The origins of randomness: Granularity, information and speed of convergence

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Abstract. The purpose of this paper is to relate the notion of randomness to the granularity of the data sampled over parametrized discrete time intervals. Indeed, data defined over infinitesimal small time intervals compared to larger ones affect the randomness due to the information embedded in their granularity: larger and smaller granular intervals, as well as limit (and parametrized) intervals necessarily affect data measurements. We discuss the impact of fractionalization and data granularity on some stochastic processes widely used in finance.

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1 Introduction: Financial modelling and the origins of randomness

Randomness and its definitions underly a finance seeking to predict the future. To do so, it nurtures the past, often contributing to a "capital" of experience

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and know-how as well as learning the underlying processes that categorize the unknown future. Engineering the future of finance is then embedded in our articulating randomness and its forms. In such contexts, randomness, as for unknown factors and origins, defines an "uncertainty" completing theories and models of finance.

The conventional financial theory based on Walras 1877 general theory of economic equilibrium [29] (albeit in a deterministic framework) provided non-random estimates of converging equilibrium prices. Arrow-Debreu extended Walras' framework to an expected estimate of future prices known as risk neutral pricing. However, theories that compensate unknown futures are necessarily incomplete, as revealed by financial data, financial crises and "Gurus", each pointing to demise the Complete Markets. Rare and unlikely events are found to be far more common than presumed, and the unpredictable seems to recur. Soros, backed by a personal financial success, related markets to nonlinearities, in particular to reflexivity and the Hola effect. Further, traditional financial systems based on financial intermediaries, globalization, dominant financial agents and large firms, among many other elements have challenged fundamental financial models. More and more, these are perceived as unsatisfactory, and their limits trace back to the origin itself of the notion of randomness defining stochastic models of finance. Traditionally, Adam Smith – challenging financial, economic dogma – prices in terms of *Needs*, *Scarcity* and *Exchange*. These elements have allowed an individual perspective for the measurement, valuation models and financial prices. Randomness intruded through the "uncertainty" described by stable normal distributions, justified by statistical assumptions such as independent sampled "errors" aggregated and made tractable by statistical rules. The current analysis of financial and Big Data awoke an awareness of the unknown, of a greater complexity raising questions not fully addressed by existing models [4], [3], [5].

The purpose of this paper is to elaborate a definition of randomness based on "*data granularity*", *information and the speed of convergence of parametrized discrete data intervals*, and their statistical limits. For example, data defined over infinitesimal small time intervals compared to larger ones affect the financial randomness due to the information that data granularity provides: larger and smaller granular intervals, as well as limit (and parametrized) intervals necessarily define data measurements. Below, we resume and introduce a partial number of origins of randomness upended to financial models. Our intent is to provide an appreciation of both.

2 Brownian Motion randomness

Brownian Motion underlies financial models such as random walks, lognormal model and others. Randomness is then defined by singular and independent normal events. They are based on simplifying assumptions, all of which are either explicit or implicit, expressing two statistical properties: the data drift and its volatility, with a Brownian Motion randomness. Similarly, the randomness of surrogate Poisson jump processes is defined by independent inter-events time distributions resulting from the time limit of counting discrete Negative Binomial Distributions (NBD). "Randomness" is is therefore the outcome of models and data defined, measured and manipulated that converge to a modeling distribution. In this sense, randomness is actually an "artificial, intelligent and incomplete" measure of randomness. To further complete it, complex and multivariable models are used to account and bridge data and models by observations we fail to better explain. These result in models that assume a recognized rationality acquired, based on theoretical econometric analyses. Such an approach is conventional, relatively simple and useful, and provides an *ex-ante* interpretation of data and of its randomness. However, *ex-post*, randomness is defined by a divergence realized by data and its expectation. For example, Bayesian analysis of stock prices is based on three distributions: a prior distribution accounting for past data, a posterior distribution accounting for an updated prior distribution by current data, and a future distribution we are mostly concerned by, which is presumed to be a function of the randomness and the vagaries of factors we cannot assert for sure. A functional assumption of randomness then simplifies their treatment. For example, let us say that the randomness of past data is defined by normal probability distributions, while an updated posterior distribution is also normal. Such an approach is therefore a simplification and a stable definition of randomness. In a temporal setting, let the Brownian Motion definition of randomness, at times t, s, $t > s$, be defined by two random events $R(t)$ and $R(s)$, stationary and independently distributed with an expectation at time t equal to $f(t) = \mathbb{E}(R(t)) - \mathbb{E}(R(0))$. Stationarity and independence imply $f(t+s) = f(t) + f(s)$, whose unique solution is $f(t) = tf(1)$. Explicitly,

$$\begin{aligned} f(t+s) &= \mathbb{E}(R(t+s) - R(0)) \\ &= \mathbb{E}(R(t+s) - R(t)) + \mathbb{E}(R(t) - R(0)) \\ &= \mathbb{E}(R(t) - R(0)) + \mathbb{E}(R(s) - R(0)) = f(t) + f(s). \end{aligned}$$

This functional equation, has a unique time linear solution which proves that such models have time linear expectations which might not be the case under alternative definitions of randomness. A similar argument using the variance of the rates of return provides a similar time linearity in the variance, a property of Brownian Motion of randomness. Such assumptions are not always confirmed by assets time series even though they underlie fundamental financial models. Stocks time series and their rates of return have statistical properties that include their mean, their volatility but also their skewness and kurtosis. Such moments negate the assumption that rates of reurns are normally distributed and therefore data need not be defined only by their mean and volatility. In real terms, this means that in a normal randomness, information is instantly reflected in the market price and reflect "all the information" that is relevant to that stock. The convergence of data to their infinitesimal limit (defining their underlying model) and its implications is not in such terms accounted for. In this sense, a price is measured relative to a filtration which changes continuously as new information access financial markets, the speed with which it converges to a definition of a limit randomness, as well as other factors, we shall elaborate in this paper. In particular, we shall consider the effects of data granularity on randomness. Such developments are based on prior publications by [26], [27] as well as on the rich literature on fractional randomness.

3 Financial BM randomness and granularity

Let S_t be the stock price at time t. Its rate of return over a time interval Δt is defined as

$$\tilde{R}_{t+\Delta t} = \frac{\tilde{S}_{t+\Delta t} - S_t}{S_t} = \frac{\Delta \tilde{S}_t}{S_t} \quad \text{or} \quad 1 + \tilde{R}_{t+\Delta t} = \frac{\tilde{S}_{t+\Delta t}}{S_t},$$

