Ca' Foscari University of Venice

Department of Economics

Working Paper

Niccolò Urbinati
Marco Li Calzi

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ISSN: 1827-3580
No. 05/WP/2022

Revised Edition

Working Papers
Department of Economics
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ISSN 1827-3580

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Niccolò Urbinati
University of Naples Federico II

Marco Li Calzi
Ca' Foscari University of Venice

First Published May 20, 2022
This Revision April 04, 2024

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JEL Codes
D02, D51, D62

Address for correspondence:
Niccolò Urbinati
Department of Economics and Statistics
University of Naples Federico II
Complesso di Monte Sant'Angelo
via Vicinale Cupa Cintia 26, 80126 Naples - Italy
e-mail: niccolo.urbinati@unina.it

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Department of Economics
Ca' Foscari University of Venice
Cannaregio 873, Fondamenta San Giobbe
30121 Venice Italy
Fax: ++39 041 2349210

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Niccolò Urbinati
Università Federico II Napoli
niccolo.urbinati@unina.it

Marco LiCalzi
Università Ca' Foscari Venezia
licalzi@unive.it

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1 Introduction

The intuitive notion of commodity is ambiguous: "should two apples of different sizes be considered two units of the same commodity?" (Genakoplos, 1989, p. 44). Economic theory relies on the idealized notion of an Arrow-Debreu commodity, whose objective description includes all relevant characteristics, including geographic and temporal location.

In principle, the Arrow-Debreu commodities of an economy are so finely identified that no further refinements can yield Pareto-improving allocations. In practice, markets for Arrow-Debreu commodities are usually so thin that they are rarely traded.

*We thank Achille Basile, Maria Gabriella Graziano, Salvatore Modica, Marialaura Pesce, and seminar audiences for their comments, as well as the referees and the associated editor for their insights. Financial support under PRIN 20222Z3CR7 is acknowledged. An Online appendix is available at http://mizar.unive.it/licalzi/MACG-Online-Appendix.pdf.

Real “commodity markets require design of both the marketplace procedures and the commodities themselves” (Roth 2018, p. 1611). After hinting at the uncountable expanse of theoretical goods, even Debreu (1959, p. 32) abruptly informs the reader that “it is assumed that there is only a *finite* number of distinguishable commodities” (emphasis added). His assumption mutes that tradable commodities are acknowledged by convention or created by design.

There are different ways to establish a commodity market. We focus on *conflation* by which “similar but distinct products are treated as identical in order to make markets thick or reduce cherry-picking” (Levin and Milgrom, 2010, p. 603). Here is an example. The people of a village cultivate a vineyard on the banks of a long river. The quality of the grapes depends on the position of the vines: one obtains different types of wine by pressing grapes grown on different vines. By custom, the villagers blend their grapes and trade at most two types of wine. Their technology separates the grapes growing on the lowlands (L) from those growing on the hills (H). The position θ of the boundary between L and H defines the conflation of grapes into two wines.

Conflation is usually achieved by setting standards, similarly to choosing the position θ in our example. Conflation (and deconflation) are techniques for the product definition that is involved in the creation or design of a commodity market (Milgrom, 2011). Relatedly, the frequent batch auctions (FBA) proposed in Budish (2015) conflate trading opportunities over a time interval to a single point in time.[1] Conflation takes place at the market level; instead, pure product bundling is the practice by which a firm makes several goods available only as a single package (Adams and Yellen, 1976).

This paper considers exchange economies where a continuum of goods are conflated and reduced into a finite number of tradable commodities. We assume that the goods are partitioned by means of a *classification* that associates each good to exactly one commodity.

We focus on classifications, assuming that every good in the endowment is conflated into a tradable commodity at no cost. Gilles and Diamantaras (2003) studies a general equilibrium model including the costly institutional arrangements required by the introduction of new tradable commodities. Stokey (1988) analyzes a dynamic general equilibrium model that sustains growth throuhg the introduction of new and

[1] Chauvin (2024) studies trade-restricted competitive equilibria.

better products. Instead, we assume that classifications are given and do not analyze how they emerge or are selected. Sprumont (2004) takes a first step towards endogenizing classifications, using an axiomatic approach where agents' preferences reveal subsets of characteristics that are sufficiently homogeneous and distinct to deserve being singled out as commodities.

We view a price-based market economy as an institution that binds agents' final allocations to the extant classification. Our main point is that prices decentralize the search for an equilibrium allocation, but the choice (or design) of a classification directly affects the outcome. Configuring the set of tradable commodities shifts the strength of agents' demands over their underlying goods, inducing significant changes over how trades equilibrate. Competitive allocations depend on the classification supporting them. In particular, one can interpret a classification as an abstract public good; see Mas-Colell (1980) and Diamantaras and Gilles (1996).

This paper studies and compares competitive allocations generated by different classifications over the same endowment of goods. The set of price-based competitive allocations in our model depends continuously on the choice of the classification, and therefore agents receive similar payoffs in economies using similar classifications. We show that, in equilibrium, the relative prices of two commodities depend on the classification of other (distinct) commodities; moreover, agents' allocations for two commodities may change even if their relative prices do not change. Importantly, if one can alter the classification underlying a competitive allocation, this latter may no longer be Pareto-optimal.

The literature on incomplete (financial) markets has shown that introducing a new security may lead to competitive equilibria that are Pareto-inferior; see Hart (1975, Section 6) or Cass and Citanna (1998). Our model, instead, demonstrates that in (real) exchange economies any refinement of a classification that repackages the available goods may lead to competitive equilibria that are never Pareto-improving, and may even reduce social welfare. The common theme is that increasing trading opportunities is not necessarily advantageous, even in the absence of transaction costs.

A price-based competitive equilibrium may be seen as "a method of creating harmony in an interactive situation with [...] self-interested agents" (Richter and Rubinstein, 2015, p. 2570). Section 5.2 takes a peek at the case where prices cannot be used, and studies an economy (without prices) where harmony can be achieved merely through selection of the appropriate classification. Consider the following example, inspired by Hylland and Zeckhauser (1979).

A College study program offers its courses over two teaching periods. The schedule must satisfy two criteria: the number of teaching hours per course is constant, and the classes are evenly distributed within each period. Every professor must teach one course and has preferences both on the period and on the concentration of his classes within each period. The school board chooses how to split the academic year in two terms, after asking each professor how much she wants to teach in each term. We consider whether there exists a division in two terms such that all teachers' requests are satisfied. We provide sufficient conditions under which, absent any price-based incentive, the choice of appropriate term lengths satisfies all professors' demands.

2 The model

We consider a society endowed with a continuum of goods to be allocated over a finite set of n agents. The agents have their own preferences on consumption and hold compatible claims on the society's endowment. It is infeasible (or impractical) to trade over a continuum, but the society can classify goods and conflate them into a finite number of commodities. The adoption of a classification by the society defines which commodities are tradable in the economy: a commodity is tradable if and only if it is acknowledged by the classification. The classification is shared by all agents and binds their choices: an agent can demand and be allocated only bundles of tradable commodities.

Goods and commodities. Let $\mathcal{I}$ be the space of goods. Each element t in $\mathcal{I}$ corresponds to a complete description of the attributes of the good. Given an algebra $\mathcal{F}$ on $\mathcal{I}$, we interpret every subset C in $\mathcal{F}$ as the *commodity* generated by the conflation of the goods in C.[2] (In the sequel, any subset C belongs to $\mathcal{F}$ and refers to a commodity.)

The endowment of goods available to the economy is described by a non-negative measure ω on $\mathcal{I}$, normalized to $\omega(\mathcal{I}) = 1$. Every positive function b in $L^1_+(\omega)$ describes

[2] From a technical point of view, this is consistent with the approach that goods aggregate characteristics (Lancaster, 1966). It suffices to redefine primitives and assume that goods are measurable sets of characteristics; see Mas-Colell (1975) and Jones (1984).

a *bundle* of goods. If the goods in C are conflated into a commodity, the total amount available is the quantity $\omega(C)$ while $\int_C b\,d\omega$ is the amount provided by the bundle b.

Agents. There are n agents. The preferences of an agent i on bundles of goods are represented by a function U_i on $L^1_+(\omega)$ that is concave, monotone, norm-continuous. Every agent i has a proportional[3] claim $\kappa_i > 0$ on the economy's endowment, and we assume $\sum_{i=1}^n \kappa_i = 1$.

Classification of goods. A *classification* π is a partition of $\mathcal{I}$ into a finite number of subsets. Each element C in π is interpreted as a conflation of all the goods in C into a tradable commodity. The conflation is technologically irreversible, as it is the case when mixing wheat in adherence to a market standard or blending grapes into a wine with protected designation of origin.[4] The partition π defines the tradable commodities and binds all the agents in the economy.

A *unit* of a commodity C consists of a bundle $\frac{1}{\omega(C)}\mathbb{1}_C$ of goods chosen uniformly from C. Given a (finite) classification π, a *tradable bundle* specifies quantities for each of the commodities in π and is described by a vector $x = (x_C)$ in $\mathbb{R}^\pi_+$, where x_C is the quantity of goods in C. Correspondingly, a tradable bundle x is represented by the simple function $\sum_{C\in\pi} \frac{x_C}{\omega(C)}\mathbb{1}_C$.

Exchange economy after a classification. Consumers can demand or exchange only tradable bundles. Thus, an instantiation of the model combines the identification of tradable commodities via π and a compatible allocation of the goods. We represent it as a configuration $\langle \pi, (x^i)\rangle$, where π is the shared classification and each x^i in $\mathbb{R}^\pi_+$ is the tradable bundle assigned to agent i. A configuration $\langle \pi, (x^i)\rangle$ is *feasible* if the total amount of allocated commodities does not exceed their initial availability: i.e., if $\sum_i x^i_C \le \omega(C)$ for every C in π.

The classification π defines an exchange economy $\mathcal{E}(\pi)$ over a (constrained) set of tradable bundles in $\mathbb{R}^\pi_+$. Every agent is endowed with the vector $e^i = (\kappa_i\omega(B))$ in

[3] This proportionality assumption implies that the exchange economies discussed in this paper may be interpreted as Fisher market models; see Vazirani (2007).

[4] This mutes concerns about cherrypicking or asymmetric information, but other interpretations are possible. A recurrent suggestion is to imagine that, after trading has closed, a lottery over the commodity unpacks its underlying goods, with agents computing expected utilities from the tradable lotteries.

$\mathbb{R}^\pi_+$ and ranks the tradable bundles by the function

$$V_i(\pi, x) = U_i\left(\sum_{C\in\pi} \frac{x_C}{\omega(C)} \mathbb{1}_C\right),$$

that is the restriction of U_i to the set of simple functions measurable with respect to π. The function $V_i(\pi, \cdot)$ inherits continuity, concavity and monotonicity from the primitive utility function U_i. Because agents' preferences do not depend on the classification, there are no spurious effects driving our results.

A competitive equilibrium in the economy $\mathcal{E}(\pi)$ is a pair $\langle p, (x^i)\rangle$ formed by a price vector $p \in \mathbb{R}^\pi_+$ and a feasible allocation (x^i) such that: (i) $p\cdot x^i \leq \kappa_i \sum_C p_C\, \omega(C)$ for all i, and (ii) if $V_i(\pi, y) > V_i(\pi, x^i)$ for some π-bundle y then $p \cdot y > \kappa_i \sum_C p_C\, \omega(C)$. We say that a configuration $\langle \pi, (x^i)\rangle$ is *competitive* if (x^i) corresponds to a competitive equilibrium in $\mathcal{E}(\pi)$.

In short, imposing a classification π reduces the economic interaction over a continuum of goods to a finite dimensional economy $\mathcal{E}(\pi)$ where agents have concave, monotone and continuous preferences and strictly positive endowments. It follows that for any classification π there exists at least a competitive equilibrium in $\mathcal{E}(\pi)$.

In the following, we make two assumptions to simplify notation and ease the exposition. First, $\mathcal{I}$ is the unit interval and $\mathcal{F}$ is the Borel σ-algebra; in contrast, Section 5.1 has an example where $\mathcal{I}$ is bidimensional. Second, the classification π consists of intervals of positive measure.

