Extended choice correspondences & an axiomatic characterization of the probabilistic Borda rule

Somdeb Lahiri

School of Petroleum Management, PDE University,
P.O. Raisan, Gandhinagar-382007, Gujarat, India.
somdeb.lahiri@gmail.com
ORCID: https://orcid.org/0000-0002-5247-3497

The common question that gets asked in business is, 'Why?'. That's a good question, but an equally valid question is, 'Why not?'. – Jeff Bezos, CEO of Amazon

Abstract. We propose an extension of the framework developed by Professor Kenneth J. Arrow, for the analysis of choice under risk by an individual, hereafter referred to as a decision maker. The framework is based on the state dependent rankings of alternatives of the decision maker. We begin by showing that the existing model of decision making under uncertainty due to Professor Edi Karni can be accommodated in our framework. We provide several examples to illustrate meaningful possibilities in the model proposed here. In a final section of the paper we provide an axiomatic characterization of the Probabilistic Borda Rule based on Anonymity and a Maximal Coherence assumption, the latter being a minor variant of a similar property due to Professor Bruno De Finetti in his philosophy of probability theory.

Keywords. Risk, State-dependent Preferences, Extended Choice Correspondence, Probabilistic Borda Rule.

M.S.C. classification. Not given by the Author.
J.E.L. classification. D01, D81.

1 Introduction

Here we propose a framework for the analysis of choice under risk by an individual, hereafter referred to as a decision maker. The framework is based on the state dependent rankings of alternatives of the decision maker. It is an extension of the seminal model of social choice theory developed by Kenneth J. Arrow which is discussed and generalized like a master craftsman by Amartya Sen in [17]. Arrow and Sen were primarily concerned with aggregating non-weighted individual/criteria/state dependent rankings of alternatives into an overall (or social) ranking of alternatives. The problem of choosing one or more alternatives from a given set of alternatives when the decision maker is given non-weighted individual/criteria/state dependent rankings of alternatives was raised and rigorously formulated for the first time in a seminal contribution on majority voting by [13]. For the classical theory of decision making under uncertainty in the state dependent case - which is the other and major motivation behind this paper- one may refer to [6]. [6] and [17]comfortably surpass the prerequisites related to decision making that is required to be able to understand the frameworks of analyses developed here. An informative overall perspective of decision theory can be found in [15]. State dependent utility functions from a more advanced perspective are discussed in chapter 8 of [4].

The initial concern that led to the frameworks discussed in this paper is that, Arrowian voting theory framework which is developed and extended in [17], does not have anything to say about the role of negotiations in group decision making and may therefore be very inadequate for our understanding of decision making in society. Does decision making in society usually take place as in Arrowian voting theory? How often are votes taken in board meetings to decide on crucial matters? Perhaps "glib talk" and "staying power in negotiation" account more than what we are willing to give them credit for, in arriving at a decision. In fact colleagues in HR would point out that good leadership (at least in the corporate sector) consists in getting everybody on board rather than scoring victories over rivals. In view of this, slight extensions of voting models as models of individual choice under risk may serve a useful purpose.

As mentioned earlier, the framework introduced here is an extension of the seminal model of social choice theory developed by Kenneth J. Arrow. A decision maker is faced with making a choice under probabilistic uncertainty (risk) in which uncertainty is with regard to a future state of nature, which is realized after the decision has been made. The decision maker is provided with (or aware of) a data profile, which is a pair whose first component is a profile of state dependent rankings over (the consequences) a non-empty finite set of alternatives and whose second component is a probability distribution over a non-empty finite states of nature. The probability distribution could be subjective, and the present author is inclined to agree with such a view. A decision support system (DSS) or decision aid is required to choose a non-empty "desirable" set of alternatives from which the final choice has to be made. The decision aid or DSS has no bias in favour of any one or more alternatives that it suggests. Such a decision support system is called an extended choice correspondence, i.e. a rule which associates with each data profile from a given set of data profiles a non-empty finite set of desirable alternatives.

The reason for our interest in state-dependent preferences are precisely the same as the ones discussed in [6], i.e. it is so obviously true that it does not need justification beyond citing trivial day-to-day examples as Karni has done in his book. Heiner (in [5]) disagrees and says: "In this regard, recall Arrow's well-known definition of a state of nature as a description of the world so complete that, if known, the full consequences from every action would be completely specified. Note the plural here (consequences). Not just part, but all of the consequences relevant to an agent's welfare would be specified from knowing each state. Not just watching a football game from the fifty-yard line, but also the complementary sensations from rain or sunshine affect an agent's enjoyment from seeing game. Not just different wealth levels, but the physical ability to use monetary wealth in particular ways affects an agent's welfare. The issue here is whether one should expand the dimensions of the outcome space to incorporate all welfare-relevant factors (as suggested by Arrow's definition) or, instead, restrict the dimensionality of the outcome space (for instance, to variations in monetary wealth) and load all other welfare altering factors into the state space. If the former is done, then there is no need to allow for state-dependent utility. Which approach is more fruitful?"

Mercifully, general equilibrium has progressed much beyond where Arrow and Debreu left it and that has been possible because both economic theorists as well applied economists who have worked on the Arrow-Debreu model were more concerned with the analytical structure of and mathematical developments in general equilibrium than with hair-splitting the definition of a state of nature, so as to render the model totally useless. Hence we can comfortably move ahead with our understanding of state-dependent preferences as in [6].

The major justification for the frameworks and the investigation within them that is presented in this paper is that the classical theory of decision making under uncertainty that rests on the assumption of maximization of expected utility (state-dependent or not) has an important limitation- i.e. the decision maker's preferences may not be available in the form of cardinal utility functions, but only as rankings. That leads to a departure from the classical theory and opens up the possibility of decision makers using other algorithms (decision aids) for the purpose of decision making under risk. That is the line of investigation pursued in this paper.

The framework of analysis with which we work here has been motivated in [7]. For instance, consider an individual who before going out to work has to decide whether to carry a hat or a raincoat along with him. His preference between the two depends on the anticipated weather during that day. On a rainy day he would prefer a raincoat to a hat and on a sunny day, his preference between the two would be reversed. If the weather forecast for the day or his own premonition suggests that there is a very high likely-hood of there being rain during the day-particularly during the hours he has to spend commuting- then he would be inclined to carry a raincoat along with him to work. On the other hand if he thinks there is a strong chance of the day being "sun-shiny", then he might choose to carry a hat along with him to work. Related axiomatic analysis when the decision maker believes that the states of nature are equiprobable is available in [8]. A full-fledged application using components of this framework to prove the existence of "preferred with probability at least half winners" has been provided in [9] and [10].

