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# Downside risk in multiperiod tracking error models

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**Abstract**
The recent crisis made it evident that replicating the performance of a benchmark is not a sufficient goal to meet the expectations of usually risk-averse investors. The manager should also consider that the investor are seeking for a downside protection when the benchmark performs poorly and thus they should integrate a form of downside risk control. We propose a multiperiod double tracking error portfolio model which combines these two goals and provide enough flexibility. In particular, the control of the downside risk is carried out through the presence of a floor benchmark with respect to which we can accept different levels of shortfall. The choice of a proper measure for downside risk leads to different problem formulations and investment strategies which can reflect different attitudes towards risk. The proposed model is tested through a set of out-of-sample rolling simulation in different market conditions.

**Keywords**: Tracking error, Downside risk, Dynamic portfolio, Stochastic programming

**JEL Codes:** C61, C63, G11

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# 1 Introduction

A major issue in portfolio management problems is the definition of a proper strategy to manage and control risk. The goal of protection from risk can be broadly defined as the objective to preserve capital and/or to mitigate and minimize losses resulting from exposure to capital markets. These objectives can, of course, be achieved through a proper diversification of the portfolio and the choice of the level of exposure to the equity market. Nevertheless there can still be need for risk protection in the context of more aggressive portfolio strategies to retain, at least partially, the possibility of upside capture while controlling the overall risk of the portfolio.

Current market uncertainties and negative sentiment have negative impact on the possibility of reaching these objectives thus enhancing the role of proper tools and strategies which could help in risk protection, like, for example, dynamic asset allocation hedging strategies. These are hedging strategies available to investors seeking to protect their portfolios from equity market downside risk, they are appealing for investors requiring downside protection but also the maintenance of a reasonable level of upside participation.

The variety and complexity of solutions offered has increased significantly in recent years and are now available also for smaller funds. Examples of portfolio protection strategies include Option-based portfolio insurance strategies (OBPI), Constant Proportion Portfolio Insurance (CPPI) and its evolutions and dynamic hedging.

The combination of a proper diversification through optimal asset allocation and the control of the overall risk are crucial also in active benchmark oriented portfolio strategies. In this framework the relaxation of the tracking error constraints may allow for a better control of the risk profile of the portfolio when capital markets experience turmoil periods.

Allowing for higher tracking error gives a manager a higher flexibility which can be used to reduce downside risk when the markets are falling. However high tracking error will only allow and not encourage a manager to produce lower downside risk, thus proper optimization strategies should be considered in order to avoid the possibility of a consistent relative underperformance with respect to the benchmark.

With reference to these goals the following questions arise: are traditional measures of risk a proper tool? How can we combine downside protection with tracking error and benchmarking? And in particular, how downside protection, tracking error and benchmarking interact? To this aim empirical evidence from the market shows that standard deviation and tracking error do not adequately measure the likelihood of left-tail events and thus do not allow a flexible control on downside risk.

We tackle this problem in the framework of an active benchmark-oriented management of an equity portfolio. Our objective is to provide an optimal built-in portfolio protection which guarantees the possibility to switch protection in-out thus avoiding the main drawbacks of rule-based protection strategies, like for the example the widely used CPPI strategy. Moreover we are interested in analyzing the effects of a ’relaxation’ of the tracking error constraint simultaneously adding other constraints to monitor the behavior of the downside risk.

The structure of the paper is as follows. Section 2 present a brief review and discuss the recent contributions in the literature on tracking error and downside protection. In section 3 we present our model while in section 4 we provide an application to out-of-sample empirical data. Finally section 5 concludes.

# 2 Literature review and model formulation

Very few contributions address the dynamic portfolio management problem when both a downside risk control and a tracking objectives are present. In particular we do not consider the point of view of an investor who wants to maximize the utility of his wealth along the planning horizon or at the end of the investment period. We consider the point of view of a manger of a fund, thus representing a collection of investors, who is responsible for the management of a portfolio which offers participation in the market movements, i.e. upside capture the risky portfolio returns, but also a protection for downward movements in the market in the form of setting a floor to the acceptable level of losses.

Maximizing the upside capture increases the total riskiness of the portfolio and this can result in a violation of the desired level of wealth. On the other side a low risk profile on the investment choices assures the achievement of the minimum return requirement, if properly designed, but leaves no opportunity for upside capture. Pursuing both goals, even if they are conflicting, is necessary in order to attract potential investors who express an interest in high stock market returns, but are not risk-seeking enough to fully accept the volatility of this investment and require a cushion. Managing downside risk is thus a crucial part of active portfolio management.

Different contributions in the literature tackled the problem of optimal portfolio choices with the presence of a downside protection both in continuous and discrete time, also from the point of view of portfolio insurance strategies, both for an European type guarantee and for an American type guarantee, see for example Deelstra et al. (2003) and (2004). If we fix a minimal guaranteed return, without any requirement on the upside capture we obtain a problem which fits the portfolio insurance framework, see, for example, Basak (1995), Brennan and Solanki (1981), Jensen and Sorensen (2001), Sorensen (1999). Consiglio et al. (2006) discuss the problem of asset and liability management for UK insurance products with guarantees. The minimum guarantee is treated as a constraints and the fund manager maximizes the Certainty Equivalent Excess Return on Equity.

A first possibility is to divide the investment decision into two steps. In the first the investor chooses the allocation strategy without taking care of the guarantee or the level of protection, while in the second step he applies a dynamic insurance strategy (see, for example, El Karoui et al., 2005). This is also the way the hedging through derivatives can be applied, introducing overlays to insure an existing portfolio. The cost and effectiveness of such strategies depend on the existence of a proper derivative instrument available in the market to hedge the underlying portfolio. In particular, the level of hedge depends on the correlation between the managed portfolio and the underlying of the derivative instrument.

A further issue is related to the timing of the protection since the hedge is linked with the maturity of the derivative and once it is activated, i.e. the derivative is bought, it is not possible to switch it off or suspend it, thus, the cost is difficult to control.

In this contribution we consider the problem of formulating and solving an optimal allocation problem including downside risk control requirements and participation in the returns generated from the risky portfolio. These goals can be achieved considering them as constraints or including them in the objective function. We choose the second approach in the context of dynamic tracking error problems and in more detail we are interested in analyzing the role of asymmetric tracking error measures.