2.1 Notable examples

The model postulates primitive preferences for bundles of goods in the infinite dimensional space $L^1_+(\omega)$, and use those to derive preferences over the finite dimensional subspaces induced by a classification. The primitive preferences over goods may be hard to describe, even if their restrictions over commodities (after classification) appear fairly simple. We exhibit some examples of preferences on $L^1_+(\omega)$ that, once adapted to a classification π, induce popular utility functions on the commodity space $\mathbb{R}^\pi_+$.

When the primitive utility function $U_i : L^1_+(\omega) \to \mathbb{R}$ is linear, we can associate it with a measure ν_i on $\mathcal{I}$ such that $U_i(b) = \int b(t)\, d\nu_i(t)$ for every bundle b. By our assumptions on U_i, the measure ν_i turns out to be absolutely continuous with respect to ω. We call ν_i the agent i's *evaluation measure* and interpret it as a description of the value that i assigns to each conflation of goods. This agent ranks bundles in $\mathbb{R}^{\pi}_{+}$ by the linear function

$$V_i(\pi, x) = \sum_{C \in \pi} \left(\frac{x_C}{\omega(C)} \right) \nu_i(C).$$

Suppose now that agent i has an evaluation measure ν_i, but her primitive utility function is $U_i(g) = \exp\left(\int \log\left[g(t)\right] \, d\nu_i(t)\right)$. Then the agent ranks bundles in $\mathbb{R}^{\pi}_{+}$ by the Cobb-Douglas utility function

$$V_i(\pi, x) = \prod_{C \in \pi} \left(\frac{x_C}{\omega(C)} \right)^{\nu_i(C)}.$$

Finally, if i's primitive utility is $U_i(g) = \left(\int g(t)^{\rho} \, d\nu_i(t)\right)^{1/\rho}$ for some $0 < \rho < 1$, then the agent ranks bundles in $\mathbb{R}^{\pi}_{+}$ by the CES-type utility function

$$V_i(\pi, x) = \left[\sum_{C \in \pi} \left(\frac{x_C}{\omega(C)} \right)^{\rho} \right]^{1/\rho}.$$

Under these utility functions, agents have well-behaved preferences in any (finite-dimensional) economy $\mathcal{E}(\pi)$ induced by a classification π. In particular, when all agents have linear (resp. Cobb-Douglas, CES) utility functions, equilibrium prices in every $\mathcal{E}(\pi)$ exist and are unique up to normalization, and thus agents receive the same utility at every competitive equilibrium in $\mathcal{E}(\pi)$. This observation generalizes as follows.

Suppose that $U_i(b) = \int u_i\left[t, b(t)\right] \, dt$ for some integrable map $u_i \colon \mathcal{I} \times \mathbb{R} \to \mathbb{R}$ where $u(t, \cdot)$ is increasing, concave, and such that

$$-v \frac{\partial^2 u(t, v)}{\partial v^2} \leq \frac{\partial u(t, v)}{\partial v}$$

with a strict inequality when the right-hand side is non-zero. (Note that the examples above have equivalent representations of this form.) By applying Leibniz integral rule, one shows that the function $V_i(\pi, \cdot)$ satisfies the conditions of Example 17.F2 in Mas-Colell et al. (1995). It follows that if all agents have preferences of this type, then in every economy $\mathcal{E}(\pi)$ their aggregate demands satisfy Gross Substitutability, and therefore equilibrium prices exist and are unique, up to normalization.

3 Classification matters

Standard models of exchange take the notion of commodity as primitive. We depart from this assumption and consider alternative classifications for the same underlying set of goods. This section studies competitive equilibria under different classifications based on the same number of commodities. For readability, proofs are relegated to the appendix.

We first show that similar classifications lead to similar exchange economies, and hence to similar sets of competitive equilibria. To formalize this intuition, we endow the set $\Pi_{(\leq k)}$ of the classifications using *at most* k intervals with a topological structure. Let $\sigma(\pi)$ denote the σ-algebra generated by a classification π and set

$$\delta_\omega(\pi, \pi') = \sup_{C \in \sigma(\pi)} \inf_{C' \in \sigma(\pi')} \omega(C \triangle C').$$

Then the function $d_\omega(\pi, \pi') = \max\{\delta_\omega(\pi, \pi'), \delta_\omega(\pi', \pi)\}$ is a Hausdorff pseudo-metric for the set of classifications, induced by the measure of the symmetric difference between sets; see Boylan (1971).

Proposition 1. *$(\Pi_{(\leq k)}, d_\omega)$ is a compact space where two classifications have zero-distance if and only if they coincide up to null sets.*

In short, two classifications are essentially equivalent when their distance is zero. Recall from Section 2 that under our assumptions the set $\mathcal{W}(\pi)$ of competitive equilibria for an economy $\mathcal{E}(\pi)$ is not empty. We show that $\mathcal{W}(\pi)$ depends continuously on the choice of the classification π.

Theorem 2. *The competitive equilibria correspondence $\mathcal{W}$ is upper-hemicontinuous on the compact space $(\Pi_{(\leq k)}, d_\omega)$.*

We obtain as a corollary that, if goods are allocated through competitive equilibria, then agents receive similar payoffs in economies based on similar classifications. Let $\Psi_i(\pi)$ be the set of utilities that an agent i may obtain across all competitive equilibria for the economy $\mathcal{E}(\pi)$.

Corollary 3. *th The competitive utilities correspondence Ψ_i is upper-hemicontinuous on the compact space $(\Pi_{(\leq k)}, d_\omega)$.*

Continuity is crucial for comparative statics. If agents' utilities depend continuously on the classification, we can study not only *if* the welfare changes with the classification, but *how* it changes.

For instance, consider an economy based on a binary classification $\pi = \{A, B\}$. One may conjecture that increasing the number of commodities by one (e.g., by splitting A into two intervals) alters the equilibrium utilities less than introducing a hundred new commodities. This is not the case. Because of continuity, if the new classification ρ is sufficiently close to π (in the topological sense), then the change in the agents' equilibrium utilities under π and ρ can be made arbitrarily small, regardless of the number of new commodities in ρ. See Section 3.3 for a simple exercise in comparative statics.

3.1 Pareto-optimality and welfare theorems

We turn to studying the efficiency of competitive configurations. We say that a configuration $\langle \pi, (x^i) \rangle$ *Pareto-dominates* another configuration $\langle \rho, (y^i) \rangle$ if $V_i(\pi, x^i) \geq V_i(\rho, y^i)$ for all i, with a strict inequality for at least an agent i. Given a set of feasible configurations $\mathcal{F}$, a configuration $\langle \pi, (x^i) \rangle \in \mathcal{F}$ is *Pareto-optimal* in $\mathcal{F}$ if there is no configuration in $\mathcal{F}$ that Pareto-dominates it.

It is a simple observation that any competitive configuration is Pareto-optimal among those based on the same classification; that is, if $\langle \pi, (x^i) \rangle$ is competitive then no feasible configuration of the type $\langle \pi, (y^i) \rangle$ Pareto-dominates it. This is a consequence of the first Welfare Theorem applied to the exchange economy $\mathcal{E}(\pi)$.

A more interesting scenario opens when we compare allocations based on different classifications: then it is no longer true that any competitive configuration is Pareto-optimal. For instance, the competitive equilibrium under the trivial classification $\pi = \{\mathcal{I}\}$ is generically Pareto-dominated by many other finer competitive configurations. for the set of classifications, In the following example, a competitive configuration is (strictly) Pareto-dominated by another competitive configuration with the same number of commodities even if the relative prices for the commodities are the same.

Example 1. *Consider a society where ω is the Lebesgue measure and there are three agents who all have the same claim $\kappa = 1/3$. Agents' preferences are linear and* *respectively based on the evaluation measures*

$$\nu_1(F) = 3\omega\left(F \cap \left[0, \frac{1}{3}\right]\right) \ \ \textit{and} \ \ \nu_2(F) = \nu_3(F) = 3\omega\left(F \cap \left[\frac{1}{3}, \frac{1}{2}\right)\right) + \omega\left(F \cap \left[\frac{1}{2}, 1\right]\right);$$

that is, agent 1 *cares only about goods in the first third of the interval (and is indifferent over them), while agents* 2 *and* 3 *value any good in* $\left[\frac{1}{3}, \frac{1}{2}\right)$ *thrice as much as those in* $\left[\frac{1}{2}, 1\right]$.

Consider the economy $\mathcal{E}(\pi)$ *under the classification formed by the intervals* $A = \left[0, \frac{1}{2}\right)$ *and* $B = \left[\frac{1}{2}, 1\right]$. *Agent* 1*'s evaluations for the two tradable commodities are respectively* 1 *and* 0, *and thus he demands only the first commodity. Agents* 2 *and* 3 *have identical evaluations for the commodities and demand whatever is cheaper. A competitive equilibrium has identical prices for the commodities: it assigns the* π*-bundle* $x^1 = \left(\frac{1}{3}, 0\right)$ *to agent* 1, *and the bundles* $x^2 = x^3 = \left(\frac{1}{12}, \frac{1}{4}\right)$ *to agents* 2 *and* 3. *At the competitive configuration* $\langle \pi, (x^i) \rangle$ *the agents have utilities:*

$$V_1(\pi, x^1) = \frac{2}{3}, \quad V_2(\pi, x^2) = \frac{1}{3}, \quad V_3(\pi, x^3) = \frac{1}{3}.$$

Consider the alternative classification $\rho = \{A', B'\}$ *where* $A' = \left[0, \frac{1}{3}\right)$ *and* $B' = \left[\frac{1}{3}, 1\right]$. *In the economy* $\mathcal{E}(\rho)$, *agent* 1 *demands only commodity* A' *and agents* 2 *and* 3 *demand only commodity* B'. *A competitive equilibrium has identical prices for the commodities: it assigns the* ρ*-bundle* $y^1 = \left(\frac{1}{3}, 0\right)$ *to agent* 1, *and the bundles* $y^2 = y^3 = \left(0, \frac{1}{3}\right)$ *to agents* 2 *and* 3. *In this case, agents have utilities:*

$$V_1(\rho, y^1) = 1, \quad V_2(\rho, y^2) = \frac{1}{2}, \quad V_3(\rho, y^3) = \frac{1}{2}.$$

Clearly, the competitive configuration $\langle \rho, (y^i) \rangle$ *Pareto-dominates the competitive configuration* $\langle \pi, (x^i) \rangle$.

This example does not describe an isolated case. Whenever agents are not all identical, there is a "rich" set O of classifications for which every competitive configuration based on a classification in O is Pareto-dominated. In short, once classifications enter the picture, the first Welfare Theorem may fail in non-pathological circumstances.

Proposition 4. *Suppose that agents are not all identical and* $k > 1$. *Then there is an open set* O *of classifications in* $\Pi_{(\leq k)}$ *whose competitive configurations are Pareto-dominated by some competitive configuration* $\langle \hat{\pi}, (\hat{x}^i) \rangle$ *with* $\hat{\pi}$ *in* $\Pi_{(\leq k)}$.

One can recast the notion of a rich set O of Pareto-dominated competitive configurations in measure-theoretic terms; see Corollary 8 in the online Appendix.

The failure of Pareto optimality for a competitive configuration $\langle \pi, (x^i) \rangle$ follows because the classification π constrains trading over goods only to commodities. Because this restriction violates the assumption of universality of markets (Arrow, 1970), the first Welfare Theorem does not hold across different classifications. Classifications (or analogous restrictions on trade) may be crucial for the design of thick markets, but they come with a risk of efficiency loss.

Nonetheless, using the continuity of agents' preferences over classifications, one can prove the existence of Pareto-optimal competitive configurations if the set of utility profiles associated with any π is a singleton. This sufficient condition is always satisfied when agents' preferences are linear or satisfy Gross Substitutability.

Theorem 5. *Assume that the competitive utilities correspondence Ψ_i is single-valued. Then the set of competitive configurations based on classifications in $\Pi_{(\leq k)}$ has a Pareto-optimal configuration.*

This result states that there exists a Pareto-optimal configuration among all the classifications using at most k commodities. In general, if there is no upper bound on the number of commodities forming a classification, there may not exist a Pareto-optimal configuration, even within set of the competitive configurations. Example 7 in the online Appendix exhibits a pathological case where every competitive configuration can be Pareto-improved by a classification based on a strictly larger number of commodities.