Here we begin by setting up the model for extended choice correspondences and show that decision making under uncertainty with state-dependent preferences fits into our framework. We illustrate the relevance of our model by invoking a rather "easy" application related to covid-19. We next provide examples of extended choice correspondences. The equi-probable version of most extended choice correspondences that we introduce here can be found in [18]. A more comprehensive and complete survey of the related literature is available in [1].

After that we proceed to discuss some decision rules introduced in [7]. Following suggestion by Professor Prasanta Pattanaik, we call the related extended choice correspondences "state-salient rules". Plausible examples of such rules are those that choose the highest ranked alternative in a state of nature which has the highest probability of occurrence. We provide (the suitably adjusted version of) an axiomatic characterization of state-salient rules due to Denicolo in [2].

Subsequently we discuss two important properties of extended choice correspondences − the Probabilistic Condorcet criterion and the Probabilistic Majority criterion. The probabilistic Condorcet criterion says if an alternative is preferred by a second alternative with probability greater than half, then the first alternative is never chose. The equiprobable version of this property is a major "mile-stone" in collective choice theory. We show that that neither the probabilistic plurality rule nor the probabilistic run-off method satisfies the probabilistic Condorcet criterion. The probabilistic majority criterion based on its equiprobable version due to Professor Janez Zerovnik and available in [8]says that if an alternative is preferred to all other alternatives with probability at least half and is not preferred by any other alternative with probability strictly greater than half, then the alternative is the unique one to be chosen. Thus, if x is ranked above y with probability exactly half and if x ties with y even once, then according to the probabilistic majority criterion, between x and y only x will be chosen. We complete our comparison of the two axioms by proving that if an extended choice correspondence satisfies the probabilistic Condorcet criterion, then it also satisfies the probabilistic majority criterion.

Among all the extended choice correspondences we introduce, one- namely, the Probabilistic Borda Rule- stands out as exceptional in a very specific sense as pointed out by [19]. It is proved there, that in the equiprobable case, the Probabilistic Borda Rule "maximizes the probability that a Condorcet" winner is chosen. A more detailed study of this rule in the equiprobable case can be found in [16]. The final section of this paper presents a complete axiomatic characterization of the Probabilistic Borda Rule using just two axioms. The first is a rather commonplace one- Anonymity- which says that the names of the states of nature do not matter in determining the set of chosen alternatives. The second is Maximal Coherence, which requires that the extended choice correspondence satisfies Coherence- a variant of a property by the same name introduced by Professor Bruno De Finetti in his seminal work on the philosophy of probability theory- and there is no other extended choice correspondence that satisfies Coherence and whose chosen sets invariably contain the chosen sets of the former. Our version of Coherence says that corresponding to each alternative and data profile there is an asset, priced at the expected rank score of the alternative at the data profile (i.e. cardinality of the set of alternatives plus one minus expected rank), such that some portfolio of assets in which no un-chosen alternative at the data profile is bought yields a "sure-loss" (i.e. a loss in every state of nature). A portfolio of assets that yields a "sure-loss" is also known as a Dutch book. Thus, Coherence requires the non-existence of a Dutch book in an asset market where each asset corresponds to a combination of an alternative and a data profile priced at the expected rank score of the alternative at the data profile and where no asset corresponding to an un-chosen alternative at the data profile, can be bought.

2 The model and some examples of extended choice correspondences

The following framework is a fairly close adaptation of the ones available in [2]and section 2.2 of [3]. Consider a decision maker (DM) faced with the problem of choosing one or more alternatives from a non-empty finite set of alternatives X. Let $\Psi(X)$ denote the set of all non-empty subsets of X. For a positive integer $n \geq 2$, let $N = \{1, 2, \ldots, n\}$ denote the set of states of nature. The satisfaction from the chosen alternative is realized only after the state of nature reveals itself.

A **preference relation/weak ranking** on X is a reflexive, complete/ connected/ total and transitive binary relation on X. Generally a preference relation is denoted by R with P and I denoting its asymmetric and symmetric parts respectively. If for $x, y \in X$, it is the case that $(x, y) \in R$, then we shall denote it by xRy and say that x is **at least as good as** y. Similarly xPy is interpreted as x is strictly preferred to y, and xIy is interpreted as there is indifference between x and y. Given a binary relation R on X, a non-empty subset Y of X and an alternative x in Y the rank of (alternative) x at/by R among Y denoted $rk^Y(x, R)$ = cardinality of $y \in Y|yPx) + 1$. Given $x, y \in Y$ where Y is a non-empty subset of X and a binary relation R on X, x is said to be **ranked higher (lower) than** y at/by R **among** Y if $rk^Y(x, R) < (>) rk^Y(y, R)$.

Given a binary relation R on X and an alternative x in X, $rkX(x, R)$ is (for the sake of simplicity) denoted by $rk(x, R)$. Let $\mathcal{W}$ denote the set of all preference relations on X. We use $\mathcal{W}$ to denote the set of preference relations, since a preference relation is what is generally known as a weak order. The set of all anti-symmetric preference relations i.e. linear orders is denoted by $\mathcal{L}$. An anti-symmetric preference relation is referred to as a **strict preference relation/strict ranking** on X.

A **preference profile** denoted R_N is a function from N to $\mathcal{W}$. R_N is represented as the array $\langle R_i | i \in N \rangle$, where R_i is the preference relation/weak ranking of the decision maker in state of nature i. The set of all preference profiles is denoted $\mathcal{W}^N$. The set of all functions from N to $\mathcal{L}$ is denoted $\mathcal{L}^N$. Clearly $\mathcal{L}^N$ is a proper subset of $\mathcal{W}^N$.

A **domain** is any non-empty subset of $\mathcal{W}^N$. We will denote a domain by $\mathcal{R}$.

In what follows we shall denote vectors in $\mathbb{R}^N$ by alphabets such as a, b, c, d etc. and when there is need for us to be explicit about (say for instance) vector a, we will write it as $(a_1, \ldots, a_n)$. $\mathbb{R}^N_+$ denotes the set $\{a \in \mathbb{R}^N | a_i \geq 0 \text{ for all } i \in N\}$.

The DM's beliefs about the possibility of the various states of nature being realized is summarized by a probability distribution, i.e. $p \in \mathbb{R}^N_+$ such that $\sum_{i=1}^n p_i = 1$. Let P^N denote the set of all probability distributions on N.

Given $p \in P^N$, the most likely states of p, denoted $ML(p) = \{j \in N | p_j \geq p_i \; \forall\, i \in N\}$ and the support of p, denoted support $(p) = \{j \in N | p_j > 0\}$. Note that support $(p) = ML(p)$ implies all states of nature in support(p) are equi-probable.