For the use of asymmetric tracking error measures in a static framework see Franks (1992), Konno and Yamazaki (1991), Rudolf et al. (1999). While for references for risk management in presence of benchmarking in a static case see Basak and Shapiro (2003); Alexander and Baptista (2006) analyze the effect of a drawdown constraint on the optimality of portfolios in a static framework.

We are interested in considering dynamic tracking error problems with a stochastic benchmark, see, for example, Barro and Canestrelli (2009), Boyle and Tian (2007), Browne (1999), Dempster and Thompson (2002), Gaivoronski et al.(2005).

Our aim is to build a protection which is optimally chosen and integrated with the management of the portfolio, this would allow to optimize the cost of the protection, i.e. minimizing it, obtaining joint optimality of the portfolio composition and protection. The cost for obtaining downside protection consists both in the direct cost related to the increasing complexity in the management of the portfolio but also in the indirect cost of sacrificing a part of the potential upside capture.

To jointly model these goals we work in the stochastic programming framework since it proved to be flexible enough to deal with many different issues which arise in the formulation and solution of these problems and formulate the two goals as a double tracking error problem.

Different risk measure for downside risk control have been proposed in the literature, for example, Chekhlov et al. (2005). Many contributions consider the use of left-tail risk measures like VaR and CVaR (see, for example, Rockafellar and Uryasev, 2000 and Alexander and Baptista, 2004, Jansen et al., 2000 and Michalowski and Ogryczak, 2001). The choice of a proper measure for downside risk leads to different problem formulations and investment strategies which can reflect different attitudes towards risk (see, for example, Krokhmal et al., 2011).

All these approaches present interesting features and allow to control different aspects of the exposure to market risk. Nevertheless we are interested in combining the downside protection with a benchmarking issue and thus we have to deal with the presence of a tracking error component in the problem.

We choose to express the downside risk control through a level of acceptable losses since, in our opinion, this is quite intuitive for the investor who can easily visualize the result of the investment in case of bad events and this provide the perception of a better control on the overall level of riskiness of the investment. Moreover the introduction of a floor on the acceptable level of losses represent a threshold which itself can be treated as a reference level in the framework of benchmarking.

# 3 Stochastic programming problem formulation

As we pointed out in previous sections, the upside capture and downside risk control represent two conflicting goals for our dynamic portfolio management problem. Indeed, maximizing the upside capture increases the total risk of the portfolio and this can be balanced by the introduction of a second goal, i.e. the minimization of the shortfall with respect to the minimum wealth level. To deal with these tasks simultaneously we propose to work in the multistage stochastic programming framework.

To describe the uncertainty in the context of a multiperiod stochastic programming problem we use a scenario tree. A set of scenarios is a collection of paths from $t = 0$ to $T$, with probabilities $\pi_{k_t}$ associated to each node $k_t$ in the path; according to the information structure assumed this collection can be represented as a scenario tree where the current state corresponds to the root of the tree and each scenario is represented as a path from the origin to a leaf of the tree.

We denote with $\phi_{k_t}(y_{k_t}, x_{k_t})$ a proper tracking error measure which accounts for the distance between the managed portfolio, $y_{k_t}$, and the risky benchmark, $x_{k_t}$, at time $t$ in node $k_t$. Moreover we denote with $\psi_{k_t}(y_{k_t}, z_t)$ a distance measure between the risky portfolio and threshold level of wealth $z_t$, which is not depend on the node $k_t$. A proper choice of $\phi_t$ and $\psi_t$ allows us to define different tracking error problems. The tracking error measures are indexed along the planning horizon and the distances between the managed portfolio and the reference benchmarks are monitored at each trading date $t$. Dempster et al. (2004) propose a model in which the monitoring of the shortfall is more frequent than the trading dates.
The resulting dynamic double tracking error problem in its arborescent form is

$$\min_{y_{k_t}} \quad \sum_{t=0}^{T} \left[ \sum_{k_t=K_{t-1}+1}^{K_t} \pi_{k_t} \phi_{k_t}(y_{k_t},\, x_{k_t}) + \sum_{k_t=K_{t-1}+1}^{K_t} \pi_{k_t} \psi_{k_t}(y_{k_t},\, z_t) \right] \tag{1}$$

$$y_{k_t} = l_{k_t} + \sum_{i=1}^{n} q_{i\,k_t} \tag{2}$$

$$q_{i\,k_t} = (1 + r_{i\,k_t}) \left[ q_{i\,f(k_t)} + a_{i\,f(k_t)} - v_{i\,f(k_t)} \right] \tag{3}$$

$$l_{k_t} = (1 + r_{l\,k_t}) \left[ l_{f(k_t)} - \sum_{i=1}^{n} \kappa^+ a_{if(k_t)} + \sum_{i=1}^{n} \kappa^- v_{if(k_t)} \right] \tag{4}$$

$$a_{i\,k_t} \geq 0 \quad v_{i\,k_t} \geq 0 \tag{5}$$

$$q_{i\,k_t} \geq 0 \quad l_{k_t} \geq 0 \tag{6}$$

$$q_{i\,0} = \bar{q}_i \quad l_0 = \bar{l} \tag{7}$$

$$i = 1, \ldots, n \quad k_t = K_{t-1} + 1, \ldots, K_t \quad t = 1, \ldots, T$$

where $q_{i\,k_t}$, $i = 1, \ldots, n$, denotes the position in the $i-$th stock; $l_{k_t}$ denotes the amount invested in the liquidity component; $r_{k_t} = (r_{1\,k_t}, \ldots, r_{n\,k_t})$ is the vector of returns of the risky assets for the period $[t-1, t]$ in node $k_t$ and $r_{l\,k_t}$ is the return on the liquidity component in node $k_t$. The vector of variables $a_{k_t} = (a_{1\,k_t}, \ldots, a_{n\,k_t})$ and $v_{k_t} = (v_{1\,k_t}, \ldots, v_{n\,k_t})$ denote the value of each asset purchased and sold at time $t$ in node $k_t$, while we account for proportional transaction costs with $\kappa^+$ and $\kappa^-$.

Equation (2) represents the portfolio composition in node $k_t$; equations (3)-(4) describe the dynamics of the amounts of stocks and liquidity component in the portfolio moving from the ancestor node $f(k_t)$, at time $t-1$, to the descendent nodes $k_t$, at time $t$, with $K_0 = 0$. Equations (7) give the initial endowments for stocks and liquidity component.

