

# EUTERPE (EUropean TERm Premium Estimation)

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## Technical Document

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# 1 The Model

## 1.1 State Variables

We assume that the economy is driven by five latent factors, three of which are country-specific factors and two are eurozone global factors. In particular, the country-specific factors are: (i) a variance factor, denoted  $v$ , that accounts for dynamics of the conditional volatility of the other two variables; (ii) a level factor  $\ell$ , that approximately corresponds to the level of yields (i.e., the first principal component of yields); (iii) a slope factor  $s$ , that proxies the slope of the yield curve (i.e., the second principal component of yields). The two global factors are represented by the eurozone instantaneous expected inflation rate,  $\pi$ , and expected output growth rate,  $\mu$ . The five state variables are collected in the state vector  $X = (X_1 \ X_2)'$ , where  $X_1$  contains the three country-specific factors,  $X_1 = (v \ \ell \ s)'$ , and  $X_2$  the global factors,  $X_2 = (\pi \ \mu)'$ . The dynamics of the state vector follows a  $A_1(5)$  process of the [Dai and Singleton \(2000\)](#) type  $dX_t = K(\Theta - X_t)dt + \Sigma\sqrt{\Xi_t}dZ_t$ , which can also be written as:

$$\begin{pmatrix} dX_{1t} \\ dX_{2t} \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} \Theta_1 - X_{1t} \\ \Theta_2 - X_{2t} \end{pmatrix} dt + \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix} \begin{pmatrix} \Omega_t & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} dZ_{1t} \\ dZ_{2t} \end{pmatrix}, \quad (1)$$

where  $K_{11}$ ,  $K_{12}$ ,  $K_{21}$  and  $K_{22}$  are full matrices,  $\Theta_1$  and  $\Theta_2$  are full vectors,  $\Sigma_{11}$  and  $\Sigma_{22}$  are diagonal matrices,  $\Omega_t$  is a diagonal matrix with all elements equal to  $\sqrt{v_t}$ ,  $I$  is an identity matrix, and  $dZ_{1t}$  and  $dZ_{2t}$  are vectors of independent Brownian motions.

In sum, we model the first factor  $v$  as a square-root process that enters the diffusion term of the other two country-specific, conditionally Gaussian factors,  $\ell$  and  $s$ . Instead, the global factors  $\pi$  and  $\mu$  in  $X_{2t}$  follow a Gaussian process and potentially interact with each other and with the country-specific factors through the drift term. We assume that they are linked to the exogenously-given price level  $p$  and the real production output  $q$  through the process:

$$dM_t = X_{2t}dt + \Sigma_M dZ_{Mt}, \quad (2)$$

where  $dM_t = \left(\frac{dp_t}{p_t} \ \frac{dq_t}{q_t}\right)$ ,  $\Sigma_M$  is a diagonal matrix and  $dZ_{Mt}$  a vector of independent Brownian motions.

We characterize the dynamics under the risk-adjusted probability measure  $\mathbb{Q}$  by using an ‘‘essentially affine’’ specification of the instantaneous market price of risk of the [Duffee \(2002\)](#) type  $\Psi_t = \sqrt{\Xi_t} (\Lambda_0 + \Lambda_1 X_t)$ , i.e.:

$$\begin{pmatrix} \Psi_{1t} \\ \Psi_{2t} \end{pmatrix} = \begin{pmatrix} \Omega_t^{-1} & 0 \\ 0 & I \end{pmatrix} \left[ \begin{pmatrix} \Lambda_{01} \\ 0 \end{pmatrix} + \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} \right], \quad (3)$$

where  $\Lambda_{01}$  is a full vector and  $\Lambda_{11}$  and  $\Lambda_{12}$  are full matrices.

We then impose the constraint that the global factors in  $X_{2t}$  are unspanned, i.e., they affect short-rate expectations and risk premia in an exactly offsetting way and, therefore, influence the dynamics of bond yields under the historical measure but not under the risk-adjusted probability measure (see, for example, [Duffee \(2011\)](#) and [Joslin et al. \(2014\)](#)). Such constraint requires (i) that the instantaneous interest rate  $r_t$  does not depend on  $X_{2t}$ :

$$r_t = \delta_0 + \begin{pmatrix} \delta'_1 & 0 \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}, \quad (4)$$

where  $\delta_0$  is a constant and  $\delta_1$  is a full vector, and that (ii):

$$\Lambda_{12} = -\Sigma_{11}^{-1}K_{12}. \quad (5)$$

The second constraint implies that in the risk-adjusted process of the state vector  $X$ , i.e.,  $dX_t = (\tilde{K}\tilde{\Theta} - \tilde{K}X_t)dt + \Sigma\sqrt{\tilde{\Xi}_t}d\tilde{Z}_t$ , where  $d\tilde{Z}_t = dZ_t + \Psi_t dt$ , the drift of the country-specific factors in  $X_{1t}$  does not depend on the global factors in  $X_{2t}$ :