Any pair $(R_N, p) \in \mathcal{W}^N \times P^N$ is said to be a data profile. A feasible set of probability distributions (about the future states of nature being realized) is a non-empty subset of P^N denoted Q. For whatever reasons, the DM's beliefs are restricted to belong to Q, and it is often (not invariably) reasonable and/or required that the probability distribution that assigns equal weight (i.e. $1/n$) to all states of nature is feasible, i.e. belongs to Q. An **extended choice correspondence** (ECC) on (the domain of the ECC) $\mathcal{R} \times Q$ is a function f from $\mathcal{R} \times Q$ to $\Psi(X)$ such that for each $(R_N, p) \in \mathcal{R} \times Q$, the decision maker chooses an alternative from $f(R_N, p)$. Note that the "domain of the ECC" $\mathcal{R} \times Q$, is different from the domain, which is $\mathcal{R}$.

3 Decision making under uncertainty with state dependent preferences: An application of extended choice correspondences

Before proceeding with the analytics we show in this section the huge width and scope of our framework to the extent that investigations based on the model of decision making under uncertainty with state dependent preferences developed in section 1.2 of [6], can be accommodated within our framework. For each state of nature $i \in N$, let $\mathcal{C}(i)$ be a non-empty finite set of **prizes** (e.g. monetary gains and losses, bar of chocolate, an ice cream cone, getting fired from a job etc.) and let $\mathcal{C} = U_{i\in N}\mathcal{C}(i)$

A **consequence/uncertain prospects** is a probability distribution over C. Since C is finite such a probability distribution must have finite support. For each $i \in N$, let $L(\mathcal{C}(i))$ denote the set of all consequences with prizes in $\mathcal{C}(i)$ and let $L = U_{i\in N}L(\mathcal{C}(i))$.

An **alternative/act** is a function $x : N \to L$ such that for all $i \in N, x(i) \in L(\mathcal{C}(i))$.

For instance an act could be a state-contingent investment decision which in each state of nature is an investment plan yielding gains and losses with a known probability distribution. The above is the gist of the formal model in section 1.2 of [6]. The interesting question is: how may these preference profiles arise? One obvious way is that the decision maker has preferences over consequences/uncertain prospects, which may be generated by the expected utility of the consequences, so that one consequence is preferred to another if and only if the first has higher expected utility than the second. Given alternatives $x, y \in X$, we may define R_i as follows: xR_iy if and only if the expected utility of $x(i)$ is no less than the expected utility of y.

It is important to notice that the utility function for gains and losses may be state-dependent and almost anything so long as it is increasing for gains, non-increasing for losses and zero at zero, so that we are not restricted to a single preference profile but have available a large class of such profiles.

Thus for each $i \in N$, let $\mathcal{U}_i = \{u : \mathcal{C}(i) \to \mathbb{R} | u$ satisfies "certain properties specific to state of nature i"$\}$, where we assume "certain properties specific to state of nature i" are such that $U_i \neq \phi$ and let $R_i = \{R | R$ is a binary relation on X satisfying the following property: there exists $u \in U_i$ such that for all $x, y \in X$, xRy if and only if $\mathbb{E}(x(i)) \geq \mathbb{E}u(y(i))\}$.

Let $\mathcal{R} = \prod_{i \in N} \mathcal{R}_i$. Clearly $\mathcal{R}$ is a non-empty subset of $\mathcal{W}^N$ and hence a domain worth considering.

Coupled with this is uncertainty over future states of nature. Hence, there is uncertainty at two levels- first in the state of nature that will unfurl itself at a future date, and second the one inherent in the consequences which are uncertain prospects.

Given a feasible set of probability distributions over future states of nature- Q- an extended choice correspondence f assigns to each pair $(R_N, p) \in \mathcal{R} \times Q$, a non-empty set of acts $f(R_N, p)$ (for instance a non-empty set of state-dependent investment opportunities- something like a non-empty set of options) from which the decision maker would be choosing one.

To illustrate the above ideas consider the following example. Consider a decision maker who is suffering from symptoms of flu, which may or may not be early symptoms of Covid 19. He can either undergo an expensive test to find out whether he is covid positive or covid negative. However, the test is not 100% accurate. Based on available data there is evidence that for approximately 30% of non-infected cases the test result turns out to be positive, and for approximately 40% infected cases the test result turns out to be negative. In case of a +ve test result the DM would undergo compulsory and quite costly hospitalization with a very high chance of recovery (particularly in the absence of co-morbidities). In case of a –ve test result the DM would be immediately discharged, which if the diagnosis turns out to be incorrect, would with very high probability lead to death. What should the DM do?

In this example there are two future (unknown states of nature)-Infected (1) or Not Infected (2)- and two alternatives – take the test (x) and not take the test (y). Suppose, p_1 is the prior probability that the decision maker assigns to being infected with covid. In this case, $\mathcal{C}(1) = \mathcal{C}(2) = \{+ve, -ve\}$.

Then $x(1)$ is the uncertain prospect of being diagnosed covid +ve with probability 7/10 and covid –ve with probability 3/10, the implication of which is recovering with probability close to 7/10 and not recovering with probability close to 3/10. In either case there is expenditure, which is higher if the test result is +ve. $y(1)$ is the uncertain prospect of not recovering with probability close to one. Hence, it can be safely concluded that in state of nature 1, x is definitely preferred to y by a reasonably well-off DM. Similarly, $x(2)$ is the uncertain prospect of being diagnosed covid +ve with probability 4/10 and covid –ve with probability 6/10, the implication of which is recovering with probability close to one but after incurring expenditure which is higher if the test result is +ve. y(2) is the uncertain prospect of not recovering with probability close to one and without incurring much expenditure.

In state of nature 2, the DM would definitely prefer y to x. Obviously, a reasonably well-off DM will take a decision depending on what the value of p_1 is. If p_1 is very close to zero, he may skip the test; if not he may decide to take the test. The DM's perception of p_1 will depend on whether before falling ill he/she has come in close contact of anyone who has tested covid +ve.

4 Examples of extended choice correspondences

Recall that given any non-empty subset Y of X, $x \in Y$ and binary relation R on X, the rank of x (in Y) with respect to R denoted $rk^Y(x, R)$ is equal to the one plus the cardinality of $\{y \in Y | yPx\}$.

Example 1. **(Probabilistic plurality rule or the Most Likely Best Alternative Rule):**

Let $R \subset \mathcal{W}^N$. Given $(R_N, p) \in \mathcal{R} \times Q$, let the **probabilistic plurality score** of x at (R_N, p) denoted

$$\text{PPlur-score}((R_N, p), x) = \begin{cases} \sum\limits_{i \in N | rk(x, R_i) = 1} p_i, & \text{if } \{i \in N | rk(x, R_i) = 1\} \neq \phi \\ 0, & \text{if } \{i \in N | rk(x, R_i) = 1\} = \phi \end{cases}$$

The **Probabilistic plurality rule** is the ECC on $\mathcal{R} \times Q$ denoted $PPlural$ such that for all $(R_N, p) \in \mathcal{R} \times Q$,

$$PPlural(R_N, p) = \{x \in X | \text{PPlur-score}((R_N, p), x) \geq \text{PPlur-score}((R_N, p), y)\} \quad \forall\, y \in X$$

The probabilistic plurality rule selects only those alternatives which have the highest probability of being ranked first.