We need to specify the value of the benchmark and the value of the threshold on the level of wealth at each time and for each node. The stochastic benchmark $x_{k_t}$ and the prices of the risky assets in the portfolio must be simulated according to given stochastic processes in order to build the corresponding scenario trees.

# 4 Empirical analysis and computational experiments

## 4.1 Model specification

Different choices of tracking error measures are possible and different trade-offs between the goals on the minimum threshold side and on the enhanced tracking error side, for the risky benchmark, are possible, too. Among different possible models, in this contribution we investigate the absolute deviation as measure of tracking error. We test different formulations of the model considering different trade-offs between the tracking goal for the risky benchmark and the risk management side of the problem. We are interested in analyzing the behavior of the optimal portfolio with respect to the resulting tracking error and the violation of the minimum level of acceptable losses. In the framework of our multiperiod tracking error problem we measure the distance between the risky benchmark and the managed portfolio at each trading date.

In more detail, we propose to consider the mean absolute deviation to measure the distance between the managed portfolio and risky benchmark. The use of absolute deviation measures presents interesting features. It allows to obtain a linear formulation of the model (see, for example, Konno and Yamazaki, 1991 and Rudolf et al., 1999) and in this framework it is quite straightforward to obtain asymmetric tracking error measures, separating positive and negative deviations from the reference level.

The distance between the managed portfolio and the risky benchmark, $\phi_{k_t}(y_{k_t}, x_{k_t})$ in node $k_t$ is thus measured as a weighted average of positive and negative deviations

$$c^+ [y_{k_t} - x_{k_t}]^+ + c^- [y_{k_t} - x_{k_t}]^- \tag{8}$$

where $[y_{k_t} - x_{k_t}]^+ = \max[y_{k_t} - x_{k_t}, 0] = \theta^+_{k_t}$ and $[y_{k_t} - x_{k_t}]^- = \max[-y_{k_t} + x_{k_t}, 0] = \theta^-_{k_t}$, and $c^+, c^-$ are non negative weights assigned to the positive and negative deviations of the managed portfolio from the risky benchmark.

With respect to the threshold level for losses we again propose to consider a weighted average of positive and negative deviations as follows

$$d^+ [y_{k_t} - z_t]^+ + d^- [y_{k_t} - z_t]^- \tag{9}$$

where $[y_{k_t} - z_t]^+ = \max[y_{k_t} - z_t, 0] = \gamma^+_{k_t}$ and $[y_{k_t} - z_t]^- = \max[-y_{k_t} + z_t, 0] = \gamma^-_{k_t}$, and $d^+, d^-$ are non negative weights for positive and negative deviations from the threshold level of wealth, respectively.

The resulting linear stochastic programming problem is

$$\min \quad \sum_{t=0}^{T} \left[ \sum_{k_t=K_{t-1}+1}^{K_t} \pi_{k_t} \left( c^+ \theta^+_{k_t} + c^- \theta^-_{k_t} \right) + \sum_{k_t=K_{t-1}+1}^{K_t} \pi_{k_t} \left( d^+ \gamma^+_{k_t} + d^- \gamma^-_{k_t} \right) \right] \quad (10)$$

$$\theta^+_{k_t} - \theta^-_{k_t} = y_{k_t} - x_{k_t} \quad (11)$$

$$\gamma^+_{k_t} - \gamma^-_{k_t} = y_{k_t} - z_t \quad (12)$$

$$y_{k_t} = l_{k_t} + \sum_{i=1}^{n} q_{i\,k_t} \quad (13)$$

$$q_{i\,k_t} = (1 + r_{i\,k_t}) \left[ q_{i\,f(k_t)} + a_{i\,f(k_t)} - v_{i\,f(k_t)} \right] \quad (14)$$

$$l_{k_t} = (1 + r_{l\,k_t}) \left[ l_{f(k_t)} - \sum_{i=1}^{n} (\kappa^+) a_{if(k_t)} + \sum_{i=1}^{n} (\kappa^-) v_{if(k_t)} \right] \quad (15)$$

$$a_{i\,k_t} \geq 0 \quad v_{i\,k_t} \geq 0 \quad (16)$$

$$q_{i\,k_t} \geq 0 \quad l_{k_t} \geq 0 \quad (17)$$

$$\theta^+_{k_t} \geq 0 \quad \theta^-_{k_t} \geq 0 \quad (18)$$

$$\gamma^+_{k_t} \geq 0 \quad \gamma^-_{k_t} \geq 0 \quad (19)$$

$$q_{i\,0} = \bar{q}_i \quad l_0 = \bar{l} \quad (20)$$

$$i = 1, \ldots, n \quad k_t = K_{t-1} + 1, \ldots, K_t \quad t = 1, \ldots, T$$

The proposed model can be considered as a generalization of the tracking error model of Dembo and Rosen (1999), who consider as objective function a weighted average of positive and negative deviations from a benchmark. In our model we propose to consider two different benchmarks and a dynamic tracking problem. The problem can be formalized as a double tracking error problem where we are interested in minimizing the tracking error for the risky benchmark while in the same time we want to minimize the downside distance from a given level of wealth. The choice of asymmetric tracking error measures allows us to properly combine the two goals. In the following subsection we discuss the results for different formulation of the problem.

## 4.2 Empirical analysis

To analyze the proposed model we consider a set of out-of sample experiments. We use a weekly dataset for the MSCI Europe Index from January 6, 1999 to May 16, 2012 and we introduce, as tracking assets, the MSCI sector indexes for the same period. The considered sectors are reported in table 1.

We carry out different experiments to analyze the behavior of the proposed model. In particular we consider a linear formulation of the problem introducing both symmetric and asymmetric tracking error measures. We test the models along different periods to compare the behavior of the protection both in raising and falling markets.