(we assume t as the current time and $\Delta t > 0$, thus $t + \Delta t$ is a future time, underlined by the use of the symbol $\tilde{}$, denoting a random variable). The standard model assumes that at all time periods, the return provided by the stock in the short time period Δt can be modeled as

$$\frac{\Delta \tilde{S}_t}{S_t} = \mu \Delta t + \sigma \tilde{\epsilon} \sqrt{\Delta t}, \quad S_0 > 0 \tag{1}$$

where μ is the expected rate of return per unit of time from the stock, σ is the volatility of the stock price and $\tilde{\epsilon}$ is a random variable drawn from a standard normal, i.e. $\tilde{\epsilon} \sim N(0,1)$. Obviously, the stochastic component $\sigma \tilde{\epsilon} \sqrt{\Delta t}$ has variance $\sigma^2 \Delta t$ and hence $\frac{\Delta \tilde{S}_t}{S_t}$ is normally distributed with mean $\mu \Delta t$ and variance $\sigma^2 \Delta t$. If we set a limit non-adapted normal random process, the Brownian motion randomness $\{W(t), t > 0\}$ results, defined by a Itô stochastic differential equation

$$\frac{d\tilde{S}_t}{S_t} = \mu dt + \sigma dW(t), \quad S_0 > 0$$

which can be written also as

$$d \ln(\tilde{S}_t) = \mu dt + \sigma dW(t), \quad S_0 > 0. \tag{2}$$

Using Itô calculus with Taylor expansion up to order 2, being the terms of order higher than $(dt)^2$ negligible, we have

$$d \ln(\tilde{S}_t) = \frac{\partial \ln(\tilde{S}_t)}{\partial t} dt + \frac{\partial \ln(\tilde{S}_t)}{\partial S_t} d\tilde{S}_t + \frac{1}{2} \frac{\partial^2 \ln(\tilde{S}_t)}{\partial S_t^2} (d\tilde{S}_t)^2 + \mathfrak{R}_2. \tag{3}$$

Since $\frac{\partial \ln(\tilde{S}_t)}{\partial t} = 0$, $\frac{\partial \ln(\tilde{S}_t)}{\partial S_t} = \frac{1}{S_t}$ and $\frac{\partial^2 \ln(\tilde{S}_t)}{\partial S_t^2} = -\frac{1}{S_t^2}$ it readily follows

$$\begin{aligned} \frac{d\tilde{S}_t}{S_t} = d \ln(\tilde{S}_t) &= \frac{1}{S_t}(\mu S_t dt + S_t \sigma dW_t) - \frac{1}{2S_t^2}(\mu S_t dt + S_t \sigma dW_t)^2 \qquad (4) \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t + \mathfrak{R}_2, \end{aligned}$$

that is, finally (since $\mathfrak{R}_2$ is negligible)

$$d\tilde{S}_t = \left(\mu - \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t \tag{5}$$

which is the familiar log-price stochastic model resulting from the application of Itô's Lemma. Notice that the volatility embedded in both the drift term and in the stochastic component "accounts" for a randomness which is not concurrent with alternative developments. In this sense, the calculus applied to a rate of return alters the randomness of such a model.

Instead of a discrete time interval $(\Delta t) < 1$, let us consider a granular parametrized interval $(\Delta t)^\alpha > (\Delta t)$ with $0 < \alpha < 1$, pointing to a greater time interval implied by the granularity $(\Delta t)^\alpha$ (see Figure 1). Such a definition then relates to a high frequency finance, based on data streams with a granularity greater than the infinitesimal time intervals implied in mathematical and continuous time finance. For example, a streaming data measured every micro second is still greater than continuous time finance.
In this granular case, setting $\Delta^\alpha \tilde{S}_t := \tilde{S}_{t+(\Delta t)^\alpha} - S_t$, the discrete time financial

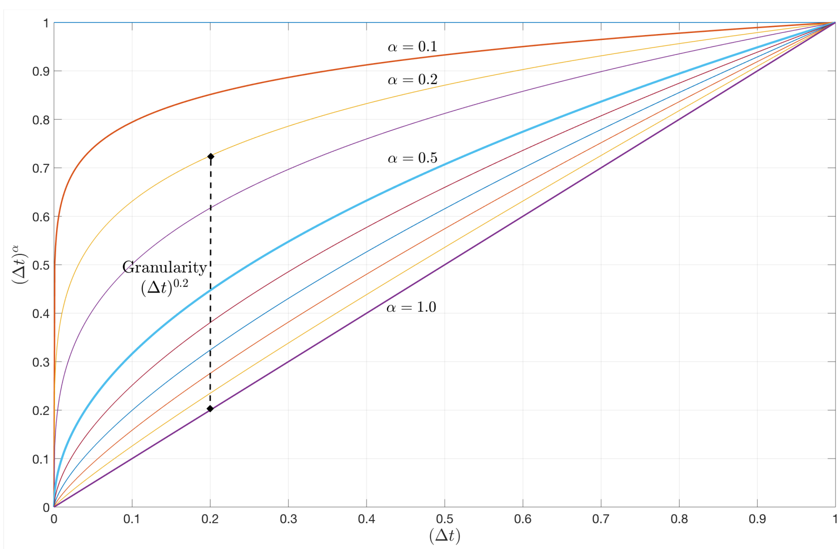

Fig. 1. Granular parametrized intervals $(\Delta t)^\alpha > (\Delta t)$ with $0 < \alpha < 1$.

model defined above becomes, with time intervals $(\Delta t)^\alpha$:

$$\frac{\Delta^\alpha \tilde{S}_t}{S_t} = \mu_\alpha (\Delta t)^\alpha + \sigma_\alpha \tilde{\epsilon}_{t+(\Delta t)^\alpha}, \quad S_0 > 0$$

When $(\Delta t) \to dt \to 0$, we note that $(\Delta t)^\alpha \to (dt)^\alpha \to 0$. In this case, the graularity of data and its theoretical infinitesimal tendency, is based on an approach that is both granular dependent as well as dependent of the speed of convergence to its infinitesimal limit. The parametric granular time interval $0 < \alpha < 1$ is therefore to be accounted for in defining a fractional randomness. Further, the parameter α has a number of implications. Essentially these include a "gain" of information as a slower convergence accounts more precisely for the residue loss which accounts for events that occur when a model is left to converge to its time limit. This information has a memory effect which contributes to the autcorrelation of a fractional Brownian motion. Further, note that parameters are indexed in the fractional parameter α with a mean and volatility μ_α and σ_α replacing μ and σ to reflect their parameter estimates. Thus, a (non-fractional) normal probability distribution, $\mathbb{E}(\tilde{\epsilon}_t) = 0$ and $\mathbb{E}(\tilde{\epsilon}_t \tilde{\epsilon}_\tau) = 0$, expresses statistical independence, while a fractional normal probability distribution, $\mathbb{E}(\tilde{\epsilon}_{t+(\Delta t)^\alpha}) = 0$ and $\mathbb{E}(\tilde{\epsilon}_{t+(\Delta_t)^\alpha} \tilde{\epsilon}_t) \neq 0$ implies a time (autocorrelation) dependence which we ascribe to a "memory". This follows the fact that a fractional derivative is defined by its non-integer (fractional) order, i.e. by the infinite polynomial

$$d^\alpha S = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} S\left(t + (\alpha - i)(\Delta t)^\alpha\right).$$

For example, its first few terms are

$$d^\alpha S = \binom{\alpha}{0} S(t+\alpha(\Delta t)^\alpha) - \binom{\alpha}{1} S(t+(\alpha-1)(\Delta t)^\alpha) + \binom{\alpha}{2} S(t+(\alpha-2)(\Delta t)^\alpha) + \ldots$$

Therefore, for a fractional index $0 < \alpha < 1$, the derivative is defined by "past events" that we may not neglet and thereby its long run memory is a function of past values

$$d^\alpha S = S(t + \alpha(\Delta t)^\alpha) - \binom{\alpha}{1} S(t - (1-\alpha)(\Delta t)^\alpha) + \binom{\alpha}{2} S(t - (2-\alpha)(\Delta t)^\alpha) + \ldots$$

The fractional index clearly defines the memory effect ascribable to past values $t - (k - \alpha)(\Delta t)^\alpha)$ for $k = 0, 1, 2, \ldots$. In the limit $(\Delta t)^\alpha$ becomes $(dt)^\alpha$, and the fact that $(dt)^\alpha > dt$ can be interporeted as a "loss of model information" due to measurements at intervals that would be greater than the presumed reference model with time intervals dt. This loss of information is compensated by its accounting of past events.

