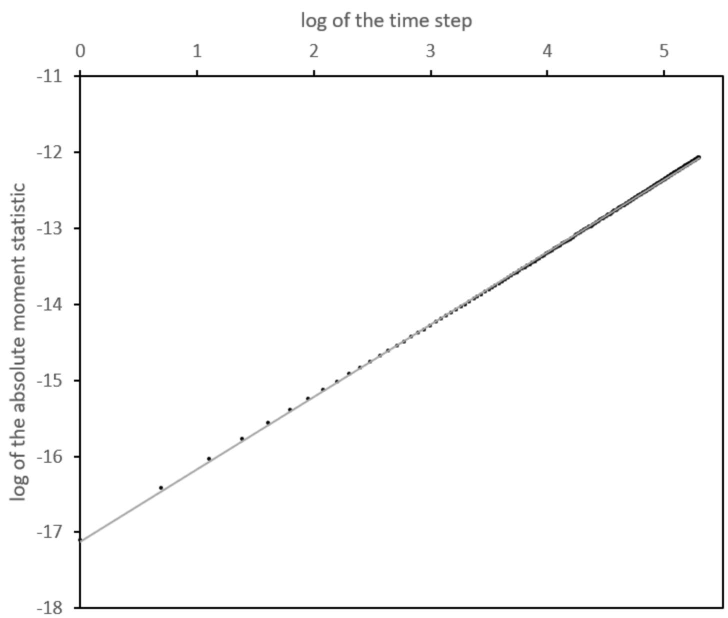

Fig. 1: Empirical log plot for the EUR/CAD series, with the corresponding affine regression in grey. We estimate the Hurst exponent by half the slope of the regression.

We explained earlier that most of the dynamics used in finance stem from the standard Bm, with adjustments taking into account some peculiarities. In the rate framework, one then use stationary models, transforming the Bm with the Ornstein-Uhlenbeck approach or the Lamperti transform. It is also possible to stationarize a fBm with similar techniques in order to make it more relevant for the rate framework [20, 21, 28, 32]. But, if the log-rates in our dataset were stationary, we should observe a flattening of the log plot for low resolutions in Figure 1. We do neither observe such a phenomenon for EUR/CAD nor for other FX pairs. This means that, given the size of the series and the considered scales, the stationarity is not strong enough to be visible.

We are now interested in the fLsm setting. While the Hurst exponent is close to 1/2, the estimated α parameters are all far from 2, which is the value corresponding to a standard Bm, as we can see in Table 1. This means that the distribution of the log-price increments has a fat tail. As a consequence,

Table 1: Estimation of the parameters of the fBm and the fLsm. We indicate the confidence level of rejecting the hypothesis of a Hurst exponent equal to 1/2 by * for 90%, ** for 95%, *** for 99%.

	Estimated H	Estimated α	Estimated m
AUD/JPY	0.505***	1.490	−0.166
AUD/USD	0.498**	1.480	−0.178
EUR/AUD	0.485***	1.543	−0.164
EUR/CAD	0.476***	1.510	−0.186
EUR/GBP	0.503**	1.467	−0.178
EUR/JPY	0.504***	1.489	−0.167
EUR/USD	0.506***	1.454	−0.182
GBP/JPY	0.496***	1.476	−0.182
GBP/USD	0.503**	1.396	−0.214
USD/JPY	0.500	1.477	−0.177

all the estimated m are significantly negative. The series of FX rates is anti-persistent. It is a striking case for which a rapid analysis would conclude that the standard Bm is not a too bad model, because Hurst exponents are not far from 1/2, whereas, regarding α and m instead of H, non-overlapping increments are strongly negatively correlated.

We can also check by other means that the distribution of log-price increments is not Gaussian. For this purpose, we conduct a Kolmogorov-Smirnov test, using only the last 100,000 log-returns of each FX pair in our dataset. The null hypothesis is the Gaussian nature of these log-returns. For EUR/CAD, the statistic of the test is 0.094. We are thus very far from a Gaussian distribution. The null hypothesis is rejected with a p-value of magnitude 10^{-16}. We obtain similar results for the other currencies.

This statistical test confirms that a Bm is not a relevant model for FX rates. The empirical fractal properties, close to the ones of a standard Bm, are thus depicted by a fat-tail distribution and an anti-persistence, in the setting of a fLsm. But we can go further and consider that the scaling rule is not as simple as in the fractional models. In the next section, we will consider scaling rules based on several parameters, that is to say a multifractal property instead of the simple monofractal property.

3 Multifractal properties

While the standard Bm has only an arbitrary fractal property, that is a scaling rule which is not necessarily consistent with the data to which we apply this model, the fBm and the fLsm generalize the standard Bm with an appropriate fractal property. It is a monofractal property because it depends on a unique parameter, the Hurst exponent. After the focus on fractional models during the 1960s, empirical evidence has suggested, since the late 1990s, that a multifractal property was leading log-prices [6, 19, 31, 46, 58]. In other words, the scaling rule of the series relies on more than one parameter.