To test the role of the presence of a minimum level of wealth as second benchmark we carried out different experiments and we consider downside deviations from

| SECTOR Indexes | Mean | Variance | Skewness | Kurtosis |
|---|---|---|---|---|
| MSCI EU ENERGY | 0.00099 | 0.00106 | -0.20102 | 2.27296 |
| MSCI EU MATERIALS | 0.00159 | 0.00132 | -0.37807 | 2.68223 |
| MSCI EU INDUSTRIALS | 0.00086 | 0.00106 | -0.40903 | 2.75302 |
| MSCI EU COML SVS/SUP | 0.00029 | 0.00092 | -0.06981 | 2.04283 |
| MSCI EU CONS DISCR | 0.00035 | 0.00112 | -0.23864 | 2.77056 |
| MSCI EU CONS STAPLES | 0.00089 | 0.00043 | -0.14500 | 3.88839 |
| MSCI EU HEALTH CARE | 0.00032 | 0.00061 | 0.34771 | 4.38549 |
| MSCI EU FINANCIALS | -0.00061 | 0.00157 | -0.06120 | 3.41035 |
| MSCI EU IT | -0.00002 | 0.00244 | -0.12961 | 2.01416 |
| MSCI EU T/CM SVS | -0.00040 | 0.00122 | 0.05330 | 2.57846 |
| MSCI EU UTILITIES | -0.00001 | 0.00062 | -0.76024 | 4.03333 |

Table 1: MSCI Sector Indexes and their statistics. Weekly dataset from January 6, 1999 to May 16, 2012.

- the risky benchmark - portfolio A;
- the minimum level of wealth - portfolio B;
- both the risky benchmark and the minimum level of wealth - portfolio AB.

We furthermore generalize the downside deviation model introducing different weights for the downside deviations penalizing more the deviations from the minimum floor with respect to the deviations from the risky benchmark. To denote these cases we use the following notation:

- portfolio AB10, downside deviations from the threshold level of wealth are penalized 10 times more than downside deviations from the risky benchmark (for example, $c^- = 1$ and $d^- = 10$ and $c^+ = d^+ = 0$);
- portfolio AB100, deviations from the threshold level of wealth are penalized 100 times more than the deviations from the risky benchmark (for example, $c^- = 1$ and $d^- = 100$ and $c^+ = d^+ = 0$).

A further set of experiments is related to a pure tracking error problem for the risky benchmark where we separate and penalize differently the upside and downside deviations. In more detail we denoted with

- portfolio C - symmetric tracking error;
- portfolio C10 - downside deviations are penalized 10 times more than upside deviations (for example, $c^+ = 1$ and $c^- = 10$, $d^+ = d^- = 0$);

- portfolio C100 - downside deviations are penalized 100 times more than upside deviations (for example, $c^+ = 1$ and $c^- = 100$, $d^+ = d^- = 0$).

Each experiment has been carried out using an out-of-sample rolling simulation for a 10-week management period. In particular, at each decision stage we generate a two-period scenario tree, we solve the optimization problem and take the first period optimal decision. This decision is then evaluated against the true realized returns and the resulting value for the managed portfolio is the new starting point for the subsequent decision.

The scenario trees are generated using an historical bootstrapping technique and there is no overlapping between the generation period and the simulation period, that is we divide the whole dataset into two part the first is used to generate the event trees while we use the second part to carry out the experiments.

We have considered three different rolling simulations with different starting points to capture different behaviors of the market also with respect to the minimum level of wealth.

For the first simulation period we have considered 10 weeks from January 18, 2012 to March 21, 2012. In this case the market experienced a positive trend and the Index is always above the minimum level.

In the second simulation period, from November 2, 2011 to January 11, 2012, the market still presents a positive result along the whole horizon but with a very huge down movement in the central parte of the management period, even below the threshold level of wealth.

Finally we have considered, as further example, a third simulation period, ranging from May 18, 2011 to July 27, 2011, in this case the market experiences a negative trend with a huge loss at the end of the period.

Tables 2-4 present the tracking results for the different periods. We compute the Root Mean Square Error (RMSE) and the Mean Absolute Percentage Error (MAPE) as tracking error statistics. As reference portfolio, we consider first the risky benchmark ($x_{k_t}$), and then the threshold level of wealth ($z_t$).

In the second case we are not interested in obtaining a good tracking performance, rather we aim at minimizing the downside distance of the managed portfolio from the threshold level of wealth when the portfolio is below the loss level. The $RMSEz_t$ has lower values for the portfolios in which we consider a penalization for downside deviations from the loss level (portfolio B, AB, AB10, AB100) while it is higher when we consider the pure tracking problem for the risky benchmark (portfolio C, C10, C100).

If we compare the results for the $RMSEx_{k_t}$ in tables 2 and 3 we can observe that there is an increase in the tracking error for the risky benchmark in particular for portfolios AB, AB10 and AB100. This result is linked with a decrease in the $RMSEz_t$ and in particular, as we can see from figures 1 and 2, the reduction in the losses during the drop in the market are obtained at the cost of a reduction also in the upside capture when the market raise again.

As we can expect, when we consider the distance from the risky benchmark, portfolio B presents the highest tracking error in all the considered periods, these results are due to the fact that the portfolios are composed minimizing the downside distance from the threshold level of wealth without any control on the tracking performance with respect to the risky benchmark (i.e. $c^+ = c^- = 0$, $d^+ = 0$ and $d^- = 1$).

In the following we consider the behavior of the proposed models with respect to the different market conditions analyzed.

In the first simulation period, see figure 1, we can observe that the risky benchmark is always above the minimum level of wealth (set to 950) and thus if we consider only the downside deviations from the threshold loss we obtain a very poor performance for the portfolio while the models which consider the tracking goal with respect to the risky benchmark ($AB$, $AB10$, $AB100$) result in a better performance. Furthermore an enforcement of penalties for the downside deviations from the risky benchmark, portfolio $C10$ and $C100$ allows to improve the performance with respect to the benchmark.

In the second simulation experiment, see figure 2, the market experiences a abrupt collapse falling even below the threshold. In this situation we can observe the effect of the protection and the portfolios which penalize more for the downside deviations from the threshold are able to limit the losses, even if not to fully avoid them; however, this is obtained at the cost of a reduced upside capture once the market start to raise again.

Finally, in the third experiment, see figure 3, we consider a market with a negative trend and a peak. Again we can observe that penalizing more for the downside deviations from the threshold loss we are able to limit the losses.

Our optimized tracking portfolios are evaluated also against the equally-weighted portfolio where all the available indexes are included with the same proportion. As we can expect, if we compare the tracking performance of the equally weighted portfolio with the case of a pure tracking goal for the risky benchmark, (portfolio C), the equally weighted portfolio performs slightly better but at the cost of including all available indexes whereas the optimized portfolio is generally composed with a smaller subset of assets; moreover no flexibility is allowed to control for risk.