Perhaps surprisingly, while the first Welfare Theorem fails for competitive configurations based over different classifications, it is possible to prove a version of the second Welfare Theorem for competitive configurations.

Proposition 6. *If $\langle \pi, (x^i) \rangle$ is a Pareto-optimal configuration such that $x^i_C > 0$ for every i and every C in π, then one can redefine agents' claims so that $\langle \pi, (x^i) \rangle$ is competitive.*

Proposition 6 ensures that an interior allocation that is Pareto-optimal can be cast as a competitive configuration after suitably modifying agents' claims. This aligns well with the classical version of the second Welfare Theorem, by which any interior Pareto-optimal allocation in an exchange economy is a competitive equilibrium for some suitable initial distribution of resources.

3.2 The relative scarcity of commodities

In a competitive equilibrium, one often interprets the ratio of the prices of two commodities as an index of relative scarcity: given preferences and endowments, the greater is the ratio, the higher is the value attributed to the first commodity. We argue that this ratio is not an intrinsic property of the two commodities, because it depends on how other distinct commodities are classified.

The next example keeps two commodities fixed and studies how the ratio of their prices varies as we change the rest of the classification. Even if agents' evaluations of the two commodities remain constant, the ratio of their prices ranges over an interval that can be made arbitrarily large. This implies that knowing the ratio of the prices of two commodities is meaningless without a full description of the whole classification.

Example 2. *Consider a society where ω is the Lebesgue measure and there are $2n$ agents who all have identical claims. There are two types of agents, forming groups of equal size. Agents' preferences are linear and based on the evaluation measures:*

$$\nu_1(F) = 2\omega\left(F \cap \left[0, \frac{1}{2}\right]\right) \quad \text{and} \quad \nu_2(F) = 2\omega\left(F \cap \left[\frac{1}{2}, 1\right]\right).$$

For every $t \in (0,1)$, let π_t be the classification formed by the four intervals:

$$A = \left[0, \frac{1}{4}\right), \quad B_t = \left[\frac{1}{4}, \frac{1+2t}{4}\right), \quad C_t = \left[\frac{1+2t}{4}, \frac{3}{4}\right), \quad D = \left[\frac{3}{4}, 1\right].$$

The two commodities A and D are acknowledged as tradable in any classification π_t, while the other two tradable commodities B_t and C_t depend on the choice of t. We claim that the ratio of equilibrium prices for the two commodities A and D depends on the threshold t.

For every t, let p_t be a competitive price system in $\mathcal{E}(\pi_t)$ and let $\varphi(t)$ denote the ratio $p_t(A)/p_t(D)$. Note that the function φ does not depend on how the p_t's are chosen, because two competitive prices in $\mathcal{E}(\pi_t)$ must be proportional to each other. Assuming $p_t(D) = 1$ for all t, we compute the equilibrium prices case by case.

Suppose $t \leq \frac{1}{2}$. Then commodities A and B_t are desirable only for agents in group 1, C_t is desirable for agents of both groups, and D is desirable only for agents in group 2. In equilibrium agents from group 1 demand commodities A, B_t and C_t as long as $t \leq \frac{1}{6}$, and demand only commodities A and B_t if $t > \frac{1}{6}$. On the other hand, *agents from group 2 demand positive amounts of commodities C_t and D. Because $p_t(D) = 1$, the resulting equilibrium prices are:*

$$p_t(A) = \begin{cases} \frac{1}{1-2t} & \text{if } t \leq \frac{1}{6}, \\ \frac{2}{1+2t} & \text{if } t > \frac{1}{6}, \end{cases} \qquad p_t(B_t) = \begin{cases} \frac{1}{1-2t} & \text{if } t \leq \frac{1}{6}, \\ \frac{2}{1+2t} & \text{if } t > \frac{1}{6}, \end{cases} \qquad p_t(C_t) = \frac{1}{2(1-t)}.$$

Suppose instead $t \geq \frac{1}{2}$. The situation is symmetric to the above. In equilibrium, agents from group 1 demand commodities A and B_t, while those from group 2 demand only commodities C_t and D if $t \leq \frac{5}{6}$, and may add commodity B_t when $t \geq \frac{5}{6}$. The resulting equilibrium prices are:

$$p_t(A) = \begin{cases} \frac{3-2t}{2}, & \text{if } t \leq \frac{5}{6}, \\ 2t-1, & \text{if } t > \frac{5}{6}, \end{cases} \qquad p_t(B_t) = \begin{cases} \frac{3-2t}{4t} & \text{if } t \leq \frac{5}{6}, \\ \frac{2t-1}{2t} & \text{if } t > \frac{5}{6}, \end{cases} \qquad p_t(C_t) = 1.$$

The function $\varphi(t)$ coincides with $p_t(A)$: it is increasing for $t \leq \frac{1}{6}$, decreasing for $\frac{1}{6} < t < \frac{5}{6}$, and increasing again for $t \geq \frac{5}{6}$. Its graph, plotted in Figure 1, ranges from $^2/_3$ to $^3/_2$.

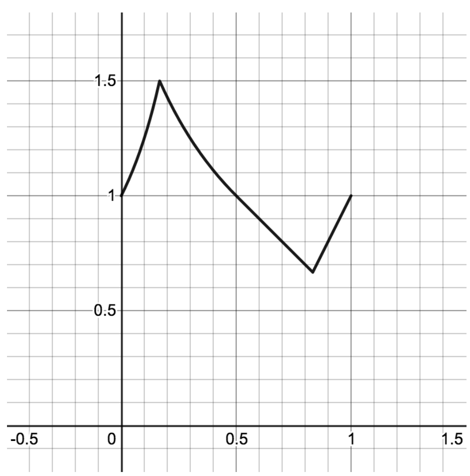

Figure 1: Ratio of the equilibrium prices for commodities A and D.

The index of relative scarcity for commodities A and D depends on the classification of tradable commodities. Moreover, for every threshold t there exists s such that $\varphi(t) = 1/\varphi(s)$, so that $p_t(A)/p_t(D) = p_s(D)/p_s(A)$. Therefore, whatever holds about the equilibrium price of A relative to D in a classification is specularly true about D relative to A under a different classification.

Note that changes in the threshold t affect the utilities that agents receive in equilibrium, and hence their relative welfare within the society (in spite of all agents having identical claims). In particular, the ratio $u^1(t)/u^2(t)$ of the equilibrium utilities that agents in the two groups $i = 1, 2$ receive in the economy $\mathcal{E}(\pi_t)$ has the same graph as the function φ plotted in Figure 1.

In Example 2, the ratio of the prices of two commodities A and D ranges over a closed interval as we change the classification and keep A and D fixed. One can modify the example and make the range of the interval arbitrarily large. However, because both agents's preferences and the function $\varphi(t)$ are continuous and the set of classifications we consider is compact, the range must remain bounded.

3.3 Economies with opposed preferences

This section demonstrates an example of comparative statics. (See Section 2 of the online Appendix for an outline of the arguments and of the computations.)

Assume that ω is the Lebesgue measure and there are two agents with identical claims and linear preferences. The evaluation measure for agent $i = 1, 2$ is given by the integral of a positive density f_i with full support over $\mathcal{I}$:

$$\nu_i(F) = \int_F f_i(t)\, dt.$$

Agents have *opposed preferences* when f_1 is decreasing and f_2 is increasing. Then there exists a single *leveling point* θ in $\mathcal{I}$ such that $\nu_1([0, \theta)) = \nu_2([\theta, 1])$.

Arrange the intervals forming the classification $\pi = \{C_1, \dots, C_k\}$ in their natural (increasing) order. Let C_{j^*} denote the commodity in π that contains the leveling point θ. Clearly, Agent 1 prefers all the commodities preceding C_{j^*} more than Agent 2, who in turn likes the commodities following C_{j^*} better than 1. Thus, computing the final equilibrium allocation under π reduces to studying how C_{j^*} is allocated between 1 and 2.

Specifically, let

$$C_\ell = \bigcup_{j<j^*} C_j \quad \text{and} \quad C_r = \bigcup_{j>j^*} C_j$$

denote the two commodities obtained by bundling the intervals on the left and on the right of C_{j^*}, respectively. Because $[\nu_1(C_j) - \nu_2(C_j)]\,(j^* - j) > 0$ for any $j \neq j^*$, in equilibrium Agent 1 is allocated all the goods in C_ℓ and none of those in C_r, while the opposite holds for Agent 2. We call C_{j^*} the *disputed commodity* because it is the only one that in equilibrium may be possibly (but not necessarily) allocated to both agents.

To characterize the quantities x_{j^*} and y_{j^*} of the disputed commodity C_{j^*} allocated to agents 1 and 2 in equilibrium, define the parameter

$$\xi = \frac{1}{2}\left[\frac{\nu_2\left(C_r\right)}{\nu_2\left(C_{j^*}\right)} - \frac{\nu_1\left(C_\ell\right)}{\nu_1\left(C_{j^*}\right)} + 1\right]. \tag{1}$$

Arranged by cases, we obtain

$$x_{j^*} = \begin{cases} 0 & \text{if } \xi \leq 0, \\ \xi\omega(C_{j^*}) & \text{if } 0 < \xi < 1, \\ \omega(C_{j^*}) & \text{if } \xi \geq 1, \end{cases} \tag{2}$$

with $y_{j^*} = \omega(C_{j^*}) - x_{j^*}$.

A simple exercise looks at how the equilibrium utility $V_1^*(\pi)$ of agent 1 changes when the classification π is perturbed. Because $V_1^*(\pi)$ depends only on the allocation of the disputed commodity (up to null sets), the relevant information is captured by the derivatives of V_1^* with respect to the two extreme points of $C_{j^*} = (\theta_1, \theta_2)$. For instance, a small change in θ_1 entails a change in the equilibrium utility for Agent 1 given by

$$\frac{\partial V_1^*(\pi)}{\partial \theta_1} = \begin{cases} f_1(\theta_1) & \text{if } \xi \leq 0, \\ \dfrac{\nu_2(C_r)}{2\nu_2^2(C_{j^*})}\left[\nu_1(C_{j^*})f_2(\theta_1) - f_1(\theta_1)\nu_2(C_{j^*})\right] & \text{if } 0 < \xi < 1, \\ 0 & \text{if } \xi \geq 1, \end{cases}$$

and this holds regardless of how $\pi \setminus \{C_{j^*}\}$ is defined, or perturbed.

For a different exercise, suppose $f_1(t) = 2(1-t)$ and $f_2(t) = 2t$ and consider the market design problem of finding the two-commodity classification that maximizes the sum of agents' utilities in equilibrium. By setting a threshold η in $(0, 1]$, the designer chooses the classification $\pi(\eta)$ formed by the two commodities $A = [0, \eta)$ and $B = [\eta, 1]$.

Because the leveling point is at $\theta = 1/2$, the disputed commodity is B for $\eta \leq 1/2$, and A otherwise. Using (1) and (2), we compute the utilities $V_1^*(\eta)$ and $V_2^*(\eta)$ that the two agents receive in equilibrium as functions of η:

$$V_1^*(\eta) = \begin{cases} \frac{1}{2} \\ 2\eta - \eta^2 \\ \frac{2-\eta}{2\eta} \end{cases} \quad \text{and} \quad V_2^*(\eta) = \begin{cases} \frac{1+\eta}{2(1-\eta)} \\ 1 - \eta^2 \\ \frac{1}{2} \end{cases} \quad \text{for} \begin{cases} 0 \leq \eta \leq 1 - \frac{\sqrt{2}}{2} \\ 1 - \frac{\sqrt{2}}{2} < \eta < \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \leq \eta \leq 1 \end{cases}$$

The sum $V_1^*(\eta) + V_2^*(\eta)$ of the equilibrium utilities is maximized at $\eta^* = \frac{1}{2}$.

4 Refining classifications

Refining a classification expands trading opportunities: goods once conflated in the original classification are acknowledged as distinct commodities. Formally, we say that a classification ρ *refines* a classification π (and write $\rho \succ \pi$) if the latter belongs to the algebra generated by ρ. When $\rho \succ \pi$, every π-bundle corresponds to a unique ρ-bundle and every feasible exchange in $\mathcal{E}(\pi)$ can also be realized within $\mathcal{E}(\rho)$, but the set of tradable commodities in ρ is larger. Therefore, allocations that are Pareto-optimal under the initial classification may cease to be so.