Example 2. **(Run-off method):**

Let $\mathcal{R} \subset \mathcal{L}^N$. Given (R_N, p), let Y_1 be the set of alternatives with least plurality scores. If $Y_1 = X$, the procedure stops. If $Y_1 \neq X$, let $X_1 = X \backslash Y_1$ and repeat the previous step on $X1$ instead of X with $rk^{X_1}(x, R_i)$ for all $x \in X_1$ and rankings $R_i, i = 1, \ldots, n$. Proceeding thus, we come to a least positive integer

K such that all alternatives in X_K have the same probabilistic plurality scores (which includes the case where X_K is a singleton). The **Probabilistic Run-off rule** is the ECC on $\mathcal{R} \times Q$ denoted PRun-off, such that for all $(R_N, p) \in \mathcal{R} \times Q$, PRun-off$(R_N, p) = X_K$. Given $x \in X$ and any binary relation R on X, we refer to the quantity [1 + cardinality of X - $rk(x, R)$] as the **rank-score** of x at R and denote it by $u(x, R)$.

Example 3. **(Probabilistic Borda or expected rank optimizing rule):**
Let $\mathcal{R} \in \mathcal{W}^N$. Given $(R_N, p) \in \mathcal{R} \times Q$, let the **Probabilistic Borda score** of x at (R_N, p) denoted

$$\text{PBorda-score}((R_N, p), x) = \sum_{i=1}^{n} p_i u(x, R_i)$$

The **Probabilistic Borda rule** is the ECC on $\mathcal{R} \times Q$ denoted PBorda such that for all $(R_N, p) \in \mathcal{R} \times Q$,

$$\text{PBorda}(R_N, p) = \{x \in X | \text{Borda-score}((R_N, p), x) \geq \text{Borda-score}((R_N, p), y)\} \quad \forall y \in X$$

The Probabilistic Borda rule can also be called the **expected rank optimizing rule**, since

$$\text{PBorda}(R_N, p) = \{x \in X | \sum_{i=1}^{n} p_i rk(x, R_i) \, \forall y \in X\}$$

Thus the Probabilistic Borda Rule chooses those alternatives with the maximum expected rank-score at (R_N, p).

For all data profiles (R_N, p) and $x \in X$, let

$$I(x, R_N, p) = k \in N | rk(x, R_k) \geq rk(x, R_h) \, \forall h \in \text{support}(p)$$

i.e. **the set of all worst states of nature for x at data profile** (R_N, p).

Let worstrk$(x, R_N, p) = rk(x, R_i)$ for all $i \in I(x, R_N, p)$. worstrk(x, R_N, p) is said to be the worst rank of x at (R_N, p). Given $(R_N, p) \times \mathcal{R} \times Q$ and $x \in X$, the **probability of the worst rank** of x at (R_N, p) denoted

$$Pr(I(x, R_N, p)) = \sum_{i \in I(x, R_N, p)} p_i$$

Example 4. **(Max-min or Pessimistic rule):** Let $\mathcal{R} \subset \mathcal{L}^N$ and for each $(R_N, p) \in \mathcal{R} \times Q$, let

$$\text{BestWorstrk}(R_N, p) = \{x \in X | \text{worstrk}(x, R_N, p) \leq \text{worstrk}(y, R_N, p) \, \forall y \in X\}$$

The Max-min rule is the ECC on $\mathcal{R} \times Q$ denoted Mm such that for all $(R_N, p) \in \mathcal{R} \times Q$,

$$Mm(R_N, p) = \{x \in \text{BestWorstrk}(R_N, p) | Pr(I(x, R_N, p)) \leq Pr(I(y, R_N, p))$$

for all $y \in$ BestWorstrk$(R_N, p)\}$

i.e. $Mm(R_N, p)$ is the set of alternatives with least total probability of securing the best worst rank at (R_N, p). Research on issues related to Example 4, but in an entirely different framework and from an entirely different perspective is available in [14].

Example 5. **(Max-max or Optimistic rule):** Let $R \in L^N$ and for each $(R_N, p) \in \mathcal{R} \times Q, x \in X$, let bestrk$(x, R_N, p) = rk(x, R_i)$ for $i \in \underset{j \in support(p)}{\text{argmin}} \; rk(x, R_j)$.

For each $(R_N, p) \in \mathcal{R} \times Q$, let

$$\text{BestBest}(R_N, p) = \{x \in X | \text{bestrk}(x, R_N, p) \leq \text{bestrk}(y, R_N, p) \, \forall y \in X\}$$

The Max-max rule is the ECC on $\mathcal{R} \times Q$ denoted MM such that for all $(R_N, p) \in \mathcal{R} \times Q$,

$$MM(RN, p) = \{x \in \text{BestBestrk}(R_N, p) | Pr(I(x, R_N, p)) \geq Pr(I(y, R_N, p)) \\ \forall y \in \text{BestBestrk}(R_N, p)\}$$

i.e. $MM(R_N, p)$ is the set of alternatives with greatest total probability of securing the best "best" rank at (R_N, p).

Note: In the context of decision making uncertainty with pre-specified probabilities with which the states of nature occur, the decision maker does not enjoy the privilege of assigning probabilities to the best and worst states of nature corresponding to an alternative. Thus the Hurwicz pessimism-optimism criteria is not easy to accommodate in our framework.

5 State salient rules

In this section we discuss some decision rules introduced in [7]. An ECC f on a domain $\mathcal{R} \times Q$ is said to be **state-salient for** $p \in Q$ **between a given pair of distinct alternatives at state of nature** i, if regardless of the state-dependent preference profile in $\mathcal{R}$, the alternative in the given pair that is ranked inferior of the two in state of nature i, is not chosen. Thus, if the two alternatives are x and y and if it is the case that (a) if x is ranked above y in state i, then y is not chosen regardless of the rankings in other states of nature, and (b) if y is ranked above x in state i, then x is not chosen regardless of the rankings in other states of nature.

An ECC f on a domain $\mathcal{R} \times Q$ is said to be **state-salient for** $p \in Q$ **at a state of nature** i, if it is state salient for $p \in Q$ between every pair of distinct alternatives at state of nature i.

An ECC f on a domain $\mathcal{R} \times Q$ is said to be a **state-salient rule** (S-SR), if for all $p \in Q$ there exists a state of nature i (possibly depending on p), such that F is state salient for p at state of nature i. An important and realistic example of an S-SR is the maximum likelihood state rule (as opposed to the maximum likelihood alternative rule, discussed in example 1).