In figure 4 we present the tracking performances for the first simulation period for portfolio $AB$ when we consider an increasing number of scenarios in the event trees, we can observe that the results are quite stable when we consistently increase the number of scenarios from 200 to 600.

A further set of experiments has been carried out to consider the effects on the optimized portfolios of an increase in the level of the threshold loss. To this aim we consider different objective functions and threshold levels on the same set of scenario trees, this allows us to highlight the effect of a change in the loss level without interaction from the generated scenarios. We consider three different objective functions:

- the first model - portfolio $CD$ - considers symmetric deviations from both the

| | RMSE $x_{k_t}$ | RMSE $z_t$ | MAPE $x_{k_t}$ | MAPE $z_t$ |
|---|---|---|---|---|
| portf B | 42.1995 | 48.9331 | 0.0339 | 0.0508 |
| portf A | 3.6247 | 88.9653 | 0.0030 | 0.0914 |
| portf AB | 2.7397 | 83.2859 | 0.0022 | 0.0857 |
| portf AB10 | 3.8526 | 82.1019 | 0.0032 | 0.0846 |
| portf AB100 | 4.0122 | 81.9328 | 0.0033 | 0.0844 |
| portf C | 4.8388 | 90.0927 | 0.0040 | 0.0925 |
| portf C10 | 4.2353 | 89.5974 | 0.0037 | 0.0920 |
| portf C100 | 3.6491 | 88.9760 | 0.0031 | 0.0914 |
| eq.weighted | 4.4122 | 82.8035 | 0.0035 | 0.0848 |

Table 2: Tracking error statistics - Distance measures between tracking portfolios and reference portfolios (risky benchmark, $x_{k_t}$ and threshold level of wealth $z_t$), over 10-week period from January 18, 2012 to March 21, 2012.

risky benchmark and the threshold loss level;

- the second model - portfolio $B$ - considers only negative deviations from the threshold loss level;

- the third model - portfolio $AB$ - considers only negative deviations from both benchmarks.

For each portfolio model we consider three different loss levels (990-970-950) and we run the experiments. We analyze all the simulation periods, but in the first period the risky benchmark is always above the three different levels of losses and the results are not particularly significant, thus, in the following, we choose to present only the case for the second and third simulation periods.

In figures (5)-(7) we present the results for the second simulation period. It is worth noting that when we consider the asymmetric tracking problem (portfolio AB) we obtain an increase in the tracking error and a reduction in the downside risk. The case of portfolio B instead, allow to better control the loss level but leave aside the possibility of upside capture when market start raising again.

Figures (8)-(10) exhibit the behavior of the models in the third simulation period. In this case we have a negative trend in the market with a sudden drop and a partial recover. Portfolio B guarantee the best protection and the lower level of losses while portfolios AB and CD are benchmark dependent and thus bear the cost of the market fall.

The obtained results suggest that the interaction between the two goals in the double tracking error problem is particularly interesting and that the cost of the protection is heavily dependent on the given market conditions.

| | RMSE $x_{k_t}$ | RMSE $z_t$ | MAPE $x_{k_t}$ | MAPE $z_t$ |
|---|---|---|---|---|
| portf B | 37.0811 | 34.8682 | 0.0289 | 0.0343 |
| portf A | 7.6397 | 57.0392 | 0.0065 | 0.0557 |
| portf AB | 26.0536 | 41.5084 | 0.0206 | 0.0397 |
| portf AB10 | 38.7349 | 32.2267 | 0.0301 | 0.0294 |
| portf AB100 | 40.3995 | 31.1690 | 0.0312 | 0.0282 |
| portf C | 7.5103 | 57.2276 | 0.0064 | 0.0559 |
| portf C10 | 8.4493 | 56.3249 | 0.0072 | 0.0551 |
| portf C100 | 8.2229 | 56.5205 | 0.0069 | 0.0552 |
| eq.weighted | 3.6249 | 61.0496 | 0.0029 | 0.0591 |

Table 3: Tracking error statistics - Distance measures between tracking portfolios and reference portfolios (risky benchmark, $x_{k_t}$ and threshold level of wealth $z_t$), over 10-week period from from November 2, 2011 to January 11, 2012.

| | RMSE $x_{k_t}$ | RMSE $z_t$ | MAPE $x_{k_t}$ | MAPE $z_t$ |
|---|---|---|---|---|
| portf B | 13.5912 | 38.0983 | 0.0114 | 0.0393 |
| portf A | 1.9460 | 29.3487 | 0.0018 | 0.0262 |
| portf AB | 2.2631 | 30.5113 | 0.0018 | 0.0286 |
| portf AB10 | 3.8304 | 30.8638 | 0.0031 | 0.0294 |
| portf AB100 | 3.7443 | 30.7144 | 0.0029 | 0.0292 |
| portf C | 2.7150 | 28.9413 | 0.0025 | 0.0255 |
| portf C10 | 2.1514 | 29.2340 | 0.0020 | 0.0260 |
| portf C100 | 1.9587 | 29.3353 | 0.0018 | 0.0262 |
| eq.weighted | 1.4617 | 30.7177 | 0.0012 | 0.0283 |

Table 4: Tracking error statistics - Distance measures between tracking portfolios and reference portfolios (risky benchmark, $x_{k_t}$ and threshold level of wealth $z_t$), over 10-week period from May 18, 2011 to July 27, 2011.

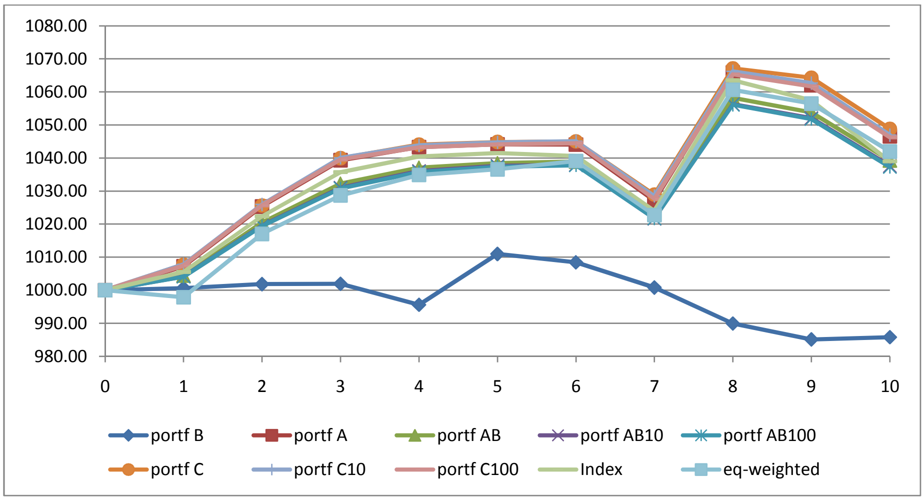


Figure 1: Comparison between optimized tracking portfolios and equally weighted portfolio, rolling simulation over 10-week period from January 18, 2012 to March 21, 2012.