When $\alpha \neq \frac{1}{2}$, a fractional normal distribution (we call from hereon, a fractional Brownian motion) is obtained.

At the continuous time limit, a fractional lognormal model results with a "fractional stochastic differential equation" with $\{W_\alpha(t), t > 0\}$ a non-adapted fractional Brownian motion, namely a normal probability distribution with variance $(dt)^{2\alpha}$ as well as an autocorrelation given by:

$$\mathbb{E}\left(W_\alpha\left(t + (\Delta t)^\alpha\right) W_\alpha(t)\right) = \frac{1}{2}\left(\left(t + (\Delta t)\right)^{2\alpha} + (t)^{2\alpha} - |(\Delta t)|^{2\alpha}\right) \neq 0, \text{ for } \alpha \neq 1$$

leading to

$$\frac{d^{\alpha}\tilde{S}_t}{S_t} = \mu_{\alpha}(dt)^{\alpha} + \sigma dW_{\alpha}(t), \quad S_0 > 0$$

where $d^{\alpha}\tilde{S}_t$ denotes the change in price in a fractional time interval when introducing the elements of fractional calculus. Note that both dt and $(dt)^{\alpha}$ tend to an infinitesimal limit. As noted earlier, the "speed" at which they do so differs, depending on the fractional parameter α. Further, the model parameters (in fact statistical estimates based on the model we define), say μ_{α}, may be specified or estimated based on the assumption regarding model time interval (which defines the data granularity). As a result, granularity alters the assumptions we make about the model parameters.

4 Example: Autoregressive processes and mean reverting models. Data based models

Financial models are defined often by financial econometric techniques including linear regression, autoregressive AR models, ARMA, ARIMA as well as ARCH and GARCH models. In such cases, financial models are based on observed time series and their models' econometric performance. For simplicity, we shall consider below a simple autoregressive AR(1) processes defined by:

$$R_t = \alpha R_{t-1} + \epsilon_t, \ |\alpha| < 1$$

where R_t is the rate of return of an asset at time t and ϵ_t is assumed i.i.d. (indipendently and identically distributed) with mean zero and variance σ^2. A general AR(n) model, accounting for past returns as well, is written similarly in terms of n parameters estimates:

$$R_t = \sum_{i=1}^{n} \alpha_i R_{t-i} + \epsilon_t, \ |\alpha_i| < 1.$$

Other models can be developed, based on a smaller number of parameters seeks to increase the explanatory power of parameter estimates. Financial models that can capture time series properties with the least number of parameters are then most efficient. The model below will outline such an approach. Say that the current rate of return R_t of a financial asset is a weighted function of past rate of return variations which we denote by $\Delta R_{t-i}, i = 1, 2, 3, \ldots$ with ΔR_t defined by a normal linear process. Setting:

$$R_t = \sum_{i=1}^{n} \mu_i \Delta R_{t-i}, \text{ with } \Delta R_{t-i} = \delta \Delta t + \sigma \Delta W(t-i)$$

where μ_i, δ and σ are a set of $n + 2$ parameters. Our intent is to reduce the number of parameters. For convenience, say that the current rate of return is a continuous time function exponentially decreasing as a function of all past rate of return variations (in discrete time, it corresponds to a geometric probability distribution):

$$R(t) = \int_{-\infty}^{t} e^{-\mu(t-\tau)} dy(\tau), \text{ with } dy(\tau) = dR(\tau)$$

An equivalent expression to this equation is found by deriving it with respect to time t, or:

$$\Delta R(t) = -\mu \Delta t \int_{-\infty}^{t} e^{-\mu(t-\tau)} dy(\tau) + \Delta y(t) = \Delta y(t) - \mu R(t) \Delta t$$

Since perturbations in rates of returns are random and given by $\Delta y(t) = \delta \Delta t + \sigma \Delta W(t)$, we obtain the following rates of returns process:

$$\Delta R(t) = \delta \Delta t + \sigma \Delta W(t) - \mu R(t) \Delta t$$

which we rewrite as a mean reversion (and normally distributed) model:

$$\Delta R(t) = \mu(\lambda - R(t))\Delta t + \sigma \Delta W(t), \quad \frac{\delta}{\mu} = \lambda$$

Such models appear in finance in various applications such as modeling interest rates (the Vasicek model) and others. Other variations of this model can be derived. If the time scale is changed, then as indicated earlier, we have the fractional difference equation.

$$\Delta^{\alpha} R(t) = \mu_{\alpha}(\lambda_{\alpha} - R_{\alpha}(t))(\Delta t)^{\alpha} + \sigma \Delta W_{\alpha}(t), \quad \alpha < 1, \quad (\Delta t)^{\alpha} > (\Delta t),$$

where meaning of parameters is defined in terms of the fractional parameter as discussed above for the lognormal model. Their limit differential equations, as discussed above, differ in the "speed" with which the limits are reached:

$$\begin{aligned} dR(t) &= \mu(\lambda - R(t))dt + \sigma dW(t), \quad \frac{\delta}{\mu} = \lambda \\ d^{\alpha} R(t) &= \mu_{\alpha}(\lambda_{\alpha} - R_{\alpha}(t))(dt)^{\alpha} + \sigma dW_{\alpha}(t), \quad \alpha < 1, \quad (dt)^{\alpha} > (dt) \end{aligned}$$

These two models, also called mean reversion models and often used to model interest rate models (and therefore price bonds with normal and mean reverting rates of returns), are not providing the same solution for the rate of return (and therefore are not providing equivalent bond prices). Both the model parameters as well as the fractional and non-fractional process evolutions will point out to different outcomes. An exercise considers the deterministic part of the equation above:

$$\frac{d^{\alpha} R(t)}{(dt)^{\alpha}} = \mu_{\alpha}(\lambda_{\alpha} - R_{\alpha}(t));$$

