EUTERPE (EUropean TERm Premium Estimation)

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Technical Document

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1 The Model

1.1 State Variables

We assume that the economy is driven by five latent factors, three of which are country-specific factors and two are eurozone global factors. In particular, the country-specific factors are: (i) a variance factor, denoted v, that accounts for dynamics of the conditional volatility of the other two variables; (ii) a level factor ℓ, that approximately corresponds to the level of yields (i.e., the first principal component of yields); (iii) a slope factor s, that proxies the slope of the yield curve (i.e., the second principal component of yields). The two global factors are represented by the eurozone instantaneous expected inflation rate, π, and expected output growth rate, μ. The five state variables are collected in the state vector $X = (X_1\ X_2)'$, where X_1 contains the three country-specific factors, $X_1 = (v\ \ell\ s)'$, and X_2 the global factors, $X_2 = (\pi\ \mu)'$. The dynamics of the state vector follows a $A_1(5)$ process of the Dai and Singleton (2000) type $dX_t = K(\Theta - X_t)dt + \Sigma\sqrt{\Xi_t}dZ_t$, which can also be written as:

$$\begin{pmatrix} dX_{1t} \\ dX_{2t} \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} \Theta_1 - X_{1t} \\ \Theta_2 - X_{2t} \end{pmatrix} dt + \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix} \begin{pmatrix} \Omega_t & 0 \\ 0 & \mathrm{I} \end{pmatrix} \begin{pmatrix} dZ_{1t} \\ dZ_{2t} \end{pmatrix}, \quad (1)$$

where K_{11}, K_{12}, K_{21} and K_{22} are full matrices, Θ_1 and Θ_2 are full vectors, Σ_{11} and Σ_{22} are diagonal matrices, Ω_t is a diagonal matrix with all elements equal to $\sqrt{v_t}$, I is an identity matrix, and dZ_{1t} and dZ_{2t} are vectors of independent Brownian motions.

In sum, we model the first factor v as a square-root process that enters the diffusion term of the other two country-specific, conditionally Gaussian factors, ℓ and s. Instead, the global factors π and μ in X_{2t} follow a Gaussian process and potentially interact with each other and with the country-specific factors through the drift term. We assume that they are linked to the exogenously-given price level p and the real production output q through the process:

$$dM_t = X_{2t}dt + \Sigma_M dZ_{Mt}, \quad (2)$$

where $dM_t' = \left(\frac{dp_t}{p_t}\ \frac{dq_t}{q_t}\right)$, Σ_M is a diagonal matrix and dZ_{Mt} a vector of independent Brownian motions.

We characterize the dynamics under the risk-adjusted probability measure $\mathbb{Q}$ by using an "essentially affine" specification of the instantaneous market price of risk of the Duffee (2002) type $\Psi_t = \sqrt{\Xi_t^-}\ (\Lambda_0 + \Lambda_1 X_t)$, i.e.:

$$\begin{pmatrix} \Psi_{1t} \\ \Psi_{2t} \end{pmatrix} = \begin{pmatrix} \Omega_t^{-1} & 0 \\ 0 & \mathrm{I} \end{pmatrix} \left[\begin{pmatrix} \Lambda_{01} \\ 0 \end{pmatrix} + \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} \right], \quad (3)$$

where Λ_{01} is a full vector and Λ_{11} and Λ_{12} are full matrices.

We then impose the constraint that the global factors in X_{2t} are unspanned, i.e., they affect short-rate expectations and risk premia in an exactly offsetting way and, therefore, influence the dynamics of bond yields under the historical measure but not under the risk-adjusted probability measure (see, for example, Duffee (2011) and Joslin et al. (2014)). Such constraint requires (i) that the instantaneous interest rate y_t does not depend on X_{2t}:

$$y_t = \delta_0 + \begin{pmatrix} \delta_1' & 0 \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}, \tag{4}$$

where δ_0 is a constant and δ_1 is a full vector, and that (ii):

$$\Lambda_{12} = -\Sigma_{11}^{-1} K_{12}. \tag{5}$$

The second constraint implies that in the risk-adjusted process of the state vector X, $dX_t = (\widetilde{K}\widetilde{\Theta} - \widetilde{K}X_t)dt + \Sigma\sqrt{\Xi_t}d\widetilde{Z}_t$, where $d\widetilde{Z}_t = dZ_t + \Psi_t dt$, the drift of the country-specific factors in X_{1t} does not depend on the global factors in X_{2t}:

$$\begin{pmatrix} dX_{1t} \\ dX_{2t} \end{pmatrix} = \left[\begin{pmatrix} K_{11}\Theta_1 + K_{12}\Theta_2 - \Sigma_{11}\Lambda_{01} \\ K_{21}\Theta_1 + K_{22}\Theta_2 \end{pmatrix} - \begin{pmatrix} K_{11} + \Sigma_{11}\Lambda_{11} & 0 \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} \right] dt + \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix} \begin{pmatrix} \Omega_t & 0 \\ 0 & \mathrm{I} \end{pmatrix} \begin{pmatrix} d\widetilde{Z}_{1t} \\ d\widetilde{Z}_{2t} \end{pmatrix} \tag{6}$$