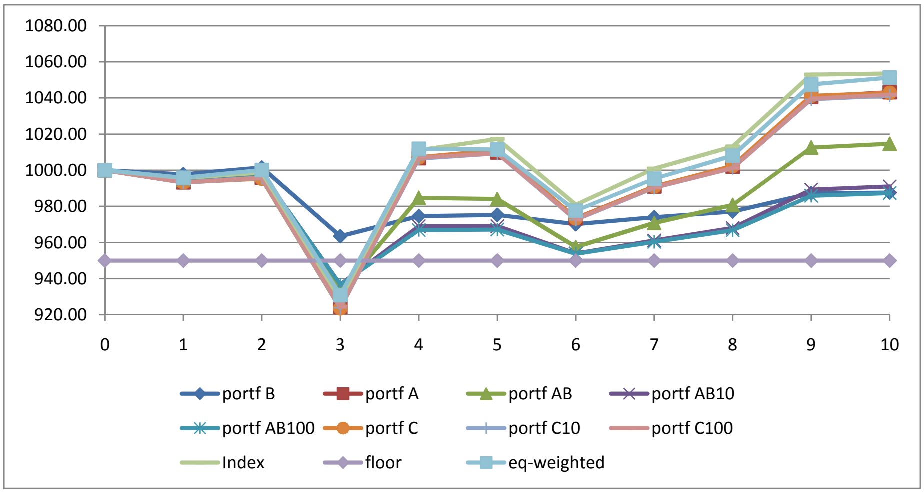


Figure 2: Comparison between optimized tracking portfolios and equally weighted portfolio, rolling simulation over 10-week period from November 2, 2011 to January 11, 2012.

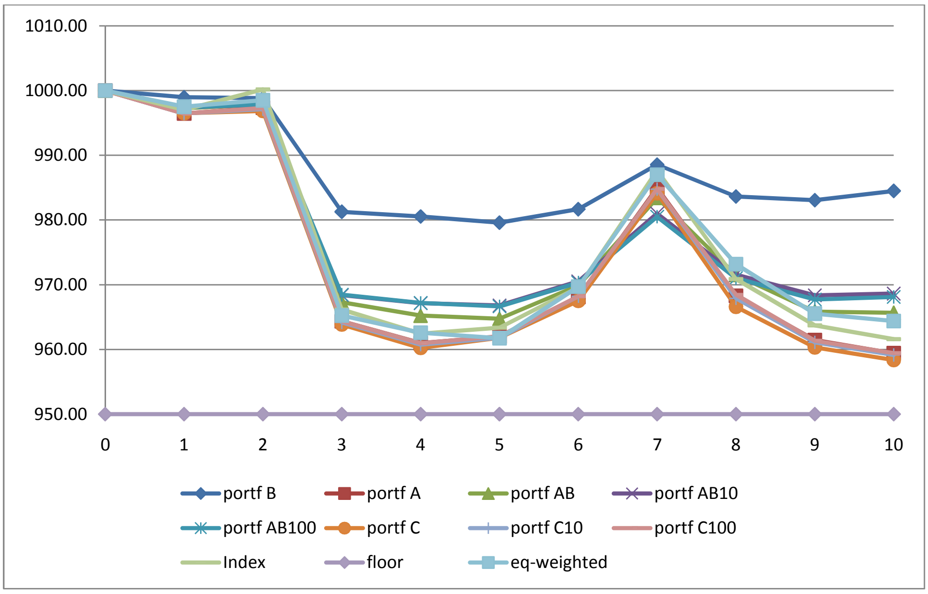


Figure 3: Comparison between optimized tracking portfolios and equally weighted portfolio, rolling simulation over 10-week period from May 18, 2011 to July 27, 2011.

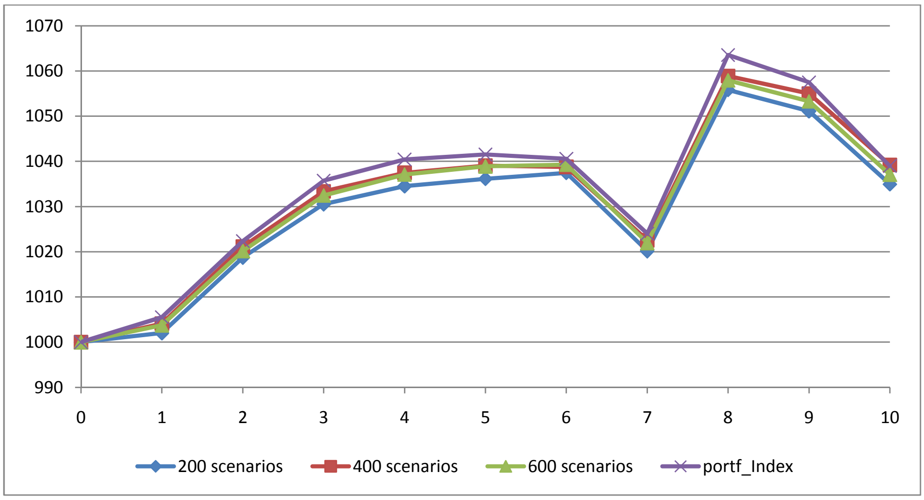


Figure 4: Comparison between optimized tracking portfolios, rolling simulation over 10-week period from January 18, 2012 to March 21, 2012 for an increasing number of scenarios.