A clear observation of the multifractality appears when the standard absolute-moment estimation methods lead either to a nonlinear log-plot or to distinct results when changing the order of the moment. The nonlinear log-plot is well known in medicine for example [54, 56, 57] but some empirical studies [32] in finance also lead to the same conclusion. The order-dependent estimator is known as the generalized Hurst exponent (GHE) [25, 26, 51]. The GHE thus provides us with a Hurst function. We can interpret its curvature as a deviation to the ideal monofractal case.

In order to follow these empirical findings, new models are proposed in the literature. Among them, the multifractional models are a natural generalisation of the fBm. We now present them, as well as the corresponding estimators and an empirical study on the same high-frequency FX dataset.

3.1 Multifractional processes

Many models have multifractal properties. In order to restrict the subject, we focus on models that are locally monofractal. This means that when we consider a series of log-prices in a small time interval, a monofractal model such as the fBm is relevant, but, in the longer run, the scaling behaviour varies over time. A straightforward generalization of the fractional models is thus possible: the mBm is an extension of the fBm in which the Hurst exponent is time-dependent [9, 52], a multifractional stable motion is an extension of the fLsm in which α and m are time-dependent [61].

A natural question arises: how do these parameter of multifractional processes evolve over time? In a first approach, we simply assume a smooth and deterministic variation. We will see in the next section another dynamic for these parameters. The mBm thus assumes a smooth evolution of the Hurst exponent. In particular, the trajectory of a mBm is square-integrable if its Hurst exponent evolves as a Hölderian function of time of order $0 < \eta \leq 1$ [22, 23]. For a mBm Y, of time-dependent Hurst exponent $t \in \mathbb{R} \mapsto H(t)$, the integral definition is [22]:

$$Y_t = \frac{\sigma\sqrt{\pi K(2H(t))}}{\Gamma\left(H(t)+\frac{1}{2}\right)} \int_{-\infty}^{\infty} \left((t-s)_+^{H(t)-1/2} - (-s)_+^{H(t)-1/2}\right) dW_s,$$

which provides us with the following covariance function [22]:

$$\mathbb{E}\{Y_t Y_s\} = \frac{\sigma^2}{2} g(H(t), H(s))(|t|^{H(t)+H(s)} + |s|^{H(t)+H(s)} - |t-s|^{H(t)+H(s)}),$$

where

$$\begin{cases} g : (x, y) \mapsto K(x+y)^{-1}\sqrt{K(2x)K(2y)} \\ K : x \mapsto \Gamma(1+x)\sin(x\pi/2)/\pi. \end{cases}$$

For s and t such that $H(t) = H(s)$, we have $g(H(t), H(s)) = 1$ and the covariance is equal to the one of a fBm.

3.2 Statistics of multifractional processes

In the mBm, increments are not stationary, and the absolute moment is thus time-dependent. Nevertheless, on a small time scale, when we assume that the Hurst exponent evolves smoothly, we can make the approximation that the local absolute moment is based on almost stationary increments. Therefore, the usual estimation method of the time-varying Hurst exponent consists in estimating it like a fBm on sliding windows [13, 22, 31]. In order to be consistent with the state of the art in financial econometrics, we instead propose a series of *realized Hurst exponents*. In a similar way to the realized volatility [7, 34, 49], we use intraday log-prices to estimate a Hurst exponent for each separate day instead of estimating it in a sliding window. We are thus working with non-overlapping time intervals.

For each day, we have much less data than in the whole sample, so instead of using a linear regression of the logarithm of the empirical absolute moments on 200 log-scales, we simply use the two finest scales. For a given day d, we observe n_d log-prices: $X_{d,0},...,X_{d,n_d-1}$. The corresponding realized Hurst exponent, that is to say the estimated Hurst exponent for day d using intraday log-prices, is thus defined by:

$$\hat{H}_{d,k} = \frac{1}{k}\log_2\left(\frac{n_d-1}{\lfloor (n_d-1)/2\rfloor}\frac{\sum_{i=1}^{\lfloor (n_d-1)/2\rfloor}|X_{d,2i}-X_{d,2(i-1)}|^k}{\sum_{i=1}^{n_d-1}|X_{d,i}-X_{d,i-1}|^k}\right).$$

In the following empirical application, we will limit the analysis to $k = 2$. In the same manner, in the stable distribution framework, we can define realized α and realized m by restricting the time interval in equation (2) to day d.

In the assumptions of the mBm, the Hurst exponent is supposed to vary smoothly with the time. It may thus be useful to smooth the series of realized Hurst exponents as a post-processing. One can achieve this with a parametric model [1], a moving average [39], or a variational smoothing [31]. In the present paper, we only work with raw Hurst exponents, so that we can conduct a statistical analysis to determine whether the evolutions of the Hurst exponent are significant. Indeed, we must beware a possible overfitting of the mBm, and check the significance of the variations of H [11].

3.3 Application to FX rates

We calculate the realized Hurst exponent series for our ten FX pairs, between December 2006 and January 2016, using log-prices sampled every minute. We can observe the evolution of the daily Hurst exponent, for example for the EUR/GBP pair, in Figure 2. We see that it is very erratic over time. We make the same observation for the dynamic m and α in the context of non-Gaussian increments.