1.2 Yields and Yield Components

The equilibrium price of a unit discount bond with time to maturity τ at time t has an exponentially affine closed-form solution, namely:

$$F_t(\tau) = \exp\left\{A(\tau) - B'(\tau)X_t\right\}. \tag{7}$$

where $A(\tau)$ and $B(\tau)$ solve a system of ordinary differential equations (see, for example, Piazzesi (2010)) and, because of the unspanned nature of the global factors, we have $B'(\tau) = (B_1'(\tau), 0)$. The term structure of interest rates is therefore affine in the country-specific factors:

$$Y_t(\tau) = a(\tau) + b'(\tau)X_t, \tag{8}$$

where $a(\tau) = -A(\tau)/\tau$ and $b'(\tau) = B'(\tau)/\tau = (b_1'(\tau), 0)$.

The diffusion term in the risk-adjusted dynamics of X is a function of v, which implies that yield volatilities are time-varying and are driven by that factor. In particular, the time t term structure of the instantaneous variance of yield changes is affine in v and is given by:

$$V_t(\tau) = b'(\tau) \left(\Sigma\, \Xi_t \Sigma'\right) b(\tau). \tag{9}$$

Equation (8) relates yields with the state vector through a linear function, whose coefficients embed both a risk-adjustment and the expectation of the future path of the short rate, plus a Jensen inequality term. To see this more formally, we follow Berardi et al. (2019) and consider the time-t instantaneous forward rate for date $t+\tau$, $f_t(\tau) = \frac{1}{F_t(\tau)} \frac{\partial F_t(\tau)}{\partial t}$, which can be expressed as:

$$f_t(\tau) = y_t + B'(\tau) K(\Theta - X_t) - B'(\tau)\Sigma(\Lambda_0 + \Lambda_1 X_t) - \frac{1}{2} B'(\tau) \left(\Sigma\, \Xi_t \Sigma'\right) B(\tau). \tag{10}$$

Combining equations (1) and (7), and taking the relevant derivatives we obtain that the instantaneous forward rate can also be written as the sum of four components, i.e., the instantaneous (i) expectation of the short rate under the $\mathbb{P}$ measure, (ii) expected excess return, (iii) convexity term, and (iv) duration adjustment term:

$$f_t(\tau) = \mathrm{E}^{\mathbb{P}}\left[y_t(\tau)\right] + e_t(\tau) + c_t(\tau) + d_t(\tau). \tag{11}$$

The analytical expressions for the four components are given, respectively, by:

$$\mathrm{E}^{\mathbb{P}}\left[y_t(\tau)\right] = y_t + B_G^{\mathbb{P}\prime}(\tau) K(\Theta - X_t), \tag{12}$$

where $B_G^{\mathbb{P}\prime}(\tau) = \delta' K(I - e^{-K\tau})$ is the expression for the $B(\tau)$ coefficient under the $\mathbb{P}$ measure in a Gaussian formulation of the dynamics of the state variables,

$$e_t(\tau) = -B'(\tau)\Sigma(\Lambda_0 + \Lambda_1 X_t), \tag{13}$$

$$c_t(\tau) = -\frac{1}{2} B'(\tau) \left(\Sigma\, S_t \Sigma'\right) B(\tau), \tag{14}$$

$$d_t(\tau) = (B'(\tau) - B_G^{\mathbb{P}\prime}(\tau)) K(\Theta - X_t). \tag{15}$$

The convexity term is proportional to the variance of yields and, therefore, is an affine function of the variance factor v:

$$c_t(\tau) = -\frac{1}{2} \tau^2 V_t(\tau). \tag{16}$$

We define the difference between the instantaneous forward rate and short rate expectation in equation (11), net of the convexity effect, as the "forward term premium":

$$FTP_t(\tau) = e_t(\tau) + d_t(\tau). \tag{17}$$

Taking the integral of both sides of equation (17) and dividing by τ, we obtain an expression for the yield term premium:

$$TP_t(\tau) = \frac{1}{\tau}\int_t^{t+\tau} FTP_t(u)du. \tag{18}$$

Therefore, taking the integral of both the left-hand side and the right-hand side of equation (11), we obtain the yield on a τ maturity zero coupon bond as the sum of the time t expectation of the average short rate, the term premium and the average convexity from t to $t+\tau$:

$$Y_t(\tau) = ES_t^{\mathbb{P}}(\tau) + TP_t(\tau) + CX_t(\tau), \tag{19}$$

where the average short rate expectations are defined as $ES_t^{\mathbb{P}}(\tau) = \frac{1}{\tau}\int_t^{t+\tau}\left[\mathrm{E}^{\mathbb{P}}\left[r_t(u)\right]\right]du$ and the average convexity as $CX_t(\tau) = \frac{1}{\tau}\int_t^{t+\tau} c_t(u)du$.

Therefore, different from other