$R(0)$ with the Laplace Transform solution

$$s^{\alpha}R^{*}(s)-s^{\alpha-1}R(0)=\frac{\mu_{\alpha}\lambda_{\alpha}}{s}-\mu_{\alpha}R^{*}(s)$$

or

$$\begin{aligned}R^{*}(s)&=\frac{s^{\alpha-1}R(0)}{s^{\alpha}+\mu_{\alpha}}+\frac{\mu_{\alpha}\lambda_{\alpha}}{s(s^{\alpha}+\mu_{\alpha})}\\&=\left(\frac{R(0)}{s^{1-\alpha}}+\frac{\mu_{\alpha}\lambda_{\alpha}}{s}\right)\frac{1}{(s^{\alpha}+\mu_{\alpha})}\end{aligned}$$

which is a convolution of $L^{-1}\left(\frac{R(0)}{s^{1-\alpha}}+\frac{\mu_{\alpha}\lambda_{\alpha}}{s}\right)$ and $L^{-1}\left(\frac{1}{s^{\alpha}+\mu_{\alpha}}\right)$ where

$$L^{-1}\left(\frac{R(0)}{s^{1-\alpha}}+\frac{\mu_{\alpha}\lambda_{\alpha}}{s}\right)=\mu_{\alpha}\lambda_{\alpha}+R(0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$$

and

$$L^{-1}\left(\frac{1}{s^{\alpha}+\mu_{\alpha}}\right)=t^{\alpha-1}E_{\alpha,\alpha}(-\mu_{\alpha}t^{\alpha})$$

where $E_{\alpha,\alpha}(-\mu_{\alpha}t^{\alpha})=\sum_{k=0}^{\infty}\frac{(-\mu_{\alpha}t^{\alpha})^{k}}{\Gamma((1+k)\alpha))}$ is the Mittag-Leffler function. Thus,

$$\begin{aligned}R(t)&=\int_{0}^{t}\left(\mu_{\alpha}\lambda_{\alpha}+R(0)\frac{t-\tau^{-\alpha}}{\Gamma(1-\alpha)}\right)\tau^{\alpha-1}E_{\alpha,\alpha}(-\mu_{\alpha}\tau^{\alpha})d\tau\\&=\int_{0}^{t}\left(\mu_{\alpha}\lambda_{\alpha}\tau^{\alpha-1}+R(0)\frac{\tau^{\alpha-1}(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\right)\sum_{k=0}^{\infty}\frac{(-\mu_{\alpha}t^{\alpha})^{k}}{\Gamma\left((1+k)\alpha\right)}d\tau\end{aligned}$$

If $R(0)=0$, then elementary manipulations yield

$$R(t)=\mu_{\alpha}\lambda_{\alpha}\sum_{k=0}^{\infty}\frac{(-\mu_{\alpha})^{k}t^{\alpha(k+1)}}{\Gamma(1+(1+k)\alpha)}$$

Let $\alpha=1$, then $R^{*}(s)=\left(R(0)+\frac{\mu\lambda}{s}\right)\frac{1}{s+\mu}$ and therefore:

$$R(t)=R(0)e^{-\mu t}+\mu\lambda\int_{0}^{t}e^{-\mu\tau}d\tau=R(0)e^{-\mu t}+\lambda\left(1-e^{-\mu t}\right)$$

Of course, if we consider random variations due to the fractional Brownian motion $\sigma dW_{\alpha}(t)$, the result is more complex.

5 The Poisson randomness and its fractional definition

The Poisson process is based on assumptions justified often in practice and used to characterize the mathematical representation of such processes. These are,

(1) events are independent and, (2) events occur one at a time. If the event rate (i.e. the number of events occurring in a given time interval) is also constant, the Poisson probability distribution results. Consider two subsequent instants of time $[t, t+dt]$ and calculate the number of events that occurs at time t. Say that we are at $[t, t+dt]$ with a count of n events. Given the Poisson process hypotheses stated above, this count can be reached if at time t there was already a count of $n-1$ occurrences and one more event was added, or if at time t there was already n event occurrences and in the subsequent time interval no event has occurred. This is represented graphically in figure 2.
If the event probability is known and given by λdt in $[t, t+dt]$, then the probability that there are n event occurrences at $t+dt$ is explicitly given by $P_n(t+dt) = P_n(t)(1-\lambda dt) + P_{n-1}(t)\lambda dt$, $n = 1, 2, 3, \ldots$ and $P_0(t+dt) = P_0(t)(1-\lambda dt)$.
These equations can be written as a systems of linear differential equations :

$$\begin{aligned} \frac{dP_n(t)}{dt} &= -P_n(t)\lambda + P_{n-1}(t)\lambda, \quad n = 1, 2, 3, \ldots, \text{ and} \\ \frac{dP_0(t)}{dt} &= -P_0(t)\lambda. \end{aligned}$$

It is a simple exercise to show that the solution of these equations is given by $P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ with cumulative distribution,

$$F_m(t) = \sum_{n=0}^{m} P_n(t) = \sum_{n=0}^{m} e^{-\lambda t}\frac{(\lambda t)^n}{n!} = 1 - \int_0^{\lambda t} \frac{x^m e^{-x}}{m!} dx$$

with mean and variance $\mathbb{E}(n) = Var(n) = \lambda t$. Further, $\int_0^{\lambda t} \frac{x^m e^{-x}}{m!} dx$ a Gamma integral.
As a result, a Poisson process concludes that it is a memoryless process since:

- At any time, after an event has occurred, the residual time for the next event to occur, has an exponential probability distribution.
- The mean residual time is equal to $\frac{1}{\lambda}$.
- This residual time is independent of the event which has occurred at time previously.

A fractional development due to Busani and Merzbach provides a kernel based approach.

5.1 The Busani-Leonenko-Merzbach fractional jump process kernel

Busani [6] and Leonenko and Merzbach [17] have proved and provided a kernel to calculate the probability of fractional distributions for a jump process. Say that $P_n(t)$ is the Poisson distribution of n events occurring in a time interval $(0, t)$ and let $P_{n,\alpha}(t)$ be the probability of a fractional Poisson probability distribution. Then