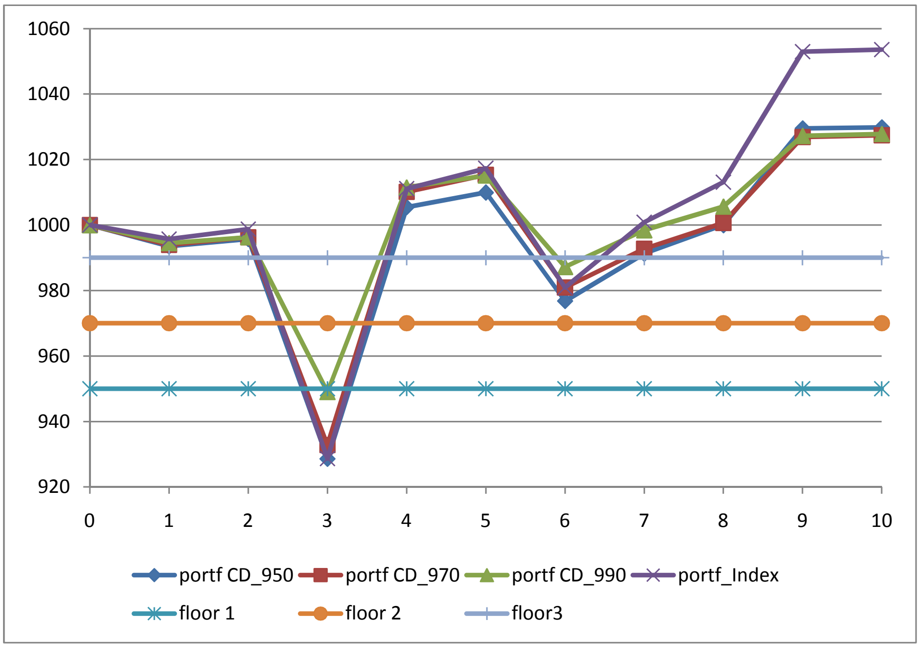


Figure 5: Optimized tracking portfolios for different levels of the loss thresholds, symmetric tracking with respect to the risky benchmark and the loss threshold. Rolling simulation over 10-week period from November 2, 2011 to January 11, 2012.

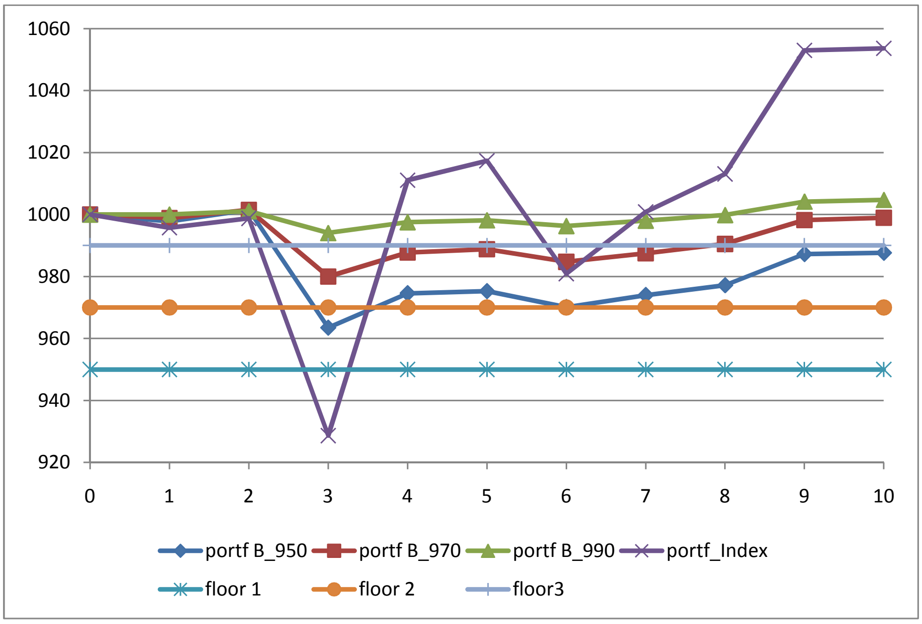


Figure 6: Optimized tracking portfolios for different levels of the loss thresholds, asymmetric tracking with respect to the loss threshold. Rolling simulation over 10-week period from November 2, 2011 to January 11, 2012.

## 5 Concluding remarks

In this contribution we propose a double tracking error model to simultaneously deal with the issues of benchmarking and providing downside protection. To control downside risk we introduce a threshold level of wealth and monitor the deviations of the managed portfolio from the threshold. This approach is flexible and allows for an easy and direct perception of the level of downside protection. Different choices of tracking error measures are possible and different trade-offs between the goals on the minimum threshold side and on the enhanced tracking error side, for the risky benchmark, are possible, too. In this contribution we do not tackle the issue of comparing different choices for tracking error measures and trade-offs in the goals with respect to the risk attitude of the investor. Among different possible models, we propose the absolute downside deviation as measure of tracking error between the managed portfolio and the threshold level of wealth, as a consequence we obtain a linear stochastic programming problem. We test the proposed model through a set of out of sample rolling simulation experiments in different market conditions. The obtained results show the effectiveness of the protection and point out that the trade-off between the level of protection and the reduction in the upside capture is heavily dependent on the given market conditions. An interesting analysis, that we leave for further research, would consider the role of the market volatility and the risk attitude of the investor.

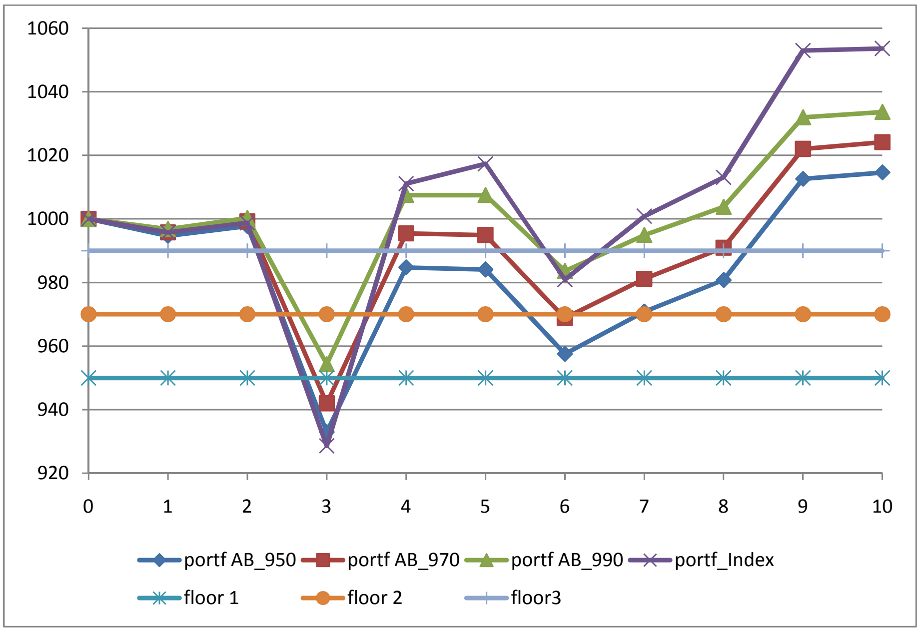


Figure 7: Optimized tracking portfolios for different levels of the loss thresholds, asymmetric tracking with respect to the risky benchmark and the loss threshold. Rolling simulation over 10-week period from November 2, 2011 to January 11, 2012.