This section compares configurations achieved by increasing the number of commodities and refining the classification underlying the economy. Our modest goal is to show that refinements may sometimes yield unexpected outcomes. We hope that this may be a first step towards more general results. We assume that goods are assigned only via competitive equilibria consistent with the given classification, except for Section 4.3 where the configuration need not be competitive.

4.1 Refinements may switch trading positions

The introduction of a new commodity may change drastically individual trading positions. An agent who is a buyer for commodity A may switch positions and become a seller for A when the underlying classification is refined. A market designer who knows only agents' demands for a given classification (but not their preferences) may not even guess the direction of their trades under a finer classification.

We compare an economy based on three commodities A, B and C against one that refines the classification by splitting C into two distinct commodities. There is an agent who consumes only commodity A in the first economy but consumes only

B in the second one. This occurs even though neither A or B are directly affected by the refinement and in spite of their relative prices staying identical.

Example 3. *Consider a society where ω is the Lebesgue measure and there are 4 agents with different claims: agents 1 and 2 have claim $\frac{1}{3}$, while 3 and 4 have claim $\frac{1}{6}$. Preferences are linear and respectively based on the evaluation measures:*

$$\nu_1(F) = \frac{3}{2}\omega\left(F \cap \left[0, \frac{2}{3}\right)\right), \qquad \nu_2(F) = \frac{3}{2}\omega\left(F \setminus \left[\frac{1}{3}, \frac{2}{3}\right)\right),$$

$$\nu_3(F) = \frac{5}{3}\omega\left(F \cap \left[\frac{1}{3}, \frac{2}{3}\right)\right) + \frac{8}{3}\omega\left(F \cap \left[\frac{2}{3}, \frac{5}{6}\right)\right),$$
$$\nu_4(F) = \frac{5}{3}\omega\left(F \cap \left[\frac{1}{3}, \frac{2}{3}\right)\right) + \frac{8}{3}\omega\left(F \cap \left[\frac{5}{6}, 1\right]\right).$$

Let $\pi = \{A, B, C\}$ be the classification with $A = \left[0, \frac{1}{3}\right)$, $B = \left[\frac{1}{3}, \frac{2}{3}\right)$ and $C = \left[\frac{2}{3}, 1\right]$.

When all commodities carry the same price, agent 1 is indifferent between A and B, agent 2 between A and C, while 3 and 4 prefer B over the other commodities. At the only equilibrium in $\mathcal{E}(\pi)$, prices are identical and the agents are allocated the bundles

$$x^1 = \left(\frac{1}{3}, 0, 0\right),\ x^2 = \left(0, 0, \frac{1}{3}\right),\ x^3 = x^4 = \left(0, \frac{1}{6}, 0\right).$$

Refine π into the classification $\rho = \{A, B, C^1, C^2\}$ by splitting C into two intervals $C^1 = \left[\frac{2}{3}, \frac{5}{6}\right)$ and $C^2 = \left[\frac{5}{6}, 1\right]$. Agents 1 and 2 value C as much as C_1 or C_2. Instead, the refinement allows 3 and 4 to differentiate between which parts of C they like more: 3 prefers C_1 while 4 prefers C_2. At the only equilibrium in $\mathcal{E}(\rho)$, the four commodities have identical prices and the agent are allocated the bundles:

$$x^1 = \left(0, \frac{1}{3}, 0, 0\right),\ x^2 = \left(\frac{1}{3}, 0, 0, 0\right),\ x^3 = \left(0, 0, \frac{1}{6}, 0\right),\quad x^4 = \left(0, 0, 0, \frac{1}{6}\right).$$

In the economy $\mathcal{E}(\pi)$ agent 1 trades away her endowments of B and C and buys A, while in $\mathcal{E}(\rho)$ she trades away A, C_1, C_2 and buys only B. Nevertheless, the commodities A and B and their relative prices are the same in both economies. Moreover, 1's switch in trading positions leaves unchanged the utility of his allocation under either classification.

4.2 Refinements may not be welfare-improving

In many situations, increasing the number of commodities allows every agent to achieve higher levels of utility. However, it is possible that introducing a new tradable commodity gives some agents more market power and thus damages others. A refinement may fail to be Pareto-improving or even welfare-improving, where the *(utilitarian) social welfare* associated to a classification π is defined as the sum of the utilities that agents receive in any competitive equilibrium of $\mathcal{E}(\pi)$.

The next example describes a society where no refinement of the initial classification yields a strictly higher social welfare. The initial classification has two commodities A and B: we show that splitting A into two (or more) commodities leaves the social welfare unchanged, while splitting B strictly reduces it. Hence, creating new commodities (in any number) is never Pareto- nor welfare-improving.

Example 4. *Let ω be the Lebesgue measure on $\mathcal{I}$. Consider a society where there are $2n$ agents with identical claims, arranged in two groups of equal size. Preferences are linear and based on the evaluation measures:*

$$\nu_1(F) = \omega\left(F \cap \left[0, \frac{1}{2}\right)\right) + \frac{3}{2}\omega\left(F \cap \left[\frac{1}{2}, \frac{3}{4}\right)\right) + \frac{1}{2}\omega\left(F \cap \left[\frac{3}{4}, 1\right]\right), \text{ and}$$
$$\nu_2(F) = 2\omega\left(F \cap \left[\frac{1}{2}, 1\right]\right).$$

Consider the classification $\pi = \{A, B\}$, with $A = \left[0, \frac{1}{2}\right)$ and $B = \left[\frac{1}{2}, 1\right]$. At the competitive equilibrium for the economy $\mathcal{E}(\pi)$, all the commodities have identical prices and each agent i is allocated $\frac{1}{2n}$ units of commodity A or B if she is in group 1 or in group 2, respectively.

We claim that no refinement of π can improve the social welfare, and splitting B actually reduces it. Let $\rho \succ \pi$ be a refinement of any size. If ρ splits A into any number of commodities $A_1, \ldots, A_m$ (while leaving B untouched), then agents remain indifferent between all the A_i's. The equilibrium is essentially unaltered and the social welfare does not change.

Suppose now that ρ also splits B into smaller intervals $B_1, \ldots, B_k$. After renaming, assume that the B_j's are ordered so that $i < j$ implies $s < j$ for every s in B_i and t in B_j. We show by contradiction that, in any equilibrium in $\mathcal{E}(\rho)$, agents in group 1 consume a positive amount of some B_j's, which were all assigned to group 2 under π. Because group 1 evaluates the B_j's less than group 2, the sum of agents' *utilities in $\mathcal{E}(\rho)$ must be strictly lower than in $\mathcal{E}(\pi)$.*

Since in equilibrium the market must clear, agents of group 1 must be allocated all the A_i's. Because they are indifferent, this requires that each A_i has the same price p, or otherwise they would only demand the cheapest one(s). But then the price of B_1 must be strictly higher than p, or otherwise agents in group 1 would prefer to demand B_1 over any of the A_i's. If only agents in group 2 demand all the B_j's, their prices must all equal the price of B_1 (because agents in 2 are indifferent over the B_j's), and hence will be strictly higher than p. But if agents in group 2 sell their whole endowment of A at price p and buy the same amount of B at a strictly higher price, this violate the budget constraints.

The intuition behind this example is that, when $\pi = \{A, B\}$, agents in group 2 do not compete against agents in group 1 and let the latter choose their best option. If a finer classification ρ splits B, the agents in group 2 compete for some goods that were previously consumed by agents of group 1, even if the latter have a higher evaluation for those goods. The increased competition induced by the refinement may strictly reduce the social welfare.

For the stronger case where every refinement of the starting classification yields a strictly lower social welfare, see Example 6 in Section 5.1, which is based on a two-dimensional space of goods. Section 3 in the online Appendix has two more related examples.

4.3 The optimal number of commodities

Limitations on the number of tradable commodities may be driven by economic considerations. Operating within economies with many commodities is costly both for the agents, who have to process more information, and for the market infrastructure, that must handle more elaborated transactions. A market designer may prefer a simpler environment when the social cost of increasing the number of commodities is higher than the social benefit from a richer set of trading opportunities.

The optimal number of commodities depends on many subtle design issues, and on agents' preferences. As a first step, we consider the simple case where the configuration maximizes the social welfare net of a classification-related cost that is proportional to the number of commodities acknowledged by the classification.

Formally, let $SW(k)$ be the maximum social welfare that can be achieved using

k commodities across any configuration $\langle \pi, (x^i) \rangle$ with $|\pi| \leq k$. We do not assume that (x^i) is a competitive allocation in $\mathcal{E}(\pi)$. The social cost of operating with k commodities is ck, with $c \in (0, 1)$. The designer faces the problem:

$$\max_{k \geq 1} SW(k) - ck. \tag{3}$$

where the function SW is bounded from above, because each agent can attain at most utility 1 by consuming all the goods available. Therefore, a solution k^* to this problem always exists. Moreover, the optimal number of commodities k^* is never larger than $\overline{k} = \frac{n-(1-c)}{c}$, because otherwise the net social welfare would be higher by choosing $k = 1$ and giving all goods to one agent.

Without restrictions on agents' preferences, it is not possible to give tighter bounds than $1 \leq k^* \leq \overline{k}$. Consider the lower bound: if all evaluation measures equal ω, then the sum of agents' utilities is constant for any configuration, and therefore it is optimal to minimize the number of commodities choosing $k^* = 1$, for any $c > 0$. The next example exhibits an economy where the upper bound $\overline{k}$ is attained, for c sufficiently small.

Example 5. *Let $c \leq \frac{1}{m+1}$ for some m in $\mathbb{N}$. Assume that ω is the Lebesgue measure and let $\pi = \{A_0, \ldots, A_{m^2-1}\}$ be a partition of $\mathcal{I}$ in m^2 intervals of identical size $\frac{1}{m^2}$. There are $n = m$ agents, with linear preferences. The evaluation measure of agent i is*

$$\nu_i(F) = \frac{m^2}{m+1} \left[\sum_{j=1}^{m} \omega\left(F \cap A_{jm-i}\right) + \omega\left(F \cap A_{im-i}\right) \right];$$

Each agent i has positive value only on the m intervals $A_{m-i}, A_{2m-i}, \ldots, A_{m^2-1}$, and this is the same for each interval, except for A_{im-i} that she values twice as much. Agents' evaluations have disjoint support, and between two intervals valuable for agent i there exists at least an interval that another agent $j \neq i$ values twice as much.

Suppose momentarily $c = 0$; because agents' evaluations have disjoint support, the highest social welfare obtains when each agent is assigned the commodity he values the most; hence, the coarsest optimal classification is π. We argue that this allocation (attained using π) remains optimal if $c \leq \frac{1}{m+1}$. Assume that one reduces the number of commodities in order to save on operating costs. Any classification with fewer commodities than π has at least one element B with size larger than $\frac{1}{m^2}$ that is valuable for at least two agents. We can adroitly split B in two new commodities and reassign *those to the agents who value them the most, increasing the social welfare by at least* $\frac{1}{m+1}$*, which is greater than the cost of introducing an additional commodity. Because the optimal partition* π *has* m^2 *commodities, replacing* $n = m$ *and* $c = \frac{1}{m+1}$ *into* $\overline{k} = \frac{n-(1-c)}{c} = m^2$ *shows that the upper bound is attained.*

5 Extensions

This work provides a theoretical framework to study how the allocation of resources within a society depends on the conflation of goods into commodities. We rely on three founding blocks: (i) there are constraints on which classifications are allowed; (ii) agents' preferences over bundles of goods induce their preferences over commodities; and (iii) goods are allocated according to some mechanism restricted to the commodities listed in the classification.

For simplicity, we added two assumptions: (i) commodities are disjoint intervals of the unit segment $[0, 1]$; and (ii) commodities are allocated via competitive equilibria. Our general framework, however, may be studied with different restrictions on the classifications, or different allocation mechanisms. The next two subsections respectively illustrate possible variations on our two simplifying assumptions.

5.1 Bi-dimensional commodities

Consider an alternative description for the space of goods and its classifications. Suppose that each type of good is described by two values x and y, where x ranges over an interval I and y over an interval J. For instance, x may indicate the position of the good and y the date at which is made available. The distribution of goods is a non-negative measure ω on $\mathcal{R} = I \times J$, normalized to $\omega(\mathcal{R}) = 1$. Bundles of goods are positive, ω-integrable functions, with the usual interpretation. Agents' preferences correspond to evaluation measures over $I \times J$.