An ECC f on a domain $\mathcal{R} \times Q$ is said to be a **maximum likelihood state rule** (MLSR) if for all $p \in Q$ there exists a state of nature i such that $pi \geq pj$ for all $j \in N \backslash \{i\}$ (i.e. $i \subset \underset{j \in N}{\operatorname{argmax}}\{p_j\}$) and further for all $R_N \in \mathcal{R}$ and $x, y \in X$: $xPiy$ implies $y \in F(R_N, p)$. In this case i is said to be **a most likely state of nature** (MLS) for p. The version of MLSR when state-dependent rankings are represented by state-dependent evaluation or utility functions is available on page 74 in chapter 4 of [12]under the name "maximum likelihood criterion". The same is also defined and discussed under a different name – "modal outcome criterion"- on page 429 (section 9.4.2) of [11]. However as correctly observed by several authors, using this criterion in a situation where several states of nature exist, with probability of occurrence nearly or exactly equal to the probability of occurrence of a most likely state of nature, may lead to serious mistakes. A problem with MLSR that ought to be taken note of is that it may depend totally on extremely unlikely events leading to absurd conclusions. Thus suppose there are two alternatives x, y, 99 states of nature with the first state having a probability of 1/50 and the remaining 98 each having a probability of 1/100. Suppose that in the first state of nature we have x preferred to y and in the remaining y is preferred to x. Then MLSR will select x and not y, though there is a 98% chance that y will be preferred to x. An ECC f on a domain $\mathcal{R} \times Q$ is said to satisfy the Weak Dominance Criterion (WDC) if for all $x, y \in X$ and $(R_N, p) \in \mathcal{R} \times Q$: $[xPiy$ for all $i \in N]$ implies $[y \notin F(R_N, p)]$.

This criterion is available and discussed in [7].

The following criterion is one based on a more general property in [2]. An ECC f on a domain $\mathcal{R} \times Q$ is said to satisfy **Independence of Irrelevant Alternatives in the sense of Denicolo** (D-IIA) if for all $R_N, R'_N \in \mathcal{R}, p \in Q$ and $x, y \in X$ with $x \neq y$: $[R_i|x, y = R'_i|x, y$ for all $i \in N, x \in f(R_N, p)$ and $y \notin f(R_N, p)]$ implies $[y \notin f(R'_N, p)]$.

An ECC f on a domain $\mathcal{R}$ is said to be resolute if for all $(R_N, p) \in \mathcal{R} \times Q$, $f(R_N, p)$ is a singleton.

The following result is based on one a theorem due to Denicolo in [2].

Theorem 1. *A resolute ECC on $\mathcal{L}^N$ satisfies WDC and D-IIA if and only if it is any resolute S-SR on $\mathcal{L}^N$.*

6 The Probabilistic Condorcet and Probabilistic Majority criterion

In this section we discuss two important properties of ECC's.

An ECC f on a domain $\mathcal{R} \times Q$ is said to satisfy Probabilistic Condorcet criterion if for all $(R_N, p) \in \mathcal{R} \times Q$ and $x, y \in X$:

$$\left[\sum_{\{i \in N | xP_iy\}} p_i > \sum_{\{i \in N | yP_ix\}} p_i\right] \implies y \notin f(R_N, p)$$

Observation 1 : It is easy to see that neither the plurality rule nor the run-off method satisfies the Probabilistic Condorcet Criterion and may hence considered to be wanting in some respect. To illustrate this, consider $N = 1, 2, 3, X = x, y, x, xP_1yP_1z, zP_2yP_2x$ and yP_3zP_3x.

Let $\epsilon > 0$ be a sufficiently small real number such that $1 > \frac{1}{3} + \epsilon > \frac{1}{3} > \frac{1}{3} - \epsilon > 0$. Let $p_1 = \frac{1}{3} + \epsilon, p_2 = \frac{1}{3}$ and $p_3 = \frac{1}{3} - \epsilon$
.

The winner set for the plurality rule is $\{x\}$ and for the run-off method it is $\{x\}$ if $\epsilon > \frac{1}{6}$, $\{x, z\}$ if $\epsilon = \frac{1}{6}$ and $\{z\}$ if $\epsilon < \frac{1}{6}$.

However, if $\epsilon < \frac{1}{6}$, then by the Probabilistic Condorcet criterion the winner set is $\{y\}$. This result agrees with the Borda rule in this particular situation. An alternative criterion based on one due to [8]is the following. An ECC f on a domain $\mathcal{R} \times Q$ is said to satisfy the Probabilistic Majority criterion if for all $(R_N, p) \in \mathcal{R} \times Q$ and $x \in X$:

$$\left[\sum_{\{i \in N | xP_iy \forall y \in X \setminus \{x\}\}} p_i \geq \frac{1}{2} \text{ and } \sum_{\{i \in N | xR_iy \forall y \in X\}} p_i > \frac{1}{2}\right] \implies [f(R_N, p) = \{x\}].$$

Proposition 1. *If an ECC satisfies Probabilistic Condorcet criterion then it satisfies the Probabilistic Majority criterion. However, the converse is not true.*

Proof. Let f be an ECC on a domain $\mathcal{R} \times Q$ that satisfies Probabilistic Condorcet criterion.

Let $(R_N, p) \in \mathcal{R} \times Q$ and suppose $x \in X$ satisfies

$$\left[\sum_{\{i \in N | xP_iy \forall y \in X \setminus \{x\}\}} p_i \geq 1/2 \text{ and } \sum_{\{i \in N | xR_iy \forall y \in X\}} p_i > 1/2\right].$$

Let $z \in X \setminus \{x\}$.
Since

$$\sum_{\{i \in N | xR_iy \forall y \in X\}} p_i > 1/2,$$

it must be the case that

$$\sum_{\{i \in N | xR_iz\}} p_i > 1/2$$

Thus,

$$\sum_{\{i \in N | zP_ix\}} p_i < \frac{1}{2} \leq \sum_{\{i \in N | xP_iz\}} p_i$$

By Probabilistic Condorcet criterion, $z \in f(R_N, p)$. This being true for all $z \in X \backslash \{x\}$ and since $f(R_N, p) \neq f$, it must be the case that $f(R_N, p) = \{x\}$. Thus f satisfies the Probabilistic Majority criterion.