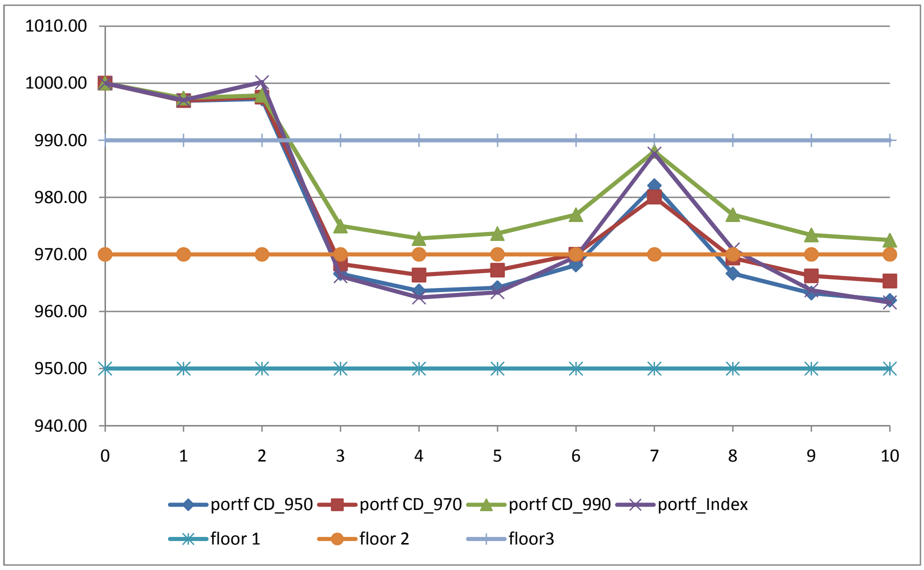


Figure 8: Optimized tracking portfolios for different levels of the loss thresholds, symmetric tracking with respect to the risky benchmark and the loss threshold. Rolling simulation over 10-week period from May 18, 2011 to July 27, 2011.

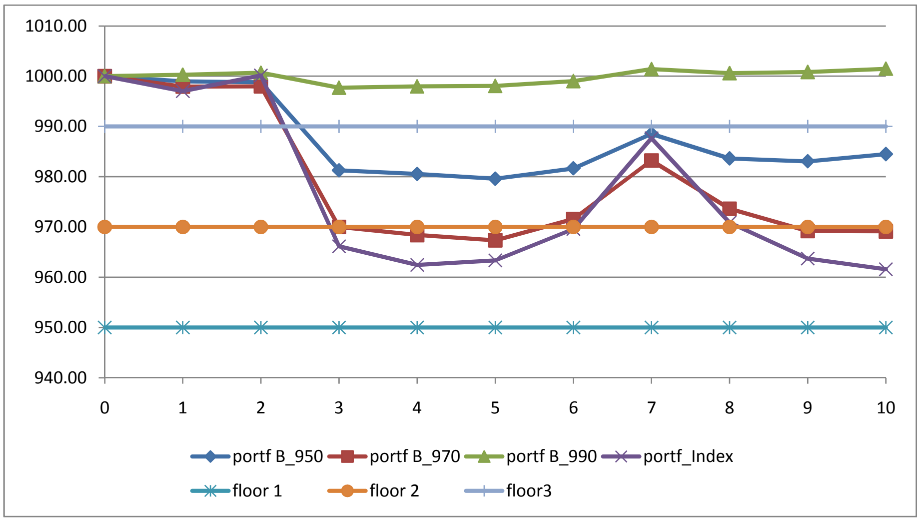


Figure 9: Optimized tracking portfolios for different levels of the loss thresholds, asymmetric tracking with respect to the loss threshold. Rolling simulation over 10-week period from May 18, 2011 to July 27, 2011.

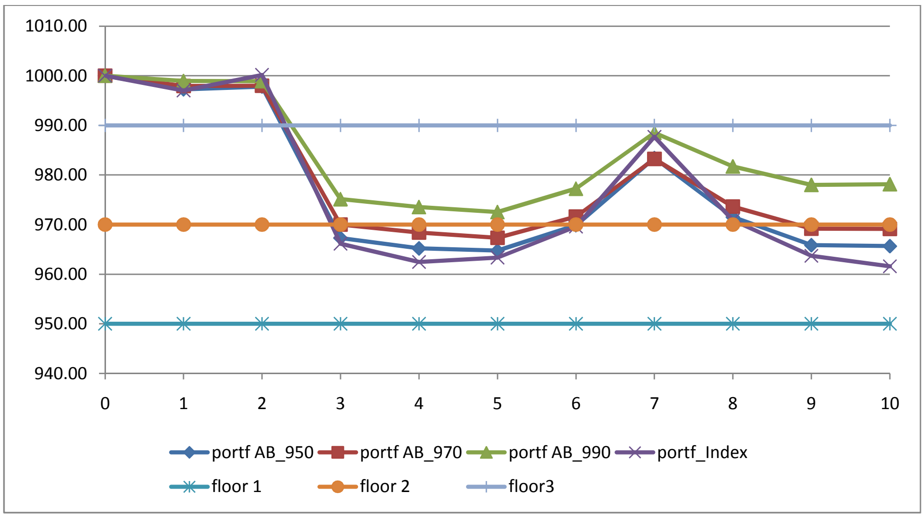


Figure 10: Optimized tracking portfolios for different levels of the loss thresholds, asymmetric tracking with respect to the risky benchmark and the loss threshold. Rolling simulation over 10-week period from May 18, 2011 to July 27, 2011.

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