We could believe that the erratic feature of time-dependent Hurst exponents, time-dependent α, and time-dependent m is an artefact caused by the absence of convergence of our estimators. When focusing on the sole Hurst exponent, we can show that this is not the case: the Hurst exponent varies significantly over

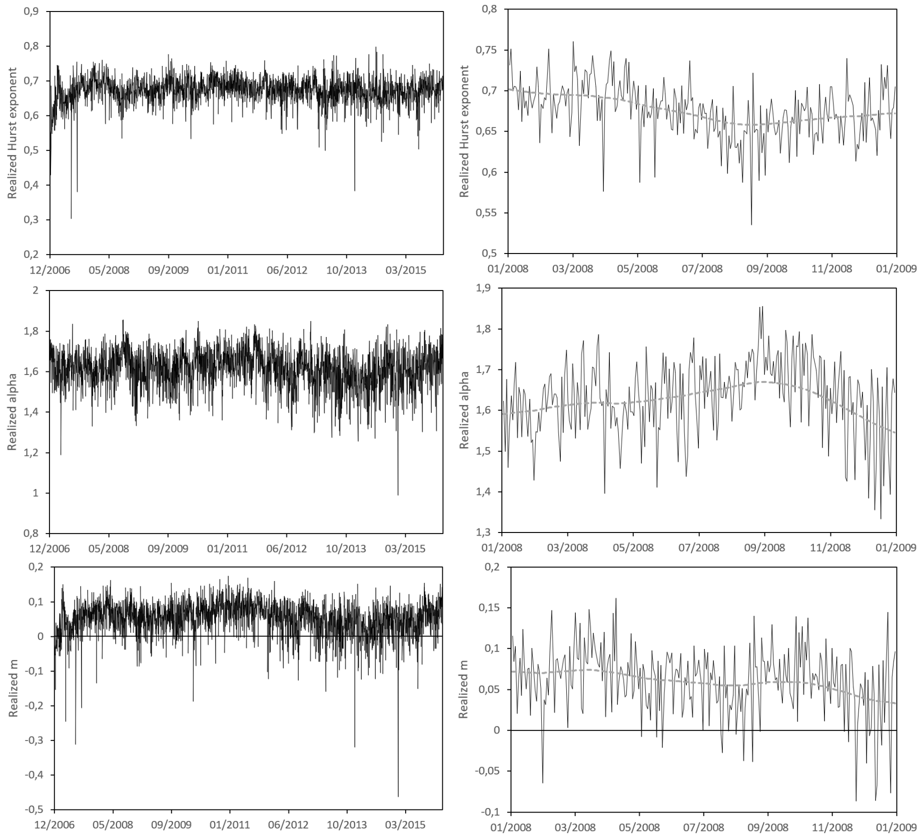

Fig. 2: Daily estimates of H, α, and m, for the EUR/GBP pair, between December 2006 and January 2016 (left), and in 2008 (right). The dotted line is a smooth approximation obtained by variational smoothing.

time. We prove it by considering the standard deviation of the series of Hurst exponents and by showing that it is much higher than what would be the standard deviation of the estimator of the Hurst exponent of a fBm. We gather all these standard deviations in Table 2. For each FX pair, we have a series of 2,327 realized Hurst exponents, each one calculated on 1,440 log-prices. The lowest standard deviation among the sample of 2,327 Hurst exponents is reached by the EUR/USD pair. We focus on this pair and apply a statistical bootstrap: we simulate, using the exact Cholesky method [33], 100 trajectories of fBm of 1,440 observations each, with an input Hurst exponent equal to the average estimated Hurst exponent for EUR/USD. We then estimate the Hurst exponent on each of the simulated trajectories. The empirical standard deviation of the 100 bootstrapped Hurst exponents is 0.018, that is to say far lower than the value, 0.300, that we observed for the series of realized Hurst exponents. This value of 0.018

is an approximation of the standard deviation of the estimator, what makes us think that the Hurst exponent is significantly not a constant. But we must prove more thoroughly that 0.018 and 0.030 are statistically significantly different. We confirm this with statistical one-tailed tests of equality of the variances. First, we compute a F-test of equality of the variances. The statistic of the test is 2.86. It corresponds to a p-value close to zero. For instance, a p-value of 0.001 would have led to a statistic of the test equal to 1.63, or, equivalently, to a standard deviation of the series of Hurst estimates of 0.023, far lower than the observed 0.030. The null hypothesis of equality of the variances is thus clearly rejected. But the F-test is relevant for Gaussian variables only. We then compute Levene's test [42] and Brown-Forsythe test [17]. The test statistic is respectively 14.18 and 14.28, with p-values 1.7×10^{-4} and 1.6×10^{-4}. The null hypothesis of equality of the variances is again rejected.

Table 2: Average and standard deviation of the estimated daily Hurst exponents.

	Average value	Standard deviation
AUD/JPY	0.679	0.040
AUD/USD	0.684	0.039
EUR/AUD	0.657	0.048
EUR/CAD	0.660	0.045
EUR/GBP	0.673	0.038
EUR/JPY	0.682	0.034
EUR/USD	0.687	0.030
GBP/JPY	0.680	0.040
GBP/USD	0.690	0.033
USD/JPY	0.677	0.036
bootstrap EUR/USD	0.687	0.018

All these results show the statistical significance of the time variation of the Hurst exponent of the EUR/USD pair. For the other pairs, the average Hurst exponent is very similar to the one of the EUR/USD pair but the standard deviation is even higher. We thus assess that the time dependence of the Hurst exponent is statistically significant for all the FX pairs we studied.