$$\begin{aligned} \begin{pmatrix} dX_{1t} \\ dX_{2t} \end{pmatrix} &= \left[ \begin{pmatrix} K_{11}\Theta_1 + K_{12}\Theta_2 - \Sigma_{11}\Lambda_{01} \\ K_{21}\Theta_1 + K_{22}\Theta_2 \end{pmatrix} - \begin{pmatrix} K_{11} + \Sigma_{11}\Lambda_{11} & 0 \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} \right] dt \\ &\quad + \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix} \begin{pmatrix} \Omega_t & 0 \\ 0 & \mathbf{I} \end{pmatrix} \begin{pmatrix} d\tilde{Z}_{1t} \\ d\tilde{Z}_{2t} \end{pmatrix} \end{aligned} \quad (6)$$

## 1.2 Yields and Yield Components

The equilibrium price of a unit discount bond with time to maturity  $\tau$  at time  $t$ ,  $P_t(\tau)$ , has an exponentially affine closed-form solution, namely:

$$P_t(\tau) = \exp \{A(\tau) - B'(\tau)X_t\}, \quad (7)$$

where  $A(\tau)$  and  $B(\tau)$  solve a system of ordinary differential equations (see, for example, [Piazzesi \(2010\)](#)) and, because of the unspanned nature of the global factors, we have  $B'(\tau) = (B'_1(\tau), 0)$ . The term structure of interest rates is therefore affine in the country-specific factors and the time- $t$  yield on a  $\tau$ -maturity zero coupon bond,  $Y_t(\tau)$ , can be expressed as:

$$Y_t(\tau) = a(\tau) + b'(\tau)X_t, \quad (8)$$

where  $a(\tau) = -A(\tau)/\tau$  and  $b'(\tau) = B'(\tau)/\tau = (b'_1(\tau), 0)$ .

The diffusion term in the risk-adjusted dynamics of  $X$  is a function of  $v$ , which implies that yield volatilities are time-varying and are driven by that factor. In particular, the time- $t$

instantaneous variance of changes in a  $\tau$ -maturity yield,  $V_t(\tau)$ , is affine in  $v$  and is given by:

$$V_t(\tau) = b'(\tau) (\Sigma \Xi_t \Sigma') b(\tau). \quad (9)$$

Equation (8) relates yields with the state vector through a linear function, whose coefficients embed both a risk-adjustment and the expectation of the future path of the short rate, plus a Jensen inequality term. To see this more formally, we follow [Berardi et al. \(2019\)](#) and consider the time- $t$  instantaneous forward rate for date  $t + \tau$ ,  $f_t(\tau) = \frac{1}{P_t(\tau)} \frac{\partial P_t(\tau)}{\partial t}$ , which can be expressed as:

$$f_t(\tau) = r_t + B'(\tau)K(\Theta - X_t) - B'(\tau)\Sigma(\Lambda_0 + \Lambda_1 X_t) - \frac{1}{2}B'(\tau) (\Sigma \Xi_t \Sigma') B(\tau). \quad (10)$$

Rearranging terms, this expression for the instantaneous forward rate can be written as the sum of four components, i.e., the instantaneous (i) expectation of the short rate at  $t + \tau$  under the  $\mathbb{P}$  measure,  $\mathbb{E}^{\mathbb{P}} [r_t(\tau)]$ , (ii) expected excess return on a  $\tau$ -maturity bond,  $e_t(\tau)$ , (iii) convexity,  $c_t(\tau)$ , and (iv) a “duration adjustment”,  $d_t(\tau)$ :

$$f_t(\tau) = \mathbb{E}^{\mathbb{P}} [r_t(\tau)] + e_t(\tau) + c_t(\tau) + d_t(\tau), \quad (11)$$

where

$$\mathbb{E}^{\mathbb{P}} [r_t(\tau)] = r_t + B'_G(\tau)K(\Theta - X_t), \quad (12)$$

$$e_t(\tau) = -B'(\tau)\Sigma(\Lambda_0 + \Lambda_1 X_t), \quad (13)$$

$$c_t(\tau) = -\frac{1}{2}B'(\tau) (\Sigma S_t \Sigma') B(\tau), \quad (14)$$

$$d_t(\tau) = [B'(\tau) - B'_G(\tau)] K(\Theta - X_t). \quad (15)$$

The term  $B_G(\tau)$  is the expression for the  $B(\tau)$  coefficient in a Gaussian framework with constant market price of risk,  $B'_G(\tau) = \delta' K^{-1}(\mathbf{I} - e^{-K\tau})$ . The difference  $(B'(\tau) - B'_G(\tau))$  can be interpreted as an adjustment in the  $B(\tau)$  “duration” coefficient due to both stochastic volatility and the essentially affine specification, which in fact goes to zero if  $S_t = \mathbf{I}$  and  $\Lambda_1 = 0$ .