$$P_{n,\alpha}(t) = \int_0^t P_n(t-\tau) f_B(\tau) d\tau,$$

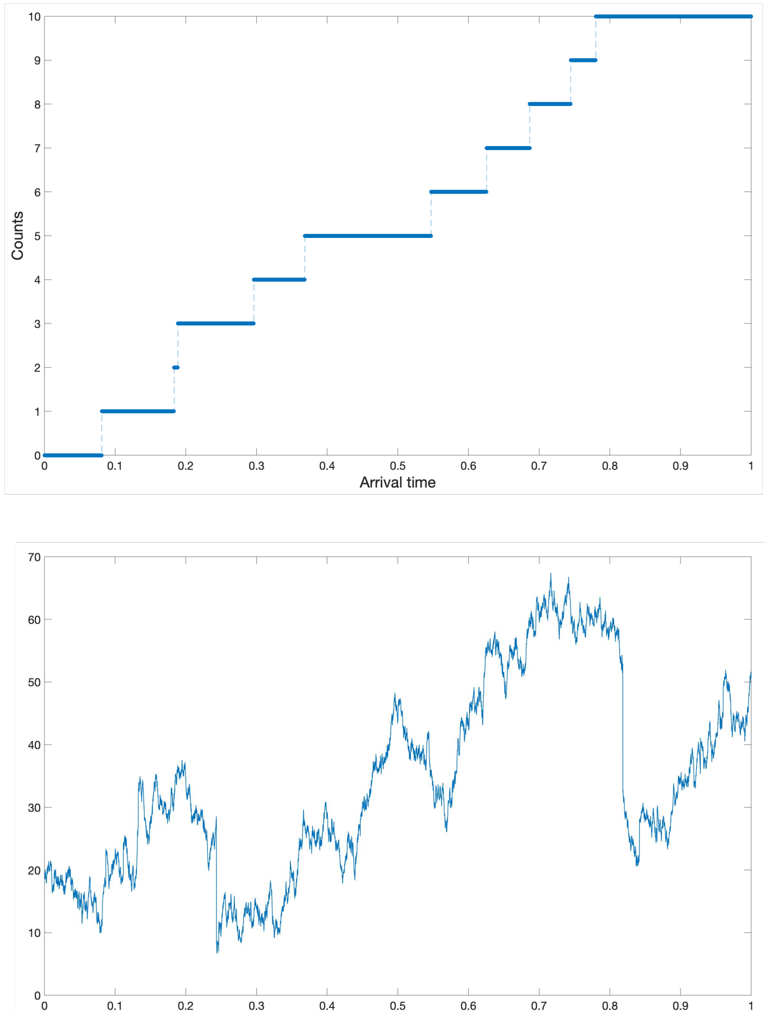

Fig. 2. Above: Example of a path of a standard Poisson process. Below: Example of superposition of Brownian motion and a compound Poisson process

or, in Laplace Transform

$$P^*_{n,\alpha}(s) = P^*_n(s) f^*_{B'}(s)$$

where $f_{B'}(t-\tau)$ is the probability measure, a Beta Prime Probability Distribution. For example, consider the counting Poisson jump process

$$P_k^{fr}(t,k) = \int_0^t P_k(t-\tau) f_{B'}(\tau) d\tau = \int_0^t (\lambda(t-\tau))^k \frac{e^{-\lambda(t-\tau)}}{k!} f_{B'}(\tau) d\tau$$

where $P_k^{fr}(t,k)$ is the fractional probability of the Poisson event k at time t. In particular, at $k=0$, we have:

$$P_k^{fr}(t,0) = \int_0^t e^{-\lambda(t-\tau)} f_{B'}(\tau) d\tau$$

However, the fractional probability of $e^{-\lambda t}$ is the Mittag-Leffler function $E_\alpha(-\lambda t^\alpha)$ defined for example, by setting $h = \lambda t^\alpha$, $E_\alpha(h) = \sum_{k=0}^{\infty} \frac{h^k}{\Gamma(1+\alpha k)}$ with the Lapace Transform $L^*\left(E_\alpha(-\lambda t^\alpha)\right) = \frac{p^{\alpha-1}}{p^\alpha+\lambda} = \frac{1}{p^{1-\alpha}(p^\alpha+\lambda)}$. Using the Mittag Leffler function

$$\begin{aligned} L^*\left(E_\alpha(-\lambda t^\alpha)\right) &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k L^*(t^{\alpha k})}{\Gamma(1+\alpha k)} = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\Gamma(1+\alpha k)} \frac{\Gamma(1+\alpha k)}{p^{1+\alpha k}} = \frac{1}{p} \sum_{k=0}^{\infty} \left(\frac{-\lambda}{p^\alpha}\right)^k \\ &= \frac{1}{p^{1-\alpha}(p^\alpha+\lambda)} \end{aligned}$$

Of course, when $\alpha = 1$, $L^*\left(E_\alpha(-\lambda t)\right) = \frac{1}{p+\lambda}$, $L^{-1}(\cdot) = e^{-\lambda t}$. As a result, the Laplace Transform is

$$P_k^{*fr}(s,0) = \frac{s^{\alpha-1}}{s^\alpha+\lambda}$$

Since the Laplace Transform of $e^{-\lambda t}$ is $\frac{1}{s+\lambda}$, by the convolution theorem we have necessarily:

$$P_k^{*fr}(s,0) = \frac{1}{s+\lambda} f_{B'}^*(s) \quad \text{or} \quad (s+\lambda)\frac{s^{\alpha-1}}{s^\alpha+\lambda} = f_{B'}^*(s).$$

Since for any n, we have $P_{n,\alpha}^{*fr}(s) = P_n^*(s) f_{B'}^*(s)$, we have also

$$\frac{P_{n,\alpha}^{*fr}(s)}{P_k^{*fr}(s,0)} = (s+\lambda)P_n^*(s) \quad \text{and} \quad P_{n,\alpha}^{*fr}(s) = (s+\lambda)\left(\frac{s^{\alpha-1}}{s^\alpha+\lambda}\right) P_n^*(s).$$

In particular, for $n=1$, $P_1^*(s) = \frac{\lambda}{(s+\lambda)^2}$ and

$$\frac{P_{1,\alpha}^{*fr}(s)}{P_k^{*fr}(s,0)} = (s+\lambda)P_1^*(s) = \frac{\lambda}{s+\lambda}, \quad \text{or} \quad P_{1,\alpha}^{*fr}(s) = \left(\frac{\lambda}{s+\lambda}\right)\left(\frac{s^{\alpha-1}}{s^\alpha+\lambda}\right).$$

Generally, since in this particular case, $P_n^*(s) = \left(\frac{\lambda}{s+\lambda}\right)^{n+1}$, the Laplace Transform of the Beta Prime probability distribution is $f_{B'}^*(s) = \frac{P_{n,\alpha}^*(s)}{P_n^*(s)}$, and therefore

$$f_{B'}^*(s) = \frac{\lambda_\alpha^n}{\lambda^{n+1}} \frac{(s^{\alpha-1}+\lambda_\alpha)(s+\lambda)^{n+1}}{(s^\alpha+\lambda_\alpha)^{n+1}}$$

Letting $\lambda = \lambda_\alpha$, this is reduced to

$$f_{B'}^*(s) = \frac{1}{\lambda}\left(\frac{s+\lambda}{s^\alpha+\lambda}\right)^{n+1} (s^{\alpha-1}+\lambda)$$

Therefore, its first two moments are defined by the following at $s = 0$:

$$\frac{\partial f^*_{B'}(s)}{\partial s} = \frac{1}{\lambda}\frac{\partial}{\partial s}\left(\frac{1}{\lambda}\left(\frac{s+\lambda}{s^\alpha+\lambda}\right)^{n+1}(s^{\alpha-1}+\lambda)\right)$$

$$\frac{\partial^2 f^*_{B'}(s)}{\partial s^2} = \frac{1}{\lambda}\frac{\partial^2}{\partial s^2}\left(\frac{1}{\lambda}\left(\frac{s+\lambda}{s^\alpha+\lambda}\right)^{n+1}(s^{\alpha-1}+\lambda)\right)$$

Alternatively,

$$P^*_n(s) = \left(\frac{\lambda}{s+\lambda}\right)^{n+1}, \quad f^*_{B'}(s) = \frac{e^{st_0}\mathit{\Gamma}(\alpha, st_0)}{\mathit{\Gamma}(\alpha)}, \ \mathit{\Gamma}(\alpha, st_0) = \int_{st_0}^{\infty} \tau^{\alpha-1}e^{-\tau}d\tau$$

and therefore,

$$P^*_{n,fr}(s) = \frac{e^{st_0}\mathit{\Gamma}(a, st_0)}{\mathit{\Gamma}(\alpha)}\left(\frac{\lambda}{s+\lambda}\right)^{n+1}.$$

These two moments can be used to derive the parameters of the Beta Prime generalized distribution specified below.