An $\mathcal{R}$-classification is a partition $\pi = \{A \times B : A \in \pi_I, B \in \pi_J\}$, where π_I and π_J are finite partitions of I and J formed by intervals. Intuitively, π may be seen as the result of cutting $\mathcal{R}$ by rows and columns drawn from π_I and π_J. If we let $\Pi_{(\leq k)}$ denote the set of $\mathcal{R}$-classifications with at most k cells, this bi-dimensional setup includes our model as special case, and all the results in Section 3 hold identically, with proofs easily mapped to the new setting.

However, this richer environment opens up more sophisticated considerations. As a way of illustration, consider a stronger version of Example 4, where any component-wise refinement of $\pi = \pi_I \times \pi_J$ strictly decreases the social welfare. In short, adding either new rows or new columns is detrimental.

Example 6. *Consider an economy where $\mathcal{R} = [0,1] \times [0,1]$ and ω is the Lebesgue measure on $\mathcal{R}$. Assume that $\mathcal{R}$ is partitioned in four equally sized quadrants:*

$$A = \left[0, \frac{1}{2}\right) \times \left[0, \frac{1}{2}\right), \; B = \left[0, \frac{1}{2}\right) \times \left[\frac{1}{2}, 1\right], \; C = \left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right], \; D = \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right),$$

and further divide B and D in two more triangles by cutting along their diagonal:

$$B^d = \{(x,y) \in B : \, x \leq 1-y\}, \; B^u = \{(x,y) \in B : \, x > 1-y\},$$

$$D^d = \{(x,y) \in D : \, x \leq 1-y\}, \; D^u = \{(x,y) \in D : \, x > 1-y\}.$$

There are $2n$ agents with identical claims, arranged in two groups of equal size. Agents have linear preferences based on the evaluation measures

$$\nu_i(F) = \int_F f_i(x,y)\, d(x,y),$$

computed using the densities

$$f_1(x,y) = \begin{cases} 1 & \text{if } (x,y) \in A \cup C, \\ \frac{3}{2} & \text{if } (x,y) \in B^d \cup D^d, \\ \frac{1}{2} & \text{if } (x,y) \in B^u \cup D^u, \end{cases} \qquad \text{and} \qquad f_2(x,y) = \begin{cases} 2 & \text{if } (x,y) \in B \cup D, \\ 0 & \text{if } (x,y) \in A \cup C. \end{cases}$$

Consider the $\mathcal{R}$-classification $\pi = \{A, B, C, D\}$. The only competitive equilibrium in $\mathcal{E}(\pi)$ assigns commodities A and C only to agents in group 1, and B and D only to those in group 2. The sum of the equilibrium utilities is $\frac{3}{2}$.

Let ρ be any refinement of π. Because ρ obtains by splitting rows or columns in π, at least one of the quadrants B and C has to be partitioned in smaller rectangles. Suppose it is B and let $B_1, \ldots, B_m$ denote the rectangular commodities obtained from B. At least one B_j is mostly contained below the diagonal, and $\omega(B_j \cap B^u) > \omega(B_j \cap B^u)$. Then agents in group 1 evaluate B_j more than A or C and yet their evaluation of B_j is lower than that of agents in group 2.

We claim that, for any equilibrium in $\mathcal{E}(\rho)$, agents in group 1 entirely consume A and C and a positive fraction of the commodities obtained from either B or D. Therefore, some goods initially assigned to group 2 under π are now allocated to 1, with a reduction in total utility: for short, any refinement of π induces a strict loss in the social welfare.

We only sketch the argument, because it is a close analog of that in Example 4. In any equilibrium in $\mathcal{E}(\rho)$, agents in group 1 demand all the commodities generated by splitting A or C, and so these must all have the same price p. At the same time, the price of commodity B_j is strictly higher than p, or otherwise agents in group 1 would demand B_j over any commodity in $A \cup C$. Because agents in group 2 are indifferent among the commodities in $B \cup D$, they demand all of them only if they come at the same price as B_j, which is strictly higher than p.

But this would imply that group 2 can trade away its endowment of commodities of type A and C at the price p and purchases the same amount of commodities of type B and D for a strictly higher price, in violation with the budget constraints.

This stronger version of Example 4 exploits the property that every refinement of the $\mathcal{R}$-classification affects at least two commodities. The uni-dimensional model has no classification with such property.

5.2 Equilibrium without prices

A competitive configuration is a price-based equilibrium where individuals demand only tradable bundles and meet specific budget constraints. Richter and Rubinstein (2020) argue that this generalizes to situations where society seek harmony through other decentralized institutions, enforcing a set of social norms that constrain agents' choices and ensure that their demands are mutually compatible.

With reference to our model, we exhibit sufficient conditions under which the equilibrium can be attained by the mere choice of an appropriate classification, absent prices. To this end, we compartmentalize the effect of the classification on individual demands by introducing a simple norm of rationing that is not price-based: the quantity of goods that an agent can demand is bounded by his claim. Thus, given a classification π, a π-bundle is *admissible for agent i* if it belongs to the set:

$$A_i(\pi) = \left\{ y \in \mathbb{R}_+^\pi : \sum_{B \in \pi} y_B \leq \kappa_i \right\}.$$

Agent i's *demand set* $D_i(\pi)$ is the collection of π-bundles that maximize i's utility among those that are admissible for i:

$$D_i(\pi) = \{x \in A_i(\pi) : V_i(\pi, x) \geq V_i(\pi, y) \text{ for every } y \in A_i(\pi)\} .$$

A configuration $\langle \pi, (x^i) \rangle$ is a *quantity-based equilibrium* if $x^i \in D_i(\pi)$ for every i. We drop the qualifier when there is no ambiguity and say that π *supports* the equilibrium.

Compare the price-based notion with the quantity-based equilibrium. The first one relies on the assumption that agents can exchange goods in different quantities. All consumers may trade up a large amount of some commodity for a smaller portion of another one they like more, when the prices allow it. This is problematic when agents are not allowed to bargain over the quantity of goods they receive. On the contrary, a quantity-based equilibrium lets agents exchange only bundles that contain the same total amount of goods. The example in the introduction about course assignments describes a situation where the total amount of commodities for each agent is fixed, making the quantity-based equilibrium look more natural than the price-based one.

A quantity-based equilibrium is associated to a standard competitive configuration where all commodities come at the same price. Therefore, any allocation in a quantity-based equilibrium is trivially envy-free, in the sense that no agent prefers the bundle assigned to another consumer with the same claim. Moreover, a quantity-based equilibrium may not lead to a Pareto-optimal configuration, as it is indirectly shown in Example 1 where all commodities have the same price in both equilibria. It does not seem obvious, however, to establish the existence of a Pareto-optimal quantity-based equilibrium.

We investigate the simpler equilibrium existence problem. The coarse partition $\pi = \{\mathcal{I}\}$ always supports a (quantity-based) equilibrium where each agent i receives a fraction κ_i of the overall endowment. We provide two conditions under which for every number of commodities there is a classification supporting an equilibrium: a) the measure ω has to be non-atomic, so that the endowment of goods is well distributed; and b) at least an agent i has a higher value when a commodity is defined by a smaller interval, exhibiting a strong preference for concentration. To formalize this last notion, we consider a larger class of preferences.

We associate every agent i with a set-function η_i defined on the subsets of $\mathcal{I}$, and called i's *evaluation capacity*. Intuitively, $\eta_i(B)$ is the utility that agent i receives from consuming commodity B; in particular, x units of commodity B yield utility $\frac{x}{\omega(B)}\eta_i(B)$. Given a classification π, the utility $V_i(\pi, x)$ of agent i for a π-bundle $x \in \mathbb{R}^{\pi}_{+}$ is the sum of the evaluations for each commodity weighted by the quantity received:

$$V_i(\pi, x) = \sum_{B \in \pi} \frac{x_B}{\omega(B)} \eta_i(B). \tag{4}$$

We assume that each evaluation capacity η_i satisfies four conditions; namely, it is:

1. *normalized*: $\eta_i(\emptyset) = 0$ and $\eta_i(\mathcal{I}) = 1$;
2. *monotone*: $\eta_i(F) \leq \eta_i(G)$ whenever $F \subseteq G$ are subsets of $\mathcal{I}$;
3. *submodular*: $\eta_i(F \cup G) + \eta_i(F \cap G) \leq \eta_i(F) + \eta_i(G)$ for every two $F, G \subseteq \mathcal{I}$;
4. *absolutely continuous*: $\eta_i(F^n) \to 0$ whenever (F^n) is a monotone sequence of subsets in $\mathcal{I}$ such that $\omega(\bigcap F^n) = \emptyset$.

The submodularity of the function η_i implies decreasing marginal evaluations: for every $F, G, G' \subseteq \mathcal{I}$ such that $G \subset G'$, we have $\eta_i(F \cup G) - \eta_i(G) \geq \eta_i(F \cup G') - \eta_i(G')$, meaning that the marginal benefit of adding F to a portion G of $\mathcal{I}$ is decreasing in the content of G. The absolute continuity of η_i with respect to ω implies that any agent's evaluation for a vanishing quantity of goods decreases to 0. Because evaluation measures are just additive evaluation capacities, Equation (4) is an extension of the standard case with linear preferences.

Another consequence of submodularity is that the ratio $\eta_i(F)/\omega(F)$ increases as $\omega(F)$ decreases. Intuitively, the average benefit from a type of goods increases as the type becomes "more concentrated". We say that agent i has a *strong preference for concentration* (*SPC* for short) if $\lim_n \eta_i(F^n)/\omega(F^n) = \infty$ for every sequence (F^n) of intervals such that $\omega(F^n) > 0$ for all $n \in \mathbb{N}$ and $\lim_n \omega(F^n) = 0$.

We state our existence result.

Theorem 7. *If ω is non-atomic and at least an agent has a strong preference for concentration, then for every k in $\mathbb{N}$ there exists an equilibrium supported by a classification with k intervals.*

The two hypotheses are: a) the measure ω is non-atomic, and b) there is an agent with SPC. The online Appendix exhibits two examples to illustrate that neither condition can be dropped, and a third example to show that they are not necessary.

Appendix: proofs

Proposition 1. $(\Pi_{(\leq k)}, d_\omega)$ *is a compact space where two classifications have zero-distance if and only if they coincide up to null sets.*

Proof. Let h be the pseudo-metric $h(F, G) = \omega(F \triangle G)$ defined on the measurable sets over $\mathcal{I}$. Then $d_\omega(\pi, \rho)$ is the Hausdorff distance between the algebras $\sigma(\pi)$ and $\sigma(\rho)$. Then $d_\omega(\pi, \rho) = 0$ if and only if $\sigma(\pi)$ and $\sigma(\rho)$ have the same closure; see Lemma 3.72, Aliprantis and Border, 2006. Equivalently, every B in $\sigma(\pi)$ corresponds to some B' in $\sigma(\rho)$ up to null sets. Because $\sigma(\pi)$ and $\sigma(\rho)$ are finite, this holds if and only if the two classifications π and ρ (respectively viewed as the generators for the two algebras) coincide up to null sets.

Let $\mathcal{J}$ be the class of intervals from $\mathcal{I} = [0, 1]$. If we identify two intervals having zero-distance, then the function ω maps isometrically $\mathcal{J}_0 = \{F \in \mathcal{J} : 0 \in F\}$ into a closed and bounded subset of $\mathcal{I}$, so that $(\mathcal{J}_0, h)$ is itself compact. Therefore, $\Pi_{(\leq k)}$ is compact because it is the image of the compact product space $\mathcal{J}_0^{k-1}$ under the continuous function

$$\varphi(F_1, \ldots, F_{k-1}) = \{F_{i+1} \setminus F_i : \omega(F_{i+1}) > \omega(F_i) \text{ and } F_k = \mathcal{I}\} .$$

□

Theorem 2. *The competitive equilibria correspondence* $\mathcal{W}$ *is upper-hemicontinuous on the compact space* $(\Pi_{(\leq k)}, d_\omega)$.