To show that the converse need not be true, consider the Plurality rule. It is easy to see that the Plurality rule satisfies the Probabilistic Majority criterion. Let X contain at least four alternatives x, y, z, w and let $N = \{1, 2, 3\}$. Suppose that:

1. R_1 ranks x uniquely first, w uniquely second, y uniquely third, z uniquely fourth and then rank the rest (if there be any) in any arbitrary ways (allowing for ties).
2. R_2 ranks z uniquely first, w uniquely second, x uniquely third, y uniquely fourth and then rank the rest (if there be any) in any arbitrary ways (allowing for ties).
3. R_3 ranks y uniquely first, w uniquely second, z uniquely third, x uniquely fourth and then rank the rest (if there be any) in any arbitrary ways (allowing for ties).

Then for any $p \in Q$ with $p_i < 1/2$ for all $i \in N$, Plural (R_N, p) is a non-empty subset of $\{x, y, z\}$ although for all $z' \in X \backslash \{w\}$,

$$\sum_{\{i \in N | w P_i z'\}} p_i > \frac{1}{2} > \sum_{\{i \in N | z' P_i w\}} p_i$$

Thus the Plurality rule violates the Probabilistic Condorcet criterion. Q.E.D.

7 An axiomatic characterization of the Probabilistic Borda Rule

Recall that given $x \in X$ and any binary relation R on X, [1 + cardinality of $X - rk(x, R)$] is the **rank-score** of x at R and denote it by $u(x, R)$. A domain $\mathcal{R}$ is said to satisfy **closure under permutation** if for all $R_N \in \mathcal{R}$, $i, j \in N$ and $R'_N \in \mathcal{R}^N$: $\left[R_k = R'_k \,\forall k \in N \backslash \{i, j\}, R_i = R'_j, R_j = R'_i\right] \implies [R'_N \in \mathcal{R}]$. The properties we invoke for our axiomatic characterization of the Probabilistic Borda Rule are not very unusual and seem plausible in the context of our analysis. However, before we proceed with the axiomatic characterization we need a few more concepts which we introduce here. Given $\mathcal{R} \times Q$ an **asset market associated with** $\mathcal{R} \times Q$ is a market where assets indexed by elements in $\mathcal{R} \times Q \times X$ can be transacted such that the net return from one unit of the asset $(R_N, p, x) \in \mathcal{R} \times Q \times X$ in state of nature i is $p_i u(x, Ri)$ to a buyer and $-p_i u(x, Ri)$ to a seller.

A **portfolio** in an asset market associated with $\mathcal{R} \times Q$ is an array of non-negative real numbers $\langle \beta(R_N, p, x) | \in \mathcal{R} \times Q \times X \rangle$ such that if $\beta(R_N, p, x) \geq 0$ then $\beta(R_N, p, x)$ indicates the number of units of asset (R_N, p, x) bought and if $\beta(R_N, p, x) \leq 0$ then -$\beta(R_N, p, x)$ indicates the number of units of asset (R_N, p, x) sold.

A **Dutch Book or Sure Loss based on** f is a **portfolio of assets** $\langle \beta(R_N, p, x)| \in \mathcal{R} \times Q \times X\rangle$ transacted in the market associated with f such that:

1. $\beta(R_N, p, x) \leq 0$ for all $x \in X \backslash f(R_N, p)$,
2. For all $i \in N$:

$$\sum_{\{(R_N,p,x):x\in f(R_N,p)\}} \beta(u, p, x)p_i u(x, R_i) + \sum_{\{(R_N,p,x):x\in X\backslash f(R_N,p)\}} \beta(R_N, p, x)p_i u(x, R_i) < 0$$

The following definition is related to the main concept introduced by Professor Bruno De Finetti in his seminal contribution to the philosophy of probability theory dated 1931.

An ECC f on $\mathcal{R} \times Q$ is said to satisfy **Coherence** (or be **Coherent**) if there does not exist a Dutch Book based on f.

Trades such as ones implied by a portfolio $\langle \beta(R_N, p, x)| \in \mathcal{R} \times Q \times X\rangle$ transacted in the market associated with f such that $\beta(R_N, p, x) \leq 0$ for all $x \in X\backslash f(R_N, p)$, should be voluntary for the decision maker, since the assets bought correspond to alternatives from among those that are chosen and all un-chosen alternatives correspond to assets that are sold, each such asset being priced at their state dependent expected returns given by the rank-score of the corresponding alternative $u(.,.)$ and nothing could be worse than a sure loss from such a portfolio regardless of the state. Coherence is the minimal requirement that such gross inconsistencies should not occur. We require a slightly stronger property for the axiomatic characterization of the Probabilistic Borda Rule. An ECC f on $\mathcal{R} \times Q$ is said to satisfy **Maximal Coherence** (or be **Maximally Coherent**) if it is Coherent and there does not exist any other Coherent ECC g on $\mathcal{R} \times Q$ such that $f(R_N, p) \subset g(R_N, p)$ for all $(R_N, p) \in \mathcal{R} \times Q$.

The following property is very intuitively appealing- the names of the states of nature do not matter.

An ECC f on $\mathcal{R}\times Q$ is said to satisfy **Anonymity** if $(R_N, p), (R'_N, q) \in \mathcal{R}\times Q$ and $i, j \in N$:

$[R_k = R'_k, p_k = q_k \forall k \in N\backslash\{i, j\}, R_i = R'_j, R_j = R'_i, p_i = q_j, p_j = q_i] \implies [f(R'_N, q) = f(R_N, p)]$.

Now we present our main result of this section.

Proposition 2. *Let $\mathcal{R} \subset \mathcal{L}^N$ satisfy closure under permutation and $Q = P^N \cap \mathbb{R}^N_{++}$ An ECC f on $\mathcal{R}\times Q$ is the Probabilistic Borda Rule if and only if f satisfies Maximal Coherence and Anonymity.*

Proof. It is easy to see from the definition of the Probabilistic Borda Rule that it satisfies the desired property. Suppose f on $\mathcal{R} \times Q$ satisfies Maximal Coherence and Anonymity. We will show first show that there exists $p \in P_N$ such that for all $(R_N, p) \in \mathcal{R} \times Q$, $f(R_N, p) = \underset{x\in X}{\text{argmax}} \sum_{i=1}^{n} \pi_i p_i u(x, R_i)$, then appeal to Anonymity to show that $\pi_i = \pi_j$ for all $i, j \in N$.

Suppose there does not exist $\pi \in \mathbb{R}^n_+$ satisfying $\sum_{i=1}^{n} \pi_i = 1$ such that for all $(R_N, p) \in \mathcal{R} \times Q, x \in f(R_N, p)$ and $y \in X \backslash \{x\}$ it is the case that

$$\sum_{i=1}^{n} \pi_i p_i [u(x, R_i) - u(y, R_i)] \geq 0.$$

Thus, the system of linear equations

$$\sum_{i=1}^{n} \pi_i p_i [u(x, R_i) - u(y, R_i)] - z(R_N, p, x, y) = 0, x \in f(R_N, p), y \in X \backslash \{x\}, (R_N, p) \in \mathcal{R} \times Q,$$

$$\sum_{i=1}^{n} \pi_i = 1$$

does not have any non-negative solution in π_i for $i = 1, \ldots, n$ and $z(R_N, p, x, y)$ where $x \in f(R_N, p)$, $y \in X \backslash \{x\}, (R_N, p) \in \mathcal{R} \times Q$.