These empirical results pave the way for the use of multifractal processes. In the definition of the mBm, the Hurst exponent varies smoothly over time. In Figure 2, it is difficult to see a clear pattern, but, when zooming on a particular year, we indeed see that the Hurst exponent, as well as the parameters m and α of the non-Gaussian extension, are oscillating around a time-varying mean, which is well depicted for example by using a variational smoothing [31]. During this year 2008, we observe a peak corresponding to the dramatic episode of the financial crisis at the end of the summer. Since then, the Hurst exponent has slightly increased, whereas the α parameter has sharply decreased, and m

has remained positive. This means that increments of prices has been positively correlated and subject to many extreme events. This is characteristic of a crisis.

In the literature, the Hurst exponent is often linked to the predictability of the series [31, 50]. In fact, it is correct in the fBm or mBm case, in which the increments are Gaussian. When we introduce the more general stable distribution, the Hurst exponent does not tell much about predictability. What is relevant in a forecast perspective is then to have a memory parameter different from 0 and preferably positive. For the studied EUR/GBP pair, this occurs frequently between 2006 and 2016. In particular, the highest values are obtained in the periods 2007-2008, 2009-2012, and 2016, during which the FX pair high-frequency increments where strongly persistent. When comparing the value of the estimated static m parameter of the fLsm model, in Table 1, with the average value of its dynamic daily version, in Table 3, we observe opposed results. Indeed, in the static case, m is negative and indicates anti-persistent increments, whereas m is positive in the dynamic case and indicates persistent increments. The origin of this difference mainly stems from differences observed between the static and dynamic Hurst exponents and that we will discuss in the following paragraphs. We are not aware of another estimator for m and we regret that the one we use, very dependent on estimators of H and α, does not really help us to determine whether the series are persistent or not. We thus resort to standard correlations so as to have an idea of which m, the static or the dynamic one, is more relevant. In Table 3, we display the correlation of adjacent increments of the log-prices. They are all strongly significantly different from zero, with a confidence level higher than 99.9%.[3] These correlations indicate a persistence in the increments, except for EUR/CAD, which is anti-persistent. We are aware that considering increments of a longer duration than one minute could lead to other results. We are facing the limits of models that try to represent in a simple manner a dependence structure which may be more complicated in reality.

The average realized Hurst exponents displayed in Table 2 are far from the global Hurst exponents estimated in the monofractal case and displayed in Table 1. In fact, the estimator is different in the two cases, since we use a regression on the 200 lowest scales for the global Hurst exponent, whereas we only used the two lowest for the realized Hurst exponent. If we apply the two-scale estimator to the global dataset, we obtain estimated Hurst exponents also far from the estimates obtained by regression on 200 scales. The two-scale estimated Hurst exponent, which is higher than the regression-based estimate, is thus even more significantly different from 1/2. We also note that the bootstrapped distributions of the two estimators are very close to each other. All this means that, despite the R^2 above 0.999, the log-plot is not linear enough to justify a monofractal model.

[3] We determined the statistical significance by making a Fisher transformation of the correlation and by making the assumption of Gaussian increments. Taking into account the number of observations, the null hypothesis of a zero correlation is always rejected.

Table 3: Average value of the estimated daily m parameter and empirical correlation of adjacent one-minute increments of the log-prices.

	Average realized m	Correlation of increments
AUD/JPY	0.107	29.8%
AUD/USD	0.095	23.4%
EUR/AUD	0.083	7.0%
EUR/CAD	0.060	−0.7%
EUR/GBP	0.051	27.4%
EUR/JPY	0.092	29.9%
EUR/USD	0.064	29.8%
GBP/JPY	0.092	8.7%
GBP/USD	0.053	31.3%
USD/JPY	0.074	28.4%

We also observe the presence of peaks in the series of realized Hurst exponents, as in Figure 2. These rare peaks correspond to strongly nonlinear log-plots. At the corresponding dates, the local monofractality of the log-prices, that the mBm approach assumes, does not hold, whereas it is a reasonable assumption for other days. Therefore, the empirical multifractality of the log-prices may be not as smooth as required by the mBm model. The aim of the next section is to take into account a rougher dynamic of multifractality. We will propose a model for this purpose. It is also worth noting that the estimated parameters of this next model are not that affected by the presence of the few peaks in the series of realized Hurst exponents.

4 Random multifractal properties

The standard Bm is a particular case of fBm, which itself is a particular case of mBm. It is again possible to define a generalization of the aforementioned processes. As the mBm assumes a smooth and determinstic evolution of the Hurst exponent, other models lead to multifractality without this assumption. For example, one can consider that the Hurst exponent is a more general function of time, with singularities, like the generalized multifractional Brownian motion [3]. One can even assume it follows a stochastic process. For example, the Hurst exponent of the log-price may be a white noise. This is the purpose of the MPRE, which Ayache and Taqqu introduced [4]. In what follows, we will focus on the MPRE.

The move from Bm towards fBm, or from fBm towards mBm was made possible by adding parameters. But, whereas the MPRE generalizes the mBm, it is in general not at the cost of additional parameters. The improvement proposed by the MPRE is thus of importance. Indeed, one of the main drawbacks of the mBm is the risk of overfitting the dynamic of Hurst exponents [11]. The MPRE

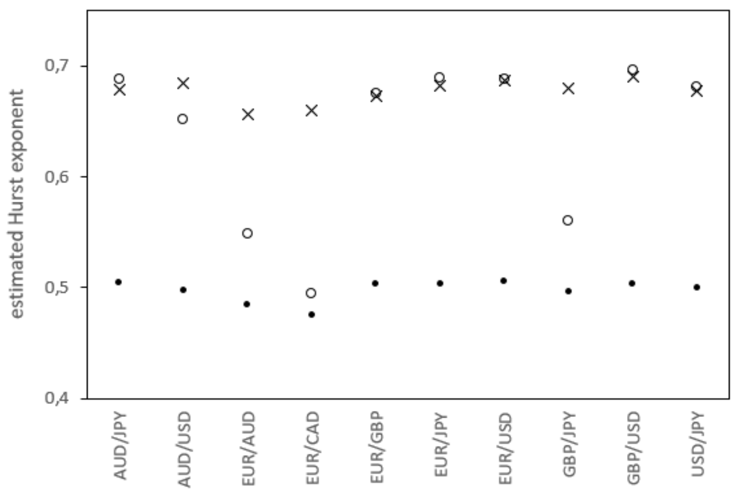

Fig. 3: Average realized Hurst exponents (crosses) and estimated global Hurst exponents using the regression on the 200 lowest scales (dots) or the 2 lowest (circles).

bypass this drawback, thanks to the definition of a stochastic dynamic for the Hurst exponents, insofar as the stochastic process used has few parameters.

The question of the choice of this stochastic process for modelling the dynamic of the Hurst exponents is thus overriding. But the MPRE is a generic model which does not specify what model one should use for the series of Hurst exponents. Only few papers document an estimation of the MPRE with a clear choice of model for the Hurst exponents. From this scarce literature, we can cite a model of i.i.d. random variables distributed according to a mixture of beta distributions [18], or a Cox-Ingersoll-Ross-type (CIR) mean-reverting process [30]. In the following empirical application, in which we estimate a MPRE, we try to justify the use of a third model. In a manner consistent with the rationale which led from Bm to fractional processes, we want to determine whether or not the dynamic of Hurst exponents follows itself a fractional model, namely a fBm.

One could then speak about a multifractional process with fractional exponent (MPFE). This nested fractionality is not intended to be esoteric in finance. Indeed, in this field, nested concepts are not rare. For example, a parameter of Bachelier's model, volatility, is a key concept of finance. Stochastic volatility models assume that this volatility follows itself a Brownian-type stochastic process, whose key parameter is nothing else than the volatility of the volatility. Therefore, defining the Hurst exponent of the Hurst exponent is not that shocking.

In the mBm paradigm, the fluctuations of the estimated Hurst exponent are largely caused by the variance of the estimation technique. In the MPRE with i.i.d. random variables for the Hurst exponents, these fluctuations are an unpredictable noise. In the MPFE, we assume that these fluctuations contain information and that the process of Hurst exponents is partially predictable.

The following empirical work provides some new evidence of the relevance of the MPFE. We develop the equations of the model, its estimation, its interpretation, and some non-Gaussian extensions.

4.1 Multifractional process with fractional Hurst exponent

Bianchi and Pantanella already mentioned the possibility for the Hurst exponents to follow a fBm [14]. A major issue with the use of a fBm to model the Hurst exponents is that the fBm is not bounded, whereas the Hurst series must remain in the interval $(0,1)$. This explains for example why the MPRE with i.i.d. random variables is based on bounded beta variables [18]. Bianchi and Pantanella thus proposed an affine scaling of the fBm. This scaling depends on the maximum and minimum values reached by the fBm used in modelling the Hurst exponents [14].

We propose a new method to bound the fBm. This method is based on the Fisher transformation, which is traditionally used to transform a correlation in an unbounded variable so as to conduct statistical tests on this correlation. We slightly modify this transformation to take into account the fact that the Hurst exponent is in $(0,1)$, whereas the correlation is in $[-1,1]$. In the MPFE, we thus define our Fisher-like transformation by:

$$\mathcal{F} : x \in (0,1) \longmapsto \frac{1}{2}\ln\left(\frac{x}{1-x}\right).$$

The following equations then define the MPFE, in which Z_t depicts the log-price at time t, of Hurst exponent $H_{0,t}$, and with H_1 the Hurst exponent of the Hurst exponents:

$$\begin{cases} Z_t &= \mu_0 + \frac{\sigma_0\sqrt{\pi K(2H_{0,t})}}{\Gamma\left(H_{0,t}+\frac{1}{2}\right)}\int_{-\infty}^{\infty}\left((t-s)_+^{H_{0,t}-1/2}-(-s)_+^{H_{0,t}-1/2}\right)dW_{0,s} \\ \mathcal{F}\left(H_{0,t}\right) &= \mu_1 + \frac{\sigma_1}{\Gamma\left(H_1+\frac{1}{2}\right)}\int_{-\infty}^{\infty}\left((t-s)_+^{H_1-1/2}-(-s)_+^{H_1-1/2}\right)dW_{1,s}, \end{cases} \tag{3}$$

where $W_{0,.}$ and $W_{1,.}$ are independent Bm. The independence of $W_{0,.}$ and $W_{1,.}$ is not mandatory in the MPRE model, but we recommend it for ease of estimation. We could also write $H_{0,t}$, for a given time t, as the inverse Fisher transformation of an underlying fBm, where $\mathcal{F}^{-1}(y) = (1+e^{-2y})^{-1}$. The major benefit of using the Fisher transformation is that the scaling of the fBm does not depend on the past values of the process of Hurst exponents. Therefore, in a practical use, with frequent updates, the scaling of the fBm will always remain the same. Moreover, for a large set of Hurst exponents, the Fisher transform is close to the identity, so that the transformation does not alter much the properties of the fBm, like its fractal property. The following empirical results are indeed very close if we use the Fisher transformation or not.

Some well-known properties of the generic MPRE with some technical assumptions, which remain true for the MPFE: it is a self-similar process and its increments are stationary, what is not the case for the mBm [4].

In addition to the representation of the Hurst exponents by a scaled fBm and by beta variables, the mean-reverting process is another solution proposed in the literature [30]. This last stochastic process proposed for the Hurst exponents is of interest. It is indeed theoretically unbounded, but, if the parameter of the mean-reversion strength is big enough, the trajectory will very likely remain close to its average value. Similarly, in the following empirical study, the Hurst exponent of the Hurst exponents is very low, around 0.02. As a consequence, the adjacent increments of realized Hurst exponents are strongly negatively correlated. It thus depicts a mean-reversion phenomenon which is strong enough to keep the trajectory very likely in the interval $(0, 1)$. In other words, using the Fisher transformation in the MPFE is a theoretical precaution but, in practice, we could do without it.

Similarly to the estimation of a rough volatility model [37], the estimation of the MPFE consists first in estimating a series of realized Hurst exponents and then to estimate a fBm on this series, or on the Fisher transformation of this series. For the first step, the realized Hurst exponents provide us with one statistic each day and must be computed using intraday prices. Therefore, two realized Hurst exponents at two different dates will use prices from non-overlapping time intervals. This avoids spurious dependence in the series of realized Hurst exponents. As already exposed, we use the two-scale estimator for each realized Hurst exponent. For the second step, we estimate the Hurst exponent of the Hurst exponents with the method of absolute moments and in particular by conducting a regression of the log-plot using the 64 finest scales. We indeed adapted the number of scales to the size of the series of realized Hurst exponents, limited to 2327 observations.

4.2 Non-Gaussian extensions

The fBm and the mBm have both their non-Gaussian counterpart. We thus also propose a non-Gaussian adaptation of the MPFE, but we will not provide much details about it in this paper. The MPFE is based on two nested fractional processes. So, the non-Gaussian feature can apply to one or two of them.

In a first approach, we could consider that the log-price follows a multifractional stable process, with a dynamic for the α and m parameters. This dynamic could be Brownian or not. In Section 3.3, we already estimated dynamic α and m parameters. But we do not explore this solution further.

In a second approach, we consider that the log-price follows a MPFE with a fLsm modelling the dynamic of Hurst exponents. We can thus estimate the α

and m parameters of the Hurst exponents. The equations of this model are:

$$\begin{cases} Z'_t &= \mu_0 + \frac{\sigma_0\sqrt{\pi K(2H_{0,t})}}{\Gamma\left(H_{0,t}+\frac{1}{2}\right)} \int_{-\infty}^{\infty} \left((t-s)_+^{H_{0,t}-1/2} - (-s)_+^{H_{0,t}-1/2}\right) dW_{0,s} \\ \mathcal{F}\left(H_{0,t}\right) &= \mu_1 + \sigma_1 \int_{-\infty}^{\infty} \left((t-s)_+^{m_1} - (-s)_+^{m_1}\right) dW_{1,s}^{\alpha_1,0}, \end{cases} \tag{4}$$

where $W_{0,.}$ and $W_{1,.}^{\alpha_1,0}$ are two independent processes, namely a standard Bm and a symmetric Lévy α_1-stable motion. We kept, for the clarity of the architecture of the model some unnecessary subscripts, for example the 1 in m_1 and α_1, which indicate the layer in the model of nested fractality.

4.3 Application to FX rates

We now estimate the MPFE, as well as its non-Gaussian extension, on our dataset. We note that an application of the MPRE to FX rates is already documented in the literature but with another model for the series of Hurst exponents and a different methodology [15].

The estimated Hurst exponent of the Hurst exponents, which we display in Table (4), is very low, around 0.02, whatever the FX pair. This confirms the mean-reverting effect documented by Frezza [30]. But it is not only mean-reverting, it is also a fractal process. The Hurst exponent of the Hurst exponents thus provides us with an interpretation about the dynamic of the realized Hurst exponent. As already exposed, we obtain very similar results, whether we use the Fisher transformation or not.

Table 4: Estimated Hurst exponent, α_1, and m_1 parameters of the dynamic of realized Hurst exponents or of their Fisher transformation (like in equations (3) and (4)).

	Dynamic of $\mathcal{F}(H_{0,.})$			Dynamic of $H_{0,.}$		
	H_1	α_1	m_1	H_1	α_1	m_1
AUD/JPY	0.045	1.804	−0.509	0.044	1.773	−0.520
AUD/USD	0.011	1.770	−0.553	0.011	1.738	−0.564
EUR/AUD	0.021	1.733	−0.556	0.026	1.696	−0.564
EUR/CAD	0.020	1.777	−0.543	0.024	1.789	−0.535
EUR/GBP	0.024	1.829	−0.522	0.024	1.821	−0.525
EUR/JPY	0.031	1.851	−0.510	0.030	1.817	−0.521
EUR/USD	0.011	1.885	−0.520	0.010	1.882	−0.521
GBP/JPY	0.019	1.879	−0.513	0.028	1.843	−0.515
GBP/USD	0.014	1.860	−0.524	0.013	1.845	−0.529
USD/JPY	0.021	1.815	−0.530	0.021	1.800	−0.534

The Hurst exponent of the Hurst exponents is so low that the relevance of this model is questionable. We thus display for some FX pairs, in Figure 4, the log-plot applied to the series of realized Hurst exponents instead of the series of log-prices. The linear shape of the plot is clear, even though the variance of the plot around the affine regression is much higher than in Figure 1. To assess the statistical significance of the Hurst exponent of the Hurst exponents, we conduct a test whose null hypothesis is that the log-prices are a MPRE with an i.i.d. random variable for the Hurst exponent. In fact, if the null hypothesis was true, we would have got a constant log-plot, reflecting a null Hurst exponent of the Hurst exponents. We conduct two bootstraps, one with uniform variables for the Hurst exponents of the log-prices, another with Gaussian variables. In both tests, we obtain a very high significance, above 99.9%, for the Hurst exponents of the Hurst exponents displayed in Table 4. The scaling phenomenon of the realized exponents is thus statistically significant. This justifies the use of the MPFE instead of a MPRE with i.i.d. random variables.

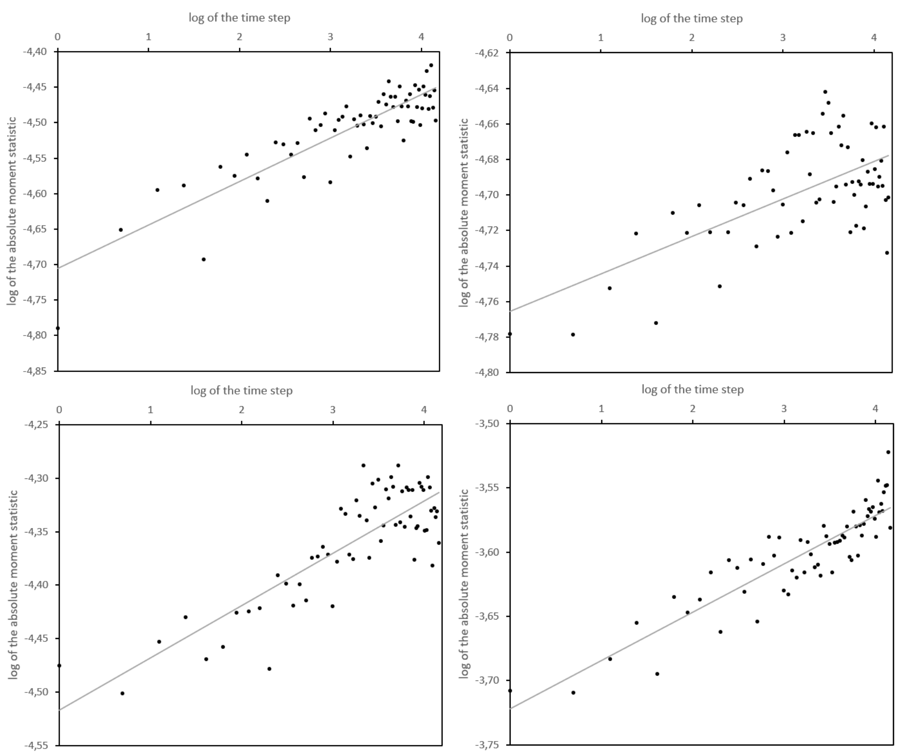

Fig. 4: Empirical log plot of the series of realized Hurst exponents for EUR/JPY (top left), EUR/USD (top right), EUR/GBP (bottom left), and GBP/JPY (bottom right). Half the slope of the corresponding affine regressions (in grey) is the estimated Hurst exponent of the Hurst exponents.

We also display in Table 4 the estimated parameters of the non-Gaussian extension of the MPFE. With no surprise, the m_1 parameter is always very strongly negative. This is relevant with the mean-reversion effect already discussed. The α_1 parameter is around 1.8, indicating a slight non-Gaussian behaviour of the realized Hurst exponents.

5 Discussion: Towards multilayer fractality

We exposed throughout this paper the rationale which led from the standard Bm to the fBm, from the fBm to the mBm, and from the mBm to the MPFE. These models always intend to reproduce more accurately some stylized facts of log-prices, in particular their fractal features. This fractality is arbitrary in the standard Bm, it is based on one parameter in the fBm, it is a deterministic function of time in the mBm, it is a stochastic one in the MPRE, and it is even a random fractal itself in the MPFE. In the MPFE approach, the log-price follows a fractional model, whose parameters follow similar dynamics. A logical continuation of all these models consists in cumulating several layers of fractality. Indeed, the fBm and the mBm have one layer of fractality, whereas the MPFE has two layers of fractality, with a Hurst exponent of the Hurst exponents. We can thus write a model with n layers of fractality:

$$\begin{cases} Z_{n,t} &= \mu_0 + \frac{\sigma_0\sqrt{\pi K(2H_{0,t})}}{\Gamma\left(H_{0,t}+\frac{1}{2}\right)} \int_{-\infty}^{\infty} \left((t-s)_+^{H_{0,t}-1/2} - (-s)_+^{H_{0,t}-1/2}\right) dW_{0,s} \\ \mathcal{F}\left(H_{0,t}\right) &= \mu_1 + \frac{\sigma_1\sqrt{\pi K(2H_{1,t})}}{\Gamma\left(H_{1,t}+\frac{1}{2}\right)} \int_{-\infty}^{\infty} \left((t-s)_+^{H_{1,t}-1/2} - (-s)_+^{H_{1,t}-1/2}\right) dW_{1,s} \\ \quad \dots & \\ \mathcal{F}\left(H_{n-2,t}\right) &= \mu_{n-1} + \frac{\sigma_{n-1}\sqrt{\pi K(2H_{n-1,t})}}{\Gamma\left(H_{n-1,t}+\frac{1}{2}\right)} \cdot \\ & \qquad \cdot \int_{-\infty}^{\infty} \left((t-s)_+^{H_{n-1,t}-1/2} - (-s)_+^{H_{n-1,t}-1/2}\right) dW_{n-1,s} \\ \mathcal{F}\left(H_{n-1,t}\right) &= \mu_n + \frac{\sigma_n}{\Gamma\left(H_n+\frac{1}{2}\right)} \int_{-\infty}^{\infty} \left((t-s)_+^{H_n-1/2} - (-s)_+^{H_n-1/2}\right) dW_{n,s}, \end{cases}$$

where $W_{0,.}, .., W_{n,.}$ are independent Bm and $H_n \in (0,1)$. In this model, $\mathcal{F}\left(H_{n-1,t}\right)$ is a fBm of Hurst exponent H_n, and $\mathcal{F}\left(H_{i,t}\right)$, for $i \leq n-1$, is a $(n-i)$-layer multifractal model.

This logical generalisation is tempting, but we are unconvinced it is the right direction for further developments of the research on multifractal models, for several reasons:

- When adding one layer to this multilayer multifractal model, one adds two parameters to the model. Therefore, a model with many layers would have many parameters. As with deep learning, one can expect a very accurate reproduction of reality while designing many layers in this model. But one can also expect bad forecast ability due to overfitting, which is already a pitfall for mBm [11]. The selection of the appropriate number of layers should therefore be based on an information criterion.

- We present this multilayer approach as the logical continuation of other fractional models presented in this paper. But the MPRE, instead of adding parameters to the mBm, bypass its high number of parameters by introducing stochasticity. Adding many parameters may thus not be that logical.
- In the MPFE, the interpretation of the Hurst exponent of the Hurst exponents may provide some insight on the dynamic of the log-prices. With a multilayer fractional model, the interpretation seems much more intricate.
- The estimation of the parameters of the multilayer multifractal model is not straightforward. In the MPFE, we estimated first a series of Hurst exponents, each defined in non-overlapping windows of 1,440 observations of the log-price, and we then analysed this series of Hurst exponents. Applying this method to the multilayer case sounds inappropriate. Indeed, if we observe a series of N log-prices and want to estimate a n-layer model, we could use only $N^{1/n}$ points[4] for the estimation of the Hurst exponents at each layer. For our dataset of roughly 3,350,000 log-prices, a 3-layer model, which contains only one more layer than the MPFE, would lead to using only 150 points to estimate a Hurst exponent. Adding layers would thus rapidly increase the estimation error.

6 Conclusion

In the fBm setting, the volatility and the Hurst exponent are two parameters whose interpretation is insightful. Following the volatility, the analysis of the fractal properties of financial time series becomes increasingly widespread. Some models even put forward the interplay between both, by analysing the fractal properties of the volatility (rough volatility, Markov-switching multifractal, HAR-RV). We note that, in the history of models in finance, volatility was successively considered constant, time-varying, stochastic, and fractional. The Hurst exponent follows the same path: it is constant in the fBm, time-varying in the mBm, stochastic in the MPRE, fractional in the MPFE, which we exposed in this paper.

The justification of any model comes from empirical findings. The econophysics literature tries to find these stylized facts that force to imagine new models. It is also the purpose of this paper, in which we display some empirical evidence of the fractality of the Hurst exponents and thus of the relevance of the MPFE approach. Further research may consist in finding new stylized facts beyond the rough fractality of both the Hurst exponents and the volatility. The econometrics literature already uses monofractality, as the fBm inspired ARFIMA [38]. But it is time for mainstream econometrics to go further, because fBm is only the first step in the fractional models and because it appears, thanks to the rise of models such as rough volatility, that modelling the fractal properties of financial time series is overriding.

[4] Points are either observations of log-prices or Hurst exponents of the previous layer.

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