Proof. We associate every π in $\Pi_{(\leq k)}$ to an auxiliary exchange economy $\tilde{\mathcal{E}}(\pi)$ with commodity space $\mathbb{R}^k_+$. Let $\tilde{\mathcal{W}}(\pi)$ be the set of competitive equilibria in $\tilde{\mathcal{E}}(\pi)$.

First, we show that the correspondence $\tilde{\mathcal{W}}$ is upper-hemicontinuous; second, we argue that $\mathcal{W}(\pi)$ is the continuous image of the set $\tilde{\mathcal{W}}(\pi)$.

For any classification $\pi = (B_1, \ldots, B_m)$ with $m \leq k$, let $\tilde{\mathcal{E}}(\pi) = \mathcal{E}(\pi)$ if $m = k$. Otherwise, if $m < k$, for each agent i define his endowment in $\tilde{\mathcal{E}}(\pi)$ as

$$\tilde{e}^i(\pi) = \kappa_i \left(\omega(B_1), \ldots, \omega(B_{m-1}), \frac{\omega(B_m)}{k-m}, \ldots, \frac{\omega(B_m)}{k-m} \right)$$

and his utility from the vector of quantities x in $\mathbb{R}^k_+$ as

$$\tilde{V}_i(\pi, x) = U_i\left(\sum_{j=1}^{m-1} \frac{x_j}{\omega(B_j)} \mathbb{1}_{B_j} + \frac{\sum_{j=m}^{k} x_j}{\omega(B_m)} \mathbb{1}_{B_m}\right).$$

Because the economies $\tilde{\mathcal{E}}(\pi)$ satisfy the assumptions of the main Theorem in Hildebrand and Mertens (1972), the equilibrium-set correspondence $\tilde{\mathcal{E}}(\pi) \mapsto \tilde{\mathcal{W}}(\pi)$ is upper-hemicontinuous. The map $\pi \mapsto \tilde{\mathcal{E}}(\pi)$ is also continuous, because $\tilde{e}^i(\pi)$ and $\tilde{V}^i(\pi, \cdot)$ change continuously with π. Therefore the whole correspondence $\tilde{\mathcal{W}}$ is upper-hemicontinuous.

Next, note that an allocation $(\tilde{x}^i)$ in $\tilde{\mathcal{E}}(\pi)$ is competitive if and only if the vectors $x^i = (\tilde{x}^i_1, \ldots, \tilde{x}^i_{m-1}, \sum_{j=m}^{k} \tilde{x}^i_j)$ constitute a competitive allocation in $\mathcal{E}(\pi)$. Thus, $\mathcal{W}(\pi)$ is a continuous image of $\tilde{\mathcal{W}}(\pi)$. □

Corollary 3. *The competitive utilities correspondence Ψ_i is upper-hemicontinuous on the compact space $(\Pi_{(\leq k)}, d_\omega)$.*

Proof. By construction, for every π in $\Pi_{(\leq k)}$ the set $\Psi_i(\pi)$ consists of all $V_i(\pi, x^i)$ with x^i ranging over the bundles assigned to agent i by the allocations in $\mathcal{W}(\pi)$. Hence, Ψ_i upper-hemicontinuous because it is the composition of a continuous function and a upper-hemicontinuous correspondence. □

Proposition 4. *Suppose that agents are not all identical and that $k > 1$. Then there is a competitive configuration $\langle \hat{\pi}, (\hat{x}^i) \rangle$ with $\hat{\pi}$ in $\Pi_{(\leq k)}$ that Pareto-dominates every competitive configuration based on a classification in an open set $O \subset \Pi_{(\leq k)}$.*

Proof. When $k > 1$ and agents do not have identical preferences, there is a configuration $\langle \hat{\pi}, (\hat{x}^i) \rangle$ that every agent prefers to the trivial configuration based on the classification $\pi_0 = \{\mathcal{I}\}$. Define the set $O = \Big\{\pi \in \Pi_{(\leq k)} : \Psi_i(\pi) < V_i(\hat{\pi}, \hat{x}^i) \text{ for every } i \leq n\Big\}$, that is not empty because it contains π_0, and it is open by the upper-hemicontinuity of each Ψ_i. By construction, every competitive configuration based on a classification in O is Pareto-dominated by $\langle \hat{\pi}, (\hat{x}^i) \rangle$. □

Theorem 5. *Assume that the competitive utilities correspondence Ψ_i is single-valued. Then the set of competitive configurations based on classifications in $\Pi_{(\leq k)}$ has a Pareto-optimal configuration.*

Proof. Because the Ψ_i's are continuous functions on the compact space $(\Pi_{(\leq k)}, d_\omega)$, there exists π^* in $\Pi_{(\leq k)}$ that maximizes $\sum \Psi_i(\pi)$. Let $(x^i) \in \mathcal{W}(\pi^*)$ and observe that a competitive configuration $\langle \pi, (y^i) \rangle$ Pareto-dominates $\langle \pi^*, (x^i) \rangle$ only if $\sum V_i(\pi, y^i) > \sum V_i(\pi^*, x^i)$, thereby violating the maximality of π^*. We conclude that $\langle \pi^*, (x^i) \rangle$ is Pareto-optimal. □

Proposition 6. *If $\langle \pi, (x^i) \rangle$ is a Pareto-optimal configuration such that $x^i_C > 0$ for every i and every C in π, then one can redefine agents' claims so that $\langle \pi, (x^i) \rangle$ is competitive.*

Proof. The second Welfare Theorem applied to the allocations in the economy $\mathcal{E}(\pi)$ implies that there exists a price system p in $\mathbb{R}^\pi_+$ that supports the allocation (x^i). Then $\langle \pi, (x^i) \rangle$ is a competitive configuration after we redefine agents' claims as:

$$\kappa_i = \sum_{B \in \pi} p_B \frac{x^i_B}{\omega(B)}.$$

□

Theorem 7. *If ω is non-atomic and at least an agent has a strong preference for concentration, then for every k in $\mathbb{N}$ there exists an equilibrium supported by a classification with k intervals.*

Proof. Some additional notation is useful. Let $\Delta = \{x \in \mathbb{R}^k_+ : \sum x_j = 1\}$ be the set of vectors in $\mathbb{R}^k_+$ whose components sum up to one, and let $\Delta^\circ \subset \Delta$ be its interior. For every classification π formed precisely by k commodities, let $\omega(\pi)$ be the vector:

$$\omega(\pi) = (\omega(C_1), \ldots, \omega(C_k))$$

where the commodities $C_1, \ldots, C_k$ in π are arranged so that $i < j$ if and only if $s < t$ for every s in C_i and t in C_j. Because every C_j has a positive measure, $\omega(\pi)$ is a vector in Δ°. Moreover, because ω is non-atomic, for every p in Δ° there is a classification π_p, unique up to null sets, such that $\omega(\pi_p) = p$. The map $p \mapsto \pi_p$ identifies isometrically the set of vectors in Δ° with the classifications formed by k intervals, up to null sets.

The proof is divided into three steps. First, we argue that the total demand for goods is a well-behaved correspondence Z from Δ° to Δ. Second, we show that a fixed point of Z corresponds to a classification with k commodities that supports an equilibrium. Third, and last, we prove that Z has a fixed point.

First step. Fix an agent i. For any p in Δ°, the set $D_i(\pi_p)$ of the maximizers of the linear function $V_i(\pi_p, \cdot)$ on the convex and compact set $\left\{x \in \mathbb{R}^k_+ : \sum x_j = \kappa_i\right\}$ is also convex and compact. Moreover, because $V_i(\pi_p, \cdot)$ depends continuously on p, the relation $p \mapsto D_i(\pi_p)$ defines an upper-hemicontinuous correspondence from Δ° to Δ.

Let $Z(p) = \sum D_i(\pi_p)$ be the total demand for goods at the classification π_p. Because each $D_i(\pi_p)$ is a convex and compact subset of $\{x \in \mathbb{R}^k_+ : \sum x_j = \kappa_i\}$, $Z(p)$ is a convex and compact subset of Δ. And, because each $p \mapsto D_i(\pi_p)$ is upper-hemicontinuous, Z is a upper-hemicontinuous correspondence from Δ° to Δ.

Second step. If Z has a fixed point p^* in Δ°, then the classification $\pi_{p^*} = (C_1, \ldots, C_k)$ supports an equilibrium, because $p^* \in Z(p^*)$ implies that there is an allocation (x^i) with $x^i \in D_i(\pi_{p^*})$ for every i and $\sum_j x^i_j = \omega(C_j)$ for each $j \leq k$.

Third step. The correspondence Z is defined on the open set Δ°, unsuitable to a straightforward fixed point argument. Nevertheless, we show that $Z(p)$ behaves well when p approaches the boundary of Δ°. Let i^* be an agent having SPC. For every sequence $(p^n) \subset \Delta^\circ$ that converges to a $p \notin \Delta^\circ$ and every (x^n) with $x^n \in D_{i^*}(\pi_{p^n})$ for each n, it must be that $\lim_n \sum\{x^n_j : p_j = 0\} = \kappa_i$. This follows because i^* prefers to spend her budget on those commodities whose size goes to zero. Therefore, Z has the following boundary behavior: for every sequence (p^n, q^n) in the graph of Z that converges to some (p, q), if $p \notin \Delta^\circ$ then $\sum\{q_j : p_j = 0\} \geq \kappa_{i^*}$.

The correspondence $\hat{Z} \colon \Delta \twoheadrightarrow \Delta$ defined by:

$$\hat{Z}(p) = \begin{cases} Z(p) & \text{if } p \in \Delta^\circ, \\ \{q \in \Delta : \sum q_j \geq \kappa_{i^*}\} & \text{if } p \notin \Delta^\circ, \end{cases}$$

extends Z and satisfies the assumptions of Kakutani's Theorem: it has convex and compact values, and its graph is closed by the boundary behavior of Z. Therefore, $\hat{Z}$ has a fixed point p^*. Moreover, for every $p \notin \Delta^\circ$, if $q \in \hat{Z}(p)$ then $q_j > 0$ for at least some j with $p_j = 0$, and thus p is not a fixed point of $\hat{Z}$. We conclude that the fixed point p^* is in Δ° and therefore it is also a fixed point of Z. □

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Market allocations under conflation of goods

Online Appendix

(available at `http://mizar.unive.it/licalzi/MACG-Online-Appendix.pdf`)

1 Pareto-dominated configurations

We show one way to formulate Proposition 4 in measure-theoretic terms. Intuitively, this requires that the endowment of goods is well-distributed, in the sense that the measure ω is non-atomic. Let $\Pi_{(=k)}$ be the set of the classifications using *exactly* k. Given π in $\Pi_{(=k)}$, rearrange it in the string of intervals $(F_1, \dots, F_k)$ where $i < j$ if and only if $s < t$ for every s in F_i and t in F_j. If we associate π with the vector $\omega(\pi) = (\omega(F_1), \dots, \omega(F_k))$ in $\mathbb{R}^k$, then the map $\pi \mapsto \omega(\pi)$ is continuous and injective. Define the measure $\lambda(O)$ for a set $O \subset \Pi_{(=k)}$ as the Lebesgue measure of the set $\{\omega(\pi) : \pi \in O\}$.

Corollary 8. *Suppose that agents are not all identical and $k > 1$. Then there is a non-null subset O of classifications in $\Pi_{(=k)}$ whose competitive configurations are Pareto-dominated by some competitive configuration $\langle \hat{\pi}, (\hat{x}^i) \rangle$ with $\hat{\pi}$ in $\Pi_{(=k)}$.*

The following example illustrates a society with two agents, where every configuration can be improved both from a Paretian and an utilitarian point of view with a suitable refinement of the underlying classification.

Example 7. *Consider an economy where ω is the Lebesgue measure and there are two agents with identical claims. Let $S \subset \mathcal{I}$ denote the Smith-Volterra-Cantor set (SVC set for short), which is a measurable set of size $\frac{1}{2}$ with the property that every non-null interval in $\mathcal{I}$ contains a non-null interval disjoint from S; see the ϵ-Cantor set in Aliprantis and Burkinshaw (1981, p. 141). Agents' of each group have linear preferences based on the evaluation measures:*

$$\nu_1(F) = 2\omega(F \cap S), \quad \nu_2(F) = 2\omega(F \setminus S).$$

We claim that the only Pareto-optimal configurations assign the 0 bundle to all agents of group 1.