By Farkas' Lemma there exist real numbers $\lambda(R_N, p, x, y)$ for each (R_N, p, x, y) where $x \in f(R_N, p), y \in X \backslash \{x\}, (R_N, p) \in \mathcal{R} \times Q$ and a real number λ such that

$$\sum_{\{(R_N,p,x,y) | x \in f(R_N,p), y \in X \backslash \{x\}\}} \lambda(R_N, p, x, y) p_i [u(x, R_i) - u(y, R_i)] + \lambda \leq 0$$

$$\text{for } i = 1, \ldots, n$$

$$-\lambda(R_N, p, x, y) \leq 0, \forall (R_N, p, x, y) \text{ where } x \in f(R_N, p), y \in X \backslash \{x\}, (R_N, p) \in \mathcal{R} \times Q,$$

and $\lambda > 0$.

Thus, there exist real numbers $\lambda(R_N, p, x, y)$ for each (R_N, p, x, y) where $x \in f(R_N, p), y \in X \backslash \{x\}, (R_N, p) \in \mathcal{R} \times Q$, and a real number λ such that

$$\sum_{\{(R_N,p,x,y) | x \in f(R_N,p), y \in X \backslash f(R_N,p)\}} \lambda(R_N, p, x, y) p_i [u(x, R_i) - u(y, R_i)] \leq -\lambda < 0$$

$$\text{for } i = 1, \ldots, n$$

$$\lambda(R_N, p, x, y) \geq 0, \forall (R_N, p, x, y), \text{ where } x \in f(R_N, p), y \in X \backslash \{x\}, (R_N, p) \in \mathcal{R} \times Q$$

For each $(R_N, p, x) \in \mathcal{R} \times Q \times X$, let

$$\beta(R_N, p, x) = \sum_{y \in X \backslash \{x\}} \lambda(R_N, p, x, y) - \sum_{y \in f(R_N,p) \backslash \{x\}} \lambda(R_N, p, y, x),$$

if $x \in f(R_N, p)$, and

$$\beta(R_N, p, x) = - \sum_{y \in X \backslash f(R_N,p)} \lambda(R_N, p, y, x),$$

if $x \in X \backslash f(R_N, p)$.

Clearly $\beta(R_N, p, x) \leq 0$ for all $(R_N, p, x) \in \mathcal{R} \times Q \times X$ with $x \in X \backslash f(R_N, p)$. Further, $\beta(R_N, p, x) \geq 0$ implies $x \in f(R_N, p)$, though the converse need not be true.

Also, for $i = 1, \ldots, n$

$$\sum_{\{(R_N,p,x):x\in f(R_N,p)\}} \beta(R_N, p, x) p_i u(x, R_i) + \sum_{\{(R_N,p,x):x\in X\backslash f(R_N,p)\}} \beta(R_N, p, x) p_i u(x, R_i) < 0,$$

Thus, the portfolio $\langle \beta(R_N, p, x) | \in \mathcal{R} \times Q \times X \rangle$ is a Dutch Book based on f, contradicting the assumption that f satisfies Coherence.

Thus, there exists $\pi \in P^N$ such that for all $(R_N, p) \in \mathcal{R} \times Q$, $x \in f(R_N, p)$ and $y \in X \backslash \{x\}$, it is the case that

$$\sum_{i=1}^{n} \pi_i p_i [u(x, R_i) - u(y, R_i)] \geq 0$$

Thus, there exists $\pi \in P^N$ such that for all $(R_N, p) \in \mathcal{R} \times Q$,

$$f(R_N, p) \subset \underset{x \in X}{\operatorname{argmax}} \sum_{i=1}^{n} \pi_i p_i u(x, R_i)$$

Let g be the ECC on $\mathcal{R} \times Q$ such that for all $(R_N, p) \in \mathcal{R} \times Q$,

$$g(R_N, p) = \underset{x \in X}{\operatorname{argmax}} \sum_{i=1}^{n} \pi_i p_i u(x, R_i)$$

By Farkas' Lemma, g is a Coherent ECC such that for all $(R_N, p) \in \mathcal{R} \times Q$,

$$f(R_N, p) \subset g(R_N, p)$$

Since f satisfies Maximal Coherence on $\mathcal{R} \times Q$, it must be the case that for all $(R_N, p) \in \mathcal{R} \times Q$,

$$f(R_N, p) = g(R_N, p) = \underset{x \in X}{\operatorname{argmax}} \sum_{i=1}^{n} \pi_i p_i u(x, R_i)$$

Suppose there exists $i, j \in N$ such that $\pi_i \neq \pi_j$ and

$$x \in f(R_N, p) = \underset{z \in X}{\operatorname{argmax}} \sum_{k=1}^{n} \pi_k p_k u(z, R_k)$$

Let $y \in X$. Thus,

$$\sum_{k=1}^{n} \pi_k p_k u(x,k) \geq \sum_{i=1}^{n} \pi_k p_k u(y,k)$$

Let $R'_N \in \mathcal{R}$ and $q \in Q$:
$[R'_{k+1} = R_k \, \forall k = 1, \ldots, n-1, R'_1 = R_n, q_{k+1} = p_k \, \forall k = 1, \ldots, n-1, q_1 = p_n]$.

By repeated application of Anonymity we get $f(R'_N, q) = f(R_N, p)$. Thus, $x \in f(R'_N, q)$ and so

$$\sum_{k=1}^{n} \pi_k q_k u(x, R'_k) \geq \sum_{i=1}^{n} \pi_k q_k u(y, R'_k)$$

Thus,

$$x \in \operatorname*{argmax}_{z \in X} \sum_{k=1}^{n} \pi_k p_k u(z, R'_k)$$

For $j \in \{1, \ldots, n\}$, let $i(j,1) = j$ and for $k \in \{2, \ldots, n\}$, let

$$i(j,k) = \begin{cases} j + k - 1, & \text{if } j + k – 1 \leq n \\ j + k - 1 - n, & \text{if } j + k – 1 > n \end{cases}$$

By repeating this argument finitely often we get that for all $j \in \{1, \ldots, n\}$,

$$\sum_{k=1}^{n} \pi_{i(j,k)} p_k u(x, R_k) \geq \sum_{k=1}^{n} \pi_{i(j,k)} p_k u(y, R_k)$$

Adding all the inequalities we get

$$\sum_{k=1}^{n} \left(\sum_{j=1}^{n} \pi_{i(j,k)} \right) p_k u(x, R_k) \geq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \pi_{i(j,k)} \right) p_k u(y, R_k)$$