The convexity term is proportional to the variance of yields and, therefore, is an affine function of the variance factor  $v$ :

$$c_t(\tau) = -\frac{1}{2}\tau^2 V_t(\tau). \quad (16)$$

The difference between the instantaneous forward rate and short rate expectation in equation (11) is generally defined as the “forward term premium” (see, for example, [Dai and Singleton \(2002\)](#) and [Kim and Wright \(2005\)](#)). However, in order to separate from the term

premium component the “mechanical” effect of bond convexity – which can be significant for long-term yields in a stochastic volatility environment –, we adopt a more restricted definition for the time- $t$  forward term premium on a  $\tau$ -maturity bond,  $FTP_t(\tau)$ :<sup>1</sup>

$$FTP_t(\tau) = e_t(\tau) + d_t(\tau). \quad (17)$$

Taking the integral of both sides of equation (17) and dividing by  $\tau$ , we obtain an expression for the yield term premium,  $YTP_t(\tau)$ :

$$YTP_t(\tau) = \frac{1}{\tau} \int_t^{t+\tau} FTP_t(u) du. \quad (18)$$

Similarly, we define the average expected short rate between time  $t$  and  $t + \tau$  as  $ESR_t(\tau) = \frac{1}{\tau} \int_t^{t+\tau} \mathbb{E}^{\mathbb{P}} [r_t(u) du]$  and the average convexity between  $t$  and  $t + \tau$  as  $CVX_t(\tau) = \frac{1}{\tau} \int_t^{t+\tau} c_t(u) du$ . The yield on a  $\tau$ -maturity zero coupon bond can thus be written as the sum of these three elements:

$$Y_t(\tau) = ESR_t(\tau) + YTP_t(\tau) + CVX_t(\tau). \quad (19)$$

Therefore, different from other term structure models (see, for example, [Kim and Wright \(2005\)](#) and [Adrian et al. \(2013\)](#)), we decompose yields by considering the convexity term as distinct from the other two components.

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<sup>1</sup>Empirically, we observe that the term  $d_t(\tau)$  in equation (17) is relatively low in size and therefore has a limited impact on the estimated forward term premium.

## 2 Data and Estimation

### 2.1 Data

The model is estimated using monthly data on country-specific yields and yield volatilities and data on eurozone macroeconomic expectations starting in January 2000. Ten countries of the euro area are considered: Germany, France, Netherlands, Austria, Finland, Belgium, Italy, Spain, Portugal and Ireland. Moreover, we provide estimates for the Euro Area as a whole. We use yields with maturities between 2 and 10 years and yield variances, which are obtained by calculating the realized within-month variance of daily changes in yields. The data source is Bloomberg for the ten countries and the ECB website for the Euro Area.<sup>2</sup> For macro expectations, we use the average 1- and 5-year ahead forecasts of annual CPI growth and annual real GDP growth rates obtained from the ECB Survey of Professional Forecasters. As these data are available on a quarterly basis, we interpolate the series with a spline technique to derive monthly observations.

### 2.2 Estimation Method

The parameters of the state-space representation of the model are estimated by the quasi maximum likelihood method, with an approximate Kalman filter algorithm being used to calculate the values of the unobserved state variables. The use of approximate linear filtering is necessary in the cases in which the state vector has affine dynamics but is not Gaussian. In this scenario, an approximate transition equation can be obtained by exploiting the existence of an analytical expression of the first two conditional moments of the state vector (see, for example, [Christoffersen et al. \(2014\)](#)).

The estimation is performed in two steps. In the first step, we estimate the model for the Euro Area by fitting fifteen series, i.e., the 2- to 10-year yields, the 5- and 10-year realized yield variances, and the average 1- and 5-year ahead forecasts of inflation and real GDP growth.

In the second step, we keep fixed the parameters estimated for the stochastic processes of the global factors, i.e., the eurozone instantaneous expected inflation rate  $\pi$  and expected output growth rate  $\mu$  (equation (1)) and the related price level  $p$  and real production output  $q$  (equation (2)). Then, we fit separately for each country the eleven country-specific variables, i.e., the 2- to 10-year yields and the 5- and 10-year realized yield variances.

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<sup>2</sup>As the ECB data start only in September 2004, for the period January 2000 to August 2004 the yield curve is calculated as the simple average of the yield curves of the ten countries considered.

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