Let $\langle \pi, (x^i) \rangle$ be a configuration and let the interval $B \in \pi$ be a commodity such that $x^1_B > 0$. By the properties of the SVC set S, there exists an interval $C \subseteq B$ such that

$C \cap S = \emptyset$, and so $\nu_1(C) = 0$ and $\nu_2(C) = 2\omega(C)$. If we label C as a new commodity, we obtain a finer classification ρ under which one can transfer all goods of type C previously assigned to 1 to agent 2, while leaving the rest of the allocation unchanged. But this benefits agent 2 without causing harm to 1 (because her evaluation of C is null), proving that $\langle \pi, (x^i) \rangle$ is Pareto-dominated.

Formally, let B^1 and B^2 be the two (possibly, empty) intervals obtained by removing C from B. Let ρ be a refinement of the classification π, where the commodity B has been replaced with B^1, B^2 and C. Consider a new allocation (y^i) in $\mathcal{E}(\rho)$ where the bundle assigned to agent 1 is

$$y^1_A = \frac{\omega(A)}{\omega(B)} x^1_B \text{ if } A \in \{B^1, B^2\}, \quad y^1_C = 0, \quad y^1_A = x^1_A \text{ otherwise;}$$

and the bundle assigned to agent 2 is:

$$y^2_A = \frac{\omega(A)}{\omega(B)} x^2_B \text{ if } A \in \{B^1, B^2\}, \quad y^2_C = \frac{\omega(C)}{\omega(B)} x^2_B + \frac{\omega(C)}{\omega(B)} x^1_B, \quad y^2_A = x^2_A \text{ otherwise.}$$

Computations shows that (y^i) is a feasible allocation in $\mathcal{E}(\rho)$ that agent 1 finds equivalent to (x^i), while agent 2 strictly prefers it to (x^i). Standard arguments based on the continuity and monotonicity of the function $V_i(\rho, \cdot)$ prove that one can modify (y^i) into a new allocation that every agent strictly prefers to (x^i).

Clearly, Example 7 relies crucially on the assumption that agents' evaluations of goods are expressed through extremely elaborated subsets of $\mathcal{I}$ (such as the SVC set) while commodities can only be defined as intervals. If we allow commodities to be arbitrary subsets of $\mathcal{I}$, then the classification $\pi = \{S, S^c\}$ would generate a Pareto-optimal configuration where all goods of type S are assigned to agent 1 and the rest to agent 2. This suggests that the stronger the exogenous constraints on the classification of goods into commodities, the further agents may be from reaching optimal allocations.

2 Comparative statics with opposed preferences

Given the classification $\pi = (C_1, \ldots, C_k)$ where intervals are naturally ordered, let $p = (p(C_i))_i$ denote the system of competitive equilibrium prices, with $x = (x(C_i))_i$ and $y = (y(C_i))_i$ being the equilibrium bundles respectively assigned to 1 and 2. We outline the main steps, including the explicit computation of $\frac{\partial V_1^*(\pi)}{\partial \theta_1}$.

1. The ratio $\nu_1(C_i)/\omega(C_i)$ decreases as i increases. Indeed, let $F(x) = \int_0^x f(t)\,dt$, so that $F'(x) = f(x)$. Assume $a < b < c$; given $A = (a, b)$ and $B = [b, c)$, by the Mean Value theorem $\nu_1(A)/\omega(A) = (F(b) - F(a))/(b - a) = f(\theta_A)$ for some θ_A in A, and similarly $\nu_1(B)/\omega(B) = (F(c) - F(b))/(c - b) = f(\theta_B)$ for some θ_B in B. When f is decreasing, $f(\theta_A) > f(\theta_B)$ and thus $\nu_1(A)/\omega(A) > \nu_1(B)/\omega(B)$. Specularly, $\nu_2(C_i)/\omega(C_i)$ increases as i increases.

2. Suppose that x is an equilibrium allocation. If

$$\frac{p(C_j)}{p(C_i)} > \frac{\nu_1(C_j)/\omega(C_j)}{\nu_1(C_i)/\omega(C_i)},$$

then $x(C_j) = 0$. Correspondingly, $x(C_i) > 0$ and $x(C_j) > 0$ requires that the above expression holds as an equality. Similar considerations hold for y and ν_2.

3. There is a j^* such that $x(C_i) = \omega(C_i)$ for $i < j^*$ and $y(C_j) = \omega(C_j)$ for $j > j^*$.

Proof. Suppose $i < j$ and $x(C_j) > 0$. Then (2) gives

$$\frac{p(C_j)}{p(C_i)} \leq \frac{\nu_1(C_j)/\omega(C_j)}{\nu_1(C_i)/\omega(C_i)} < 1,$$

where the last inequality follows from (1). Using (1) again gives

$$\frac{p(C_j)}{p(C_i)} < 1 < \frac{\nu_2(C_j)/\omega(C_j)}{\nu_2(C_i)/\omega(C_i)}$$

and thus $y(C_i) = 0$, or (2) would be violated. Therefore, because the market clears in equilibrium, $x(C_i) = \omega(C_i)$. $\square$

4. Let $C^* = C_{j^*}$ be the disputed commodity; define $C_\ell = \bigcup_{i<j^*} C_i$ and $C_r = \bigcup_{j>j^*} C_j$. If $x(C^*) > 0$ and $y(C^*) > 0$, then $x(C^*) = \xi\omega(C^*)$, where

$$\xi = \frac{1}{2}\left[\frac{\nu_2(C_r)}{\nu_2(C^*)} - \frac{\nu_1(C_\ell)}{\nu_1(C^*)} + 1\right]. \qquad (*)$$

Proof. By (1), we have $x(C_i) = \omega(C_i)$ for $i < j^*$ and $x(C_j) = 0$ for $j > j^*$. Applying (2) gives

$$\frac{p(C^*)}{p(C_i)} = \frac{\nu_1(C^*)/\omega(C^*)}{\nu_1(C_i)/\omega(C_i)} \quad \text{for every } i < j^*,$$

from which we obtain

$$\omega(C_i)p(C_i) = \left[\frac{\omega(C^*)p(C^*)}{\nu_1(C^*)}\right]\nu_1(C_i) \quad \text{for every } i < j^*.$$

Given the system of prices p, the worth of the bundle x for Agent 1 is:

$$\sum_{i<j^*} \omega(C_i)p(C_i) + \xi\omega(C^*)p(C^*) = \left[\frac{\omega(C^*)p(C^*)}{\nu_1(C^*)}\right] \cdot \left[\nu_1(C_\ell) + \xi\nu_1(C^*)\right].$$

Similarly, the worth of the bundle y for Agent 2 is:

$$\sum_{j>j^*} \omega(C_j)p(C_j) + (1-\xi)\omega(C^*)p(C^*) = \left[\frac{\omega(C^*)p(C^*)}{\nu_2(C^*)}\right] \cdot \left[\nu_2(C_r) + (1-\xi)\nu_2(C^*)\right],$$

Because in equilibrium x and y must have the same worth at p, we have:

$$\frac{\nu_1(C_\ell) + \xi\nu_1(C^*)}{\nu_1(C^*)} = \frac{\nu_2(C_r) + (1-\xi)\nu_2(C^*)}{\nu_2(C^*)}$$

from which (*) follows. □

5. By a standard continuity argument:

$$x_{j^*} = \begin{cases} 0 & \text{if } \xi \le 0, \\ \xi\omega(C_{j^*}) & \text{if } 0 < \xi < 1, \\ \omega(C_{j^*}) & \text{if } \xi \ge 1, \end{cases}$$

with $y_{j^*} = \omega(C_{j^*}) - x_{j^*}$.

6. Suppose $C^* = (\theta_1, \theta_2)$ and denote by V_1 the utility that agent 1 obtains from the bundle x. Using (5), we have

$$V_1^* = \begin{cases} \nu_1(C_\ell) & \text{if } \xi \leq 0, \\ \nu_1(C_\ell) + \xi \nu_1(C^*) & \text{if } 0 < \xi < 1, \\ \nu_1(C_\ell \cup C^*) & \text{if } \xi \geq 1. \end{cases}$$

In particular, when $0 < \xi < 1$, substituting (*) from (4) gives:

$$V_1^* = \frac{\nu_1(C_\ell) + \nu_1(C^*)}{2} + \frac{\nu_1(C^*)\nu_2(C_r)}{2\nu_2(C^*)} = \frac{\nu_1(C_\ell \cup C^*)}{2} + \frac{\nu_1(C^*)\nu_2(C_r)}{2\nu_2(C^*)}.$$

Recall that $C_\ell = [0, \theta_1]$, $C^* = (\theta_1, \theta_2)$ and $C_r = [\theta_2, 1]$. Therefore:

$$\frac{\partial \nu_1(C_\ell \cup C^*)}{\partial \theta_1} = 0, \qquad \frac{\partial \nu_1(C^*)}{\partial \theta_1} = -f_1(\theta_1), \qquad \frac{\partial \nu_2(C^*)}{\partial \theta_1} = -f_2(\theta_1).$$

Then the derivative of V_1^* with respect to θ_1 when $0 < \xi < 1$ is:

$$\frac{\partial V_1^*}{\partial \theta_1} = \frac{\nu_2(C_r)}{2\nu_2^2(C_{j^*})} \left[\nu_1(C_{j^*}) f_2(\theta_1) - f_1(\theta_1) \nu_2(C_{j^*}) \right].$$

3 Refinements may not be welfare-improving

The next example exhibits an economy and a classification π with the following property: for every (finer) classification ρ that splits a commodity from π into two commodities, there is an agent who strictly prefers every competitive allocation in $\mathcal{E}(\pi)$ to any competitive allocation in $\mathcal{E}(\rho)$. In short, adding a new commodity damages at least one agent and therefore is not a Pareto-improvement for the society.

Example 8. *Consider an economy where ω coincides with the Lebesgue measure. There are 4 agents with identical claims and linear preferences based on the evaluation measures:*

$$\nu_1(F) = 2\omega\left(F \setminus \left[\frac{1}{4}, \frac{3}{4}\right]\right), \quad \nu_2(F) = 2\omega\left(F \cap \left[\frac{1}{4}, \frac{3}{4}\right]\right),$$

$$\nu_3(F) = 2\omega\left(\left[0, \frac{1}{2}\right]\right), \quad \nu_4(F) = 2\omega\left(\left[\frac{1}{2}, 1\right]\right).$$

Let π be the classification formed by the two intervals $A = \left[0, \frac{1}{2}\right]$ and $B = \left(\frac{1}{2}, 1\right]$.

In the exchange economy $\mathcal{E}(\pi)$ agent 3 cares only about commodity A, agent 4 only about B, and agents 1 and 2 are indifferent between them. An equilibrium is achieved when the two commodities have the same price and agents demand, for example, the π-bundles:

$$x^1 = x^3 = \left(\frac{1}{4}, 0\right), \; x^2 = x^4 = \left(0, \frac{1}{4}\right).$$

We claim that for every refinement ρ of π formed by 3 tradable commodities there is an agent that strictly prefers (x^i) to any competitive allocation in $\mathcal{E}(\rho)$. Precisely, we assume that ρ is obtained by splitting A into two commodities A_1 and A_2 and we prove that, in equilibrium, agent 3 cannot afford $\frac{1}{4}$ units of goods of type A_1 or A_2, implying that 3 receives a strictly lower utility under ρ. The same strategy shows that if ρ is obtained by splitting B then agent 4 strictly prefers π to ρ.

Assume $t \in \left(0, \frac{1}{2}\right)$ such that $\omega(A_1) = t$ and $\omega(A_2) = \frac{1}{2} - t$. Let p be a competitive price in $\mathcal{E}(\rho)$ normalized so that $p(B) = 1$ and let w be agent 3's wealth at p. We assume that $p(A_1) \leq p(A_2)$ (the other case is treated identically) so that agent 3 demands exactly:

$$\frac{w}{p(A_1)} = \frac{1}{4}\left[t + \frac{p(A_2)}{p(A_1)}\left(\frac{1}{2} - t\right) + \frac{1}{2p(A_1)}\right]$$

units of commodity A_1.