But $\sum_{j=1}^{n} \pi_{i(j,k)} = 1$ for all $k \in \{1, \ldots, n\}$
Thus,

$$\sum_{k=1}^{n} p_k u(x, R_k) \geq \sum_{i=1}^{n} p_k u(y, R_k)$$

This being true for all $x \in f(R_N, p)$ and $y \in X$, we get

$$f(R_N, p) = \operatorname*{argmax}_{z \in X} \sum_{k=1}^{n} \pi_k p_k u(z, R_k) \subset \operatorname*{argmax}_{z \in X} \sum_{k=1}^{n} p_k u(z, R_k)$$

Towards a contradiction suppose

$$\operatorname*{argmax}_{z\in X}\sum_{k=1}^{n} p_k u(z,R_k)\setminus \operatorname*{argmax}_{z\in X}\sum_{k=1}^{n}\pi_k p_k u(z,R_k)\neq \phi$$

Let

$$x\in \operatorname*{argmax}_{z\in X}\sum_{k=1}^{n} p_k u(z,R_k)\setminus \operatorname*{argmax}_{z\in X}\sum_{k=1}^{n}\pi_k p_k u(z,R_k)$$

and

$$y\in f(R_N,p)=\operatorname*{argmax}_{z\in X}\sum_{k=1}^{n}\pi_k p_k u(z,R_k)$$

Thus, $x\notin f(R_N,p)$

By repeated application of Anonymity we get, that for all $j\in\{1,\ldots,n\}$,

$$\sum_{k=1}^{n}\pi_{i(j,k)} p_k u(x,R_k)\geq \sum_{k=1}^{n}\pi_{i(j,k)} p_k u(y,R_k)$$

Adding all the inequalities and applying

$$\sum_{j=1}^{n}\pi_{i(j,k)}=1 \text{ for all } k\in\{1,\ldots,n\},$$

we get

$$\sum_{k=1}^{n} p_k u(y,R_k)>\sum_{i=1}^{n} p_k u(x,R_k)$$

contradicting our assumption that $x\in \operatorname*{argmax}_{z\in X}\sum_{k=1}^{n} p_k u(z,R_k)$

Thus,

$$f(R^N,p)=\operatorname*{argmax}_{z\in X}\sum_{k=1}^{n} p_k u(x,R_k)$$

Q.E.D. □

Acknowledgement

Idea behind this paper began with an essay entitled "Reinterpreting the Arrow Sen Framework of Social Choice" (available at https://www.academia.edu/40224623/Re-interpreting_the_Arrow-Sen_Framework_of_Social_Choice) which resulted after much discussion with Prasanta Pattanaik and many of the concepts- both here and there- being provided by him. I am immensely grateful to him for all help rendered along the way. I would also like to thank my friends and colleagues- Sankarsan Basu, Surajit Borkotokey, Subhodip Chakraborty, Sujoy Chakraborty, Utteeyo Dasgupta, Itzhak Gilboa, T Madhavan, Kunal Saha and Soumyen Sikdar- for comments, suggestions for improvement and acting as virtual sounding boards for much that is discussed here as well as in related research. I would also like to put on record my immense gratitude for much useful insights and discussions to Jerzy Michnik, Krystoff Targiel, Tadeusz Trzsaskalik and Janez Zerovnik. Finally, I would like to thank an anonymous referee of this journal for several valuable comments, which I believe has improved the quality of this paper. However, responsibility for residual errors in the paper rests with the author and none other.

References

1. Brams S.J., P.C. Fishburn P.C.: Voting procedures. In: Arrow K.J., Sen A.K., Suzumura K. (eds) Handbook of social choice and welfare 1-4. Elsevier, Amsterdam (2002) 173–236
2. Denicolo, V.: Fixed Agenda Social Choice Theory: Correspondence and Impossibility Theorems for Social Choice Correspondences and Social Decision Functions. Journal of Economic Theory 59-2 (1993) 324-332
3. Endriss, U.: Logic and Social Choice Theory. In A. Gupta and J. van Benthem (eds.), Logic and Philosophy Today. College Publications (2011)
4. Gilboa, I.: Lecture Notes for Introduction to Decision Theory. (unpublished) 2013
5. Heiner, R.A.: Book review of "Decision making under uncertainty: The case of state-dependent preferences : Edi Karni, (Harvard University Press, Cambridge, 1985)". Journal of Economic Behavior & Organization 8-2 (1987) 321-323
6. Karni, E.: Decision Making Under Uncertainty: The Case of State-Dependent Preference. Harvard University Press (1985)
7. Lahiri, S.: State Salient Decision Rules For Choice Under Uncertainty. AIMS International Journal of Management 13-3 (2019) 101-105
8. Lahiri, S.: The No-spoiler Condition for Choice Correspondences. Multiple Criteria Decision Making 14 (2019) 60-74
9. Lahiri, S.: Generalized Sen-Coherence and Existence of Preferred With Probability At Least Half Winners. International Journal of Operations Research 17-3 (2020) 93-99
10. Lahiri, S.: Pattanaik's Axioms and Existence of Preferred With Probability At Least Half Winners. (Forthcoming in) Operations Research and Decisions (2021)
11. Monahan, G.: Management Decision Making: Spreadsheet Modelling, Analysis, and Application. Cambridge University Press (2000)
12. Mc Kenna, C.: Quantitative Methods for Public Decision Making (First Edition). Mc Graw Hill, New York (1980)

13. Pattanaik, P. K.: Sufficient Conditions for the Existence of a Choice Set under Majority Voting. Econometrica 38-1 (1970) 165-170
14. Puppe, C., K. Schlag.: Choice Under Complete Uncertainty When Outcome Spaces Are State-dependent. Theory and Decision 66 (2009) 1-16
15. Resnik, M.D.: Choices: An Introduction to Decision Theory. University of Minnesota Press, Minneapolis (1987)
16. Saari, D.G.: Which is better: the Condorcet or Borda Winner? Social Choice and Welfare 26 (2006) 107-129
17. Sen, A.K.: Collective Choice and Social Welfare. Holden Day, San Francisco (1970)
18. Taylor, Alan D.: Mathematics and Politics: Strategy, Voting, Power and Proof. Springer-Verlag, New York (1995)
19. Van Newenhizen, J.: The Borda Method Is Most Likely to Respect the Condorcet Principle. Economic Theory 2-1 (1992) 69-83

Mathematical Methods in Economics and Finance – m^{2}ef
Vol. 15/16, No. 1, 2020/2021
ISSN print edition: 1971-6419 – ISSN online edition: 1971-3878
Web page: `http://www.unive.it/m2ef/` – E-mail: `m2ef@unive.it`