The Gamma probability distribution (G) is the inverse counting time distribution for n events to occur in a Poisson process. Their $n+1$ convolutions have the Laplace Transform:

$$\lambda\int_0^\infty e^{-(s+\lambda)x}dx = \frac{\lambda}{s+\lambda} \text{ and therefore } L^*(G) = \left(\frac{\lambda}{s+\lambda}\right)^{n+1}$$

As a result, its fractional randomness (its distribution) is as stated above:

$$P^*_{G,fr}(s) = \frac{e^{st_0}\mathit{\Gamma}(\alpha, st_0)}{\mathit{\Gamma}(\alpha)}\left(\frac{\lambda}{s+\lambda}\right)^{n+1}$$

6 The Beta Prime probability distribution and its fractional randomness

The Beta Prime distribution is defined as the odds of a Beta probability distribution. Explicitly, let the Beta probability distribution be $f_B(y;\alpha,\beta)$, the Beta Prime distribution, also called the inverted Beta or Beta of the second kind, is then defined by the probability distribution of $x = \frac{y}{1-y}$:

$$f_{B'}(x;\alpha,\beta) = \frac{x^{\alpha-1}(1+x)^{-(\alpha+\beta)}}{B(\alpha,\beta)}$$

while the mean and the variance are

$$\mathbb{E}_{B'}(x) = \frac{\alpha}{\beta-1}; \quad \mathrm{Var}_{B'}(x) = \left(\frac{\alpha}{\beta-1}\right)\left(\frac{\alpha+\beta-1}{(\beta-1)(\beta-2)}\right), \quad \beta > 2$$

The k-th moment is defined for $-\alpha < k < \beta$ by

$$\mathbb{E}_{B'}(x^k) = \prod_{i=1}^{k} \left(\frac{\alpha + i - 1}{\beta - i} \right).$$

The cumulative distribution is also given in terms of a Gauss hypergeometric function. A generalized Beta Prime distribution can be generalized further by setting:

$$f_{B'}(x; \alpha, \beta, h) = \frac{p \left(\frac{x}{h}\right)^{\alpha p - 1} \left(1 + \left(\frac{x}{h}\right)^p\right)^{-(\alpha+\beta)}}{hB(\alpha, \beta)}$$

with

$$\mathbb{E}_{B'}(x) = \frac{h\Gamma\left(\alpha + \frac{1}{p}\right) \Gamma\left(\beta - \frac{1}{p}\right)}{\Gamma(\alpha)\Gamma(\beta)}, \quad \beta p > 1$$

The fractional kernel for a jump process is then given for $p = 1$ by:

$$P_{n,\alpha}(t) = \frac{1}{hB(\alpha, \beta)} \int_0^t P_n(t - \tau) \left(\frac{\tau}{h}\right)^{\alpha - 1} \left(1 + \frac{\tau}{h}\right)^{-(\alpha+\beta)} d\tau$$

while its Laplace Transform is given by

$$\int_0^\infty e^{-sx} f_{B'}(x; \alpha, \beta, h) dx = \frac{p}{hB(\alpha, \beta)} \int_0^\infty e^{sx} \left(\frac{x}{h}\right)^{\alpha p - 1} \left(1 + \left(\frac{x}{h}\right)^p\right)^{-(\alpha+\beta)} dx$$

Consider the integral

$$\int_0^\infty e^{-sx} \left(\frac{x}{h}\right)^{\alpha p - 1} \left(1 + \left(\frac{x}{h}\right)^p\right)^{-(\alpha+\beta)} dx$$

which is a confluent hypergeomeric function (see [1]). A development by Ofer Busani indicates then that the Laplace Transform is

$$f^*_{B'}(s) = \frac{e^{sT}\Gamma(\alpha, sT)}{\Gamma(\alpha)}, \quad \text{with } \lim_{\alpha \to 1} \frac{e^{sT}\Gamma(\alpha, sT)}{\Gamma(\alpha)} = 1$$

where $\Gamma(\alpha, sT)$ is the incomplete Gamma function: $\Gamma(\alpha, sT) = \int_{sT}^{\infty} e^{-\tau}\tau^{\alpha-1}d\tau$, with

$$\frac{\partial \Gamma(\alpha, sT)}{\partial s} = -T^\alpha s^{\alpha-1} e^{-sT}, \quad \text{and} \quad \frac{\partial^2 \Gamma(\alpha, sT)}{\partial s^2} = (1 - \alpha + sT)\, T^\alpha s^{\alpha-2} e^{-sT}$$

which we use to calculate the mean, the second moment and the third moment of the fractional distribution.
For the exponential function $\lambda e^{-\lambda t}$, whose Laplace Transform is $\frac{\lambda}{s+\lambda}$, we have its fractional Laplace Transform:

$$f^*_{Exp,\alpha}(s) = \frac{\lambda}{s + \lambda} \frac{e^{sT}\Gamma(\alpha, sT)}{\Gamma(\alpha)}$$

By the same token, the fractional Transform of the Gamma distribution is:

$$f^*_{G,\alpha}(s) = \left(\frac{\lambda}{s+\lambda}\right)^m \frac{e^{sT}\Gamma(\alpha, sT)}{\Gamma(\alpha)}$$

Therefore

$$\frac{\partial f^*_{G,\alpha}(s)}{\partial s} = \frac{\lambda^m e^{sT}}{\Gamma(\alpha)}\left(\frac{1}{s+\lambda}\right)^m \left[-m\left(\frac{1}{s+\lambda}\right) + T + \frac{\partial\Gamma(\alpha, sT)}{\Gamma(\alpha, sT)\partial s}\right]$$

or

$$\frac{\partial f^*_{G,\alpha}(s)}{\partial s} = \frac{\lambda^m e^{sT}}{\Gamma(\alpha)}\left(\frac{1}{s+\lambda}\right)^m \left[-m\left(\frac{1}{s+\lambda}\right) + T - \frac{T^\alpha s^{\alpha-1}e^{-sT}}{\Gamma(\alpha, sT)}\right]$$

with

$$\frac{\partial f^*_{G,\alpha}(0)}{\partial s} = \frac{1}{\Gamma(\alpha)}\left(-\frac{m}{\lambda} + T\right)$$

7 The origins of the Brownian fractional bridge randomness

Brownian Motion anomalous models properties have been observed in many areas and applications. In financial trading and pricing models, models' granularity and its implications are often neglected, including the financial risks they entail. For example, models risks measurements based on High Frequency models or on Day data models that do not imply the same risks. This is due to the effects of the fractional operators applied that alters the definition of their underlying probability distribution as well as their associated risks. In [25], [26], [27] we demonstrated that fractional probability distributions need be complete and therefore leading for the most part to fat tail distributions.
Mandelbrot in many applications suggested and applied a fractional Brownian Motion defined by its fractional variance [19], [20]. In [26] and [27] we demonstrated that fractional randomness depending on its fractional index may lead instead to a randomness defined by a fractional bridge when its index H is defined in $\frac{1}{2} < H < 1$ while randomness is defined by alpha stable distributions defined in $0 < H < \frac{1}{2}$. These results have proved that randomness has its origin in the granularity of models and thus the granularity of data. These results are practically important as they imply that financial models may mislead investors and high frequency traders (or traders in general), as they depend on the granularity of the models and the data they access to. For example, a common High Frequency trading strategy may have access to data that other traders may not, due to their trading granularity. Providing thereby opportunities for arbitrage. In a non-fractional environment, statistical arguments and Kolmogorov-Smirnov theorem, led to a Brownian Motion randomness while in a fractional model and data environment, we have shown that randomness is far more varied, and a function of the fractional index. The financial mathematical literature has emphasized the

definition of randomness by a Brownian Motion while Mandelbrot, in a multitude of studies (for example, [20] and others) has pointed to fractional volatility Brownian Motion. It has led to numerous opportunities to financial traders. For example, [7], [8], [9] indicates that granularity introduces in financial models a potential for arbitrage, [12] provided a solution to a fractional Black and Scholes model to be a pricing martingale and therefore complete since future prices are always known (and under the martingale probability measure, equal the current price). While this is an appropriate mathematical result, it is not clear whether this martingale is also a true market pricing martingale as in any model, there may be several martingales but only one may be a pricing martingale. Further, martingale pricing defines a current price based on the expectation of a future price defined by appropriate (and price relative) probability measures. In a financial context, both the future distribution and the probability measure have to account for their granularity that renders the definition of a fractional financial pricing model far more questionable. Other developments are due to [10]. [11] provide a general fractional white-noise randomness based on the Wicks-Itô-Skorohod (WIS) calculus resulting in integrating the Itô and Stratonovich calculus. [13], [16] and [15] instead applies and modifies the Riemann-Liouville derivative and fractional Taylor series to non-differentiable functions ([14], [18]). A statistical and fractional development ([26] and [27]) application to granular data leads to alternative measures of randomness. As a reference, we outline the Riemann-Liouville fractional operators (1832) to a PDF, $f(\tau)$ at time τ over a time interval $[0,t]$ applied to a probability distribution ([25]). Define a distribution FCDF and its FPDF with the notations CDF be $F(t)$, FCDF be $F_H(t)$ and $f_H(t)$ the FPDF, defined by the Riemann-Liouville convolution integrals:

$$\begin{aligned} f_H(t) &= \frac{1}{\Gamma(1-H)}\int_0^t f(\tau)(t-\tau)^{-H}d\tau \\ &= \frac{1}{\Gamma(1-H)}\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{1-H}d\tau,\ t\geq 0,\ \text{or} \\ f_H(t) &= \frac{1}{\Gamma(1-H)}(f*g)(t);\ g=\tau^{-H}1_{\tau\geq 0} \end{aligned}$$

$$F_H(t) = I^H f(t) = \frac{1}{\Gamma(H)}\int_0^t f(\tau)(t-\tau)^{H-1}d\tau,\ t\geq 0.$$

with $(f*g)$ denoiting the convolution operator.
We let $0<H<1$ and $F:[0,\infty[\to[0,\infty[$, a fractional probability distribution is then defined by the fractional derivative of its Cumulative Distribution Function (CDF): $f_H := D^H F$. Application of the Liouville operator points as well to the following definition: $f_H := D^H I^1 f = D^H(I^H I^{1-H})f = I^{1-H}f$. The fractional distribution that results then, is not as stated previously, a conventional distribution since $\int_0^\infty g(\tau)d\tau = +\infty$. While the fractional cumulative distribution function (FCDF) is defined by the fractional integration of its density function $f(t)$ (FDF). It is easy to prove that $\lim_{t\to\infty} F_H(t) \neq 1$ and therefore the

fractional distribution is not a conventional one. A statistical approach to a fractional distribution will provide instead a fractional randomness, a function of the fractional index.

8 The Fractional Brownian Bridge and randomness

Assume the PDF $f(\tau)$ and τ a random variable. In a time interval $[0, t]$, the probability of such events occurring is $F(t)$ while its FCDF is defined as stated above by the convolution integral (2.3). Let τ be defined in $1_{\tau\geq 0}$, then:

$$(t-\tau)_+^{H-1} = \begin{cases} (t-\tau)^{H-1}, & \text{if } t > \tau \\ 0, & \text{otherwise} \end{cases}$$

while its truncated expectation with respect to the PDF $f(\tau)$ is:

$$F_H(t) = \mathbb{E}\{(t-\tau)_+^{H-1}\}, \quad t \geq \tau \geq 0$$

Consider a large population of events (τ_i), all of which are identically and independently distributed. Then, under the Kolmogorov-Smirnov theorem for a Central Limit Convergence and since samples $(t-\tau_i)_+^{H-1}, i = 1, 2, \ldots, n$ is integrable, we have:

$$\frac{1}{n\Gamma(H)} \sum_{i=1}^{n} \left((t-\tau_i)_+^{H-1}\right) \xrightarrow[n\to+\infty]{\text{a.s.}} F_H(t)$$

Several limit cases arise corresponding to the fractional index H. The standard, non-fractional case $H = 1$ with τ_k defined by a uniform distribution in $[0, 1]$, and i.i.d. events at time, $(\tau_k)_{k\geq 1}$. Then, $\tau_1 \overset{d}{=} \tau$, $1_{\{\tau_1 \leq t\}}$ defines a random variable denoting the probability of an event occurring at time τ_1. Its CDF in a time interval $[0, t]$ has thus a Bernoulli probability $F_1(t)$ given by:

$$\lim_{n\to\infty} \left(\frac{1}{n}\sum_{i=1}^{n} 1_{\{\tau_1 \leq t\}}\right) \xrightarrow{\text{a.s.}} P(\tau_1 \leq t) = F_1(t), \quad 0 \leq t \leq 1.$$

In other words, $1_{\{\tau_1 \leq t\}} \implies F_1(t)$. As a result, the central limit of its sum leads to a Binomial distribution:

$$\sum_{i=1}^{n} 1_{\{\tau_1 \leq t\}} \sim B(n, F_1(t)), \quad t \geq 0$$

while its central limit leads to a normal probability distribution with Bernoulli variance $F_1(t)(1 - F_1(t))$

$$\Lambda_{1,n}(t) = \frac{\sum_{i=1}^{n} 1_{\{\tau_1 \leq t\}} - nF_1(t)}{\sqrt{n}}, \quad t \geq 0, \text{ and}$$

$$\begin{aligned} \mathrm{Var}(\Lambda_{1,n}(t)) &= \frac{1}{n}\mathrm{Var}\left(\sum_{i=1}^{n} 1_{\{\tau_1 \leq t\}} - nF_1(t)\right) = \frac{1}{n}\mathrm{Var}\left(\sum_{i=1}^{n} 1_{\{\tau_1 \leq t\}}\right) \\ &= F_1(t)(1 - F_1(t)) \end{aligned}$$

Two cases arise, $\frac{1}{2} < H < 1$ and $0 < H < \frac{1}{2}$.

8.1 The Fractional Brownian Bridge, $\frac{1}{2} < H < 1$

Let U be a uniform random distribution in $0 \le t \le 1$. Then the standardized limit $\Lambda_n^U(t)$ has in law, a Brownian Bridge distribution, $BB_n(t)$, where $t \to BB_n(t)$ is a function of its finite variation and by definition $BB_n(t) \to W(t) - tW(1) : \Lambda_n^U(t) = \frac{1}{\sqrt{n}}\left(\sum_{k=1}^n 1_{U_k \le t} - t\right)$ where at the limit, $\Lambda_n^U(t) \sim BB_n(t)$ with a variance and covariance: $\mathrm{Var}(BB(t)) = t(1-t)$ and $\mathrm{Cov}(BB(t)BB(\tau)) = t\tau$. For its fractional distribution, we have:

$$\Lambda_{H,n}^U(t) = \frac{1}{\sqrt{n}}\left\{\sum_{k=1}^n (t-U_k)_+^{H-1} - \frac{nt^H}{H}\right\}, \quad t \in [0,1]$$

At the limit it admits a normal probability distribution with mean null and variance $t^{2H-1}\left(\frac{1}{2H-1} - \frac{t}{H^2}\right)$ and

$$\Lambda_H^U(t) \sim N\left(0, t^{2H-1}\left(\frac{1}{2H-1} - \frac{t}{H^2}\right)\right)$$

Proof in [26].

The implication of this result, indicates that for $\frac{1}{2} < H < 1$, the variance is nonlinear while at $H = 1$, it is reduced to the variance of the standard Brownian Bridge $t(1-t)$. Further, for $H = 0.6$, $t^{0.2}\left(5 - \frac{t}{0.36}\right) > t(1-t)$, $t \in [0,1]$ (see Figure 3), which points to a substantial increase in the variance compared to the case $H = 1$. In other words, a fractional random distribution has a far greater variance. By the same token, we note that the fractional uniform distribution is in fact auto-correlated as it is the case for Fractional Brownian Motion (which has also a self similar distribution).

Further, the **Covariation of a Fractional Uniform Distribution** $\Lambda_H^U(t)$ defined above is autocorrelated with an autocovariance given by:

$$\mathrm{Cov}(\Lambda_H^U(t), \Lambda_H^U(s)) = \frac{1}{[\Gamma(H)]^2}\left(\Psi(s,t) - \frac{s^H}{H}\frac{t^H}{H}\right), \quad 0 < s \le t \le 1$$

where $\Psi(s,t)$ is given by

$$\begin{aligned}\Psi(s,t) &= \int_0^s (t-\tau)_+^{H-1}(s-\tau)_+^{H-1} d\tau \\ &= s^H (t-s)^{H-1} \frac{\Gamma(H)}{\Gamma(1+H)} F\left(1-H, H; 1+H; \frac{-s}{t-s}\right)\end{aligned}$$

with

$$F\left(1-H, H; 1+H; \frac{z^k}{\Gamma(1+k)}\right) = \frac{1}{\Gamma(1-H)}\sum_{k\ge 0} \frac{\Gamma(1-H+k)}{H+k}\frac{z^k}{\Gamma(1+k)}, \; z = \frac{-s}{t-s}$$

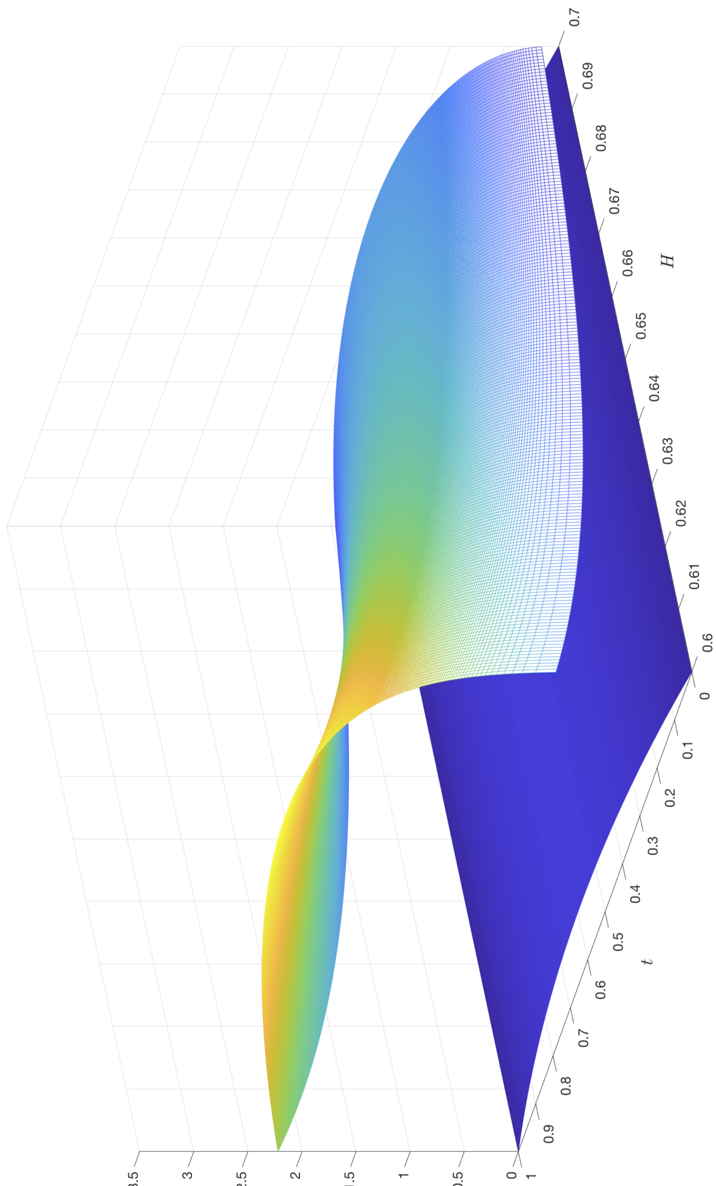

Fig. 3. Variance of the limit fractional normal distribution $t^{2H-1}\left(\frac{1}{2H-1} - \frac{t}{H^2}\right)$ compared with the non fractional case $t(1-t)$ (blue surface), for $H \in (0.6, 0.7)$

Again, for $H = 1$, $\mathrm{Cov}(\Lambda_1^U(t), \Lambda_1^U(s)) = s(1-t); 0 < s \leq t \leq 1$ (proof in [25]). Thus, the case $\frac{1}{2} < H < 1$ implies that randomness is defined by a Brownian Bridge and therefore the events and their outcomes described by a financial granular model (or its data granularity) with a fractional index in this interval are defined by such a randomness.

9 Conclusion

Randomness origins are many, we have considered in this paper the effects of data granularity, information and speed of convergence, all three factors dependent on one another. The standard case led to the Fractional Brownian Motion while the fractional index defined in $0 < H < \frac{1}{2}$ implies an α-stable randomness while for $\frac{1}{2} < H < 1$ implies a Brownian Bridge randomness. Much further research is needed however to assess the implications of the origins of randomness and their implications to the models we constructs as theoretical guides to the study of stochastic processes.

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