Let us assume by contradiction that $w/p(A_1)$ is greater than $\frac{1}{4}$. There are two possible cases:

- *if $p(A_1) = p(A_2) \leq 1$, then each of the agents 1, 2 and 3 demands $\frac{1}{4}$ units of commodity A_1 or A_2. This creates an excess of demand and thus p cannot be an equilibrium price. On the other hand, if $p(A_1) = p(A_2) > 1$ then $w/p(A_1)$ is strictly less than $\frac{1}{4}$.*

- *If $p(A_1) < p(A_2)$ then agents 1 and 3 demand A_1 instead of A_2. Therefore, $p(A_2) \leq 2$, or no agents would demand A_2. At the same time, it must be that $p(A_1) \geq \frac{1}{2t}$ or agent 1 would demand only A_1, leaving 3 with strictly less than $\frac{1}{4}$ units of A_1. Combining these two inequalities we obtain:*

$$\frac{w}{p(A_1)} = \frac{1}{4}\left[t + \frac{p(A_2)}{p(A_1)}\left(\frac{1}{2} - t\right) + \frac{1}{2p(A_1)}\right] \leq \frac{1}{4}\left[t + 2t(1-2t) + t\right] = t - t^2$$

  which is strictly smaller than $\frac{1}{4}$ for every $t < \frac{1}{2}$.

The above example is based on refinements of π formed only by 3 intervals. If we allow for richer classifications, then we can find refinements of π that are strictly preferred to π by every agent in the society. As a way of illustration, let ρ be formed by the intervals:

$$A = \left[0, \frac{1}{4} - \varepsilon\right), \quad B = \left[\frac{1}{4} - \varepsilon, \frac{1}{2}\right], \quad C = \left(\frac{1}{2}, \frac{3}{4} - \varepsilon\right], \quad D = \left(\frac{3}{4} - \varepsilon, 1\right]$$

with $\varepsilon \in \left(0, \frac{1}{4}\right)$. For ε sufficiently small, an equilibrium in $\mathcal{E}(\rho)$ is achieved when all commodities have identical prices and each agent consumes the whole of a commodity (1 gets A, 2 gets C, 3 gets B, and 4 gets D). This leaves every agent with a utility strictly larger than the one they received with the allocation (x^i).

The following example refines both Example 4 in the main text and Example 8 above by describing an economy where every refinement of the starting classification gives a strictly lower social welfare. The setup is similar to Example 4, but the set of feasible classifications is curtailed by assuming that the commodity B is an atom, so that some tradable commodities cannot be split into smaller parts.

Example 9. *Let λ be the Lebesgue measure on $\mathcal{I}$ and $\delta_{\{1\}}$ denote the Dirac measure for the singleton $\{1\}$. We consider a society where there are $2n$ agents with identical claims and the measure ω is given by:*

$$\omega(F) = \frac{1}{2}\left(\lambda(F) + \delta_{\{1\}}(F)\right).$$

There are only two types of agents, forming groups of equal size. Agents have linear preferences based on the evaluation measures:

$$\nu_1(F) = \lambda(F) \quad \textit{and} \quad \nu_2(F) = \frac{1}{4}\lambda\left(F \cap \left[0, \frac{1}{2}\right]\right) + \frac{3}{4}\lambda\left(F \cap \left[\frac{1}{2}, 1\right]\right) + \frac{1}{2}\delta_{\{1\}}(F).$$

Intuitively, agents of type 1 value all types of goods identically, while those of type 2 care more about goods in $\left[\frac{1}{2}, 1\right)$ and especially about those labelled with 1.

Let π be the classification formed by the commodities $A = [0, 1)$ and $B = \{1\}$. At the competitive equilibrium, A and B have the same prices, with every agent from group 1 consuming $\frac{1}{2n}$ units of commodity A and every agent from group 2 consuming $\frac{1}{2n}$ units of B.

We prove that, if $\rho \succ \pi$, then every competitive allocation in $\mathcal{E}(\rho)$ assigns a *positive amount of goods of type A to agents in group 2. Because the utility received from goods of type A is higher for agents in group 1, this implies that the sum of agents' utilities in $\mathcal{E}(\rho)$ must be strictly lower than in $\mathcal{E}(\pi)$.*

Suppose by contradiction that there exists a refinement ρ of π and a competitive allocation in $\mathcal{E}(\rho)$ such that agents in group 1 consume all goods of type A and those in group 2 all goods of type B. Because B is an atom, ρ can refine π only by splitting A into smaller intervals and leaving B intact. We write $\rho = \{A_1, \dots, A_m, B\}$ where $i < j$ implies $s < t$ for all $s \in A_i$ and $t \in A_j$. Because agents from group 1 demand all commodities $A_1, \dots, A_m$, these must have all equal prices (otherwise agents of group 1 would demand only the cheapest ones). At the same time, A_m must cost strictly more than B, otherwise agents in group 2 would rather demand A_m than B. Hence, the average price of the commodities A_j's is strictly greater than the price of B, implying that each agent in group 2 can demand more than $\frac{1}{2n}$ units of B. This leads to an excess of demand for B, which contradicts the assumption that prices are competitive.

4 Equilibrium without prices

The next two examples illustrate that neither sufficient condition in Theorem 7 can be dropped. A third following example shows that they are not necessary.

Example 10 (A society where ω is non-atomic but no agent has SPC)**.** *There are n agents and ω is the Lebesgue measure. Every agent i has linear preferences with an evaluation measure defined by:*

$$\eta_i(F) = \int_F u_i \, d\omega.$$

for some strictly increasing density u_i, so that no agent exhibits SPC. We claim that no classification based on $k \geq 2$ intervals can support an equilibrium.

Take any classification $\pi = (B_1, \dots, B_k)$ and let $0 = \theta_0 < \theta_1 < \cdots < \theta_k = 1$ be such that θ_{j-1} and θ_j are the extreme points of the interval B_j. An agent i maximizes the utility $V_i(\pi, x)$ by demanding positive amounts only for the tradable commodities B_j for which the ratio

$$\frac{\eta_i(B_j)}{\omega(B_j)} = \frac{\int_{\theta_{j-1}}^{\theta_j} u_i \, d\omega}{(\theta_j - \theta_{j-1})}$$

is maximized. On the other hand, because u_i is an increasing function, the map $t \mapsto \int_0^t u_i \, d\omega$ is convex and so:

$$\frac{\int_{\theta_{k-1}}^{\theta_k} u_i \, d\omega}{(\theta_k - \theta_{k-1})} > \frac{\int_{\theta_{k-2}}^{\theta_{k-1}} u_i \, d\omega}{(\theta_{k-1} - \theta_{k-2})} > \cdots > \frac{\int_{\theta_0}^{\theta_1} u_i \, d\omega}{(\theta_1 - \theta_0)}.$$

Because every agent demands exclusively the same k-th tradable commodity, there is a positive excess of demand under any classification π with $k \geq 2$. We conclude that no such classification can support an equilibrium.

In Example 10 agents have additive evaluation capacities: their demands are not affected by the width of the intervals in the classification π. This no longer holds if a consumer exhibits SPC, because that consumer is attracted to sufficiently smaller cells.

Example 11 (A society where every agent has SPC but ω is atomic)**.** *Consider an economy where half of the total amount of goods correspond to the point 0 and the other half correspond to 1. Then the measure ω has two atoms and assigns to each $F \subseteq \mathcal{I}$ the value*

$$\omega(F) = \frac{1}{2}\delta_{\{0\}}(F) + \frac{1}{2}\delta_{\{1\}}(F)$$

There are n agents, with linear preferences based on the evaluation measure $\eta_i(F) = \delta_{\{1\}}(F)$; thus, every agent exhibits SPC.

We claim that no classification π based on $k \geq 2$ intervals can support an equilibrium. Given any π, every agent prefers the cell B containing 1 over any other cell and therefore demands only this commodity. This implies a positive excess of demand for B, and the conclusion follows.

Example 11 shows how the presence of large chunks of identical goods can make agents' demands insensitive to changes in the classification. This cannot occur when the measure ω is non-atomic, because the amount of goods labelled with the same $t \in \mathcal{I}$ is negligible.

Example 12 (A society where ω is atomic and no agent has SPC, but equilibrium exists)**.** *There are three agents. The measure ω is defined by*

$$\omega(F) = \lambda\left(F \cap \left[0, \frac{2}{3}\right]\right) + \frac{1}{3}\delta_{\{1\}}(F),$$

where λ denotes the Lebesgue measure. Assume that the three agents have linear preferences based on the evaluation measures:

$$\eta_1(F) = \int_F 2t\,dt, \qquad \eta_2(F) = \eta_3(F) = \int_F 2(1-t)\,dt.$$

Note that ω has the atom $\{1\}$ and that no agent has SPC.

Consider the classification $\pi = \{[0, 2/3], (2/3, 1]\}$. Then agent 1 demands the π-bundle $x^1 = (0, 1)$, while agents 2 and 3 demand the π-bundle $x^2 = x^3 = (1/2, 0)$. Because $\langle \pi, (x^a) \rangle$ is an allocation, we conclude that π supports an equilibrium.

One can relax some assumptions on the model in Subsection 5.2 without compromising the existence result of Theorem 7. We illustrate two possible extensions.

Measure space for the goods' characteristics. We assume that the space of goods' characteristics is a totally ordered set and that commodities are defined as intervals. This can be relaxed to an abstract measure space for the goods, where commodities are defined by measurable subsets. We sketch the main features of this more general approach.

Let $(X, \mathcal{F})$ be a measurable space. We interpret each element t in X as a complete description of a good and each F in $\mathcal{F}$ as a commodity. A non-negative measure ω on $\mathcal{F}$ describes the availability of goods. A *classification of goods* is a partition π of X formed by finitely many sets in $\mathcal{F}$ with positive ω-measure. The definitions for bundles, agents' evaluations and equilibrium are naturally adapted to this more general setup.

Even in this broader setting, there exists a non-trivial classification supporting an equilibrium if ω is non-atomic and at least an agent has SPC. In fact, one can define a family of classifications with similar properties to those formed by intervals of $\mathcal{I}$ and have almost identical proofs. The main intuition is to choose an increasing family of sets, and then mimic a "moving-knife procedure" to define partitions similar to those formed by intervals in $\mathcal{I}$.

Formally, let $\mathcal{C} = \{C_t : t \in \mathcal{I}\} \subseteq \Sigma$ be a monotone chain such that $\omega(C_t) = t$ for all $t \in \mathcal{I}$. Such a chain exists by the non-atomicity of ω. A set J is a $\mathcal{C}$*-interval* if there exists $t < s$ in $\mathcal{I}$ such that $J = C_s \setminus C_t$. Let $\Pi^{\mathcal{C}}_{(\leq k)}$ be the set of classifications formed by at least a number $k \geq 2$ of $\mathcal{C}$-intervals. One may extend the proof of Theorem 7 with respect to $\mathcal{C}$-intervals in X instead of intervals in $\mathcal{I}$.

Note that this more general setting has a much larger class of classifications than $\mathcal{I}$ has. Therefore, although an equilibrium exists, other results may no longer hold. For example, our proof that there exists a Pareto-optimal configuration within the set of competitive configurations cannot be directly extended.

Weaker form of SPC. The assumption that at least an agent has SPC is restrictive, because it requires that there is an agent that will drastically change his choice whenever he is offered a sufficiently concentrated commodity. From a technical viewpoint, however, this assumption is used only to show that the aggregate demand correspondence meets some standard boundary conditions. Therefore, it can be relaxed into a local requirement: if the interval defining a commodity is sufficiently small, then there is at least one consumer who prefers it to all the other commodities. More precisely, consider the following assumption of *distributed SPC*:

> If $\pi^n = (C_1^n, \ldots, C_i^n, \ldots C_k^n)$ is a sequence of classifications in $\Pi_{(\leq k)}$ and $\omega(C_i^n) \to 0$ as $n \to \infty$, then there exists an agent whose demand for commodity C_i^n goes to infinity as $n \to \infty$.

Under distributed SPC, the proof of Theorem 7 holds unchanged.

Compare the import of SPC versus distributed SPC to appreciate the greater realism of this latter. For the explanatory example in the introduction, SPC requires that there is an agent who, given any classification, might change his choice if he is offered another type of wine using a purer selection of grapes; distributed SPC requires only that, for any classification, there is some agent willing to.

References

[1] C.D. Aliprantis and O. Burkinshaw (1998), *Principles of Real Analysis*, London: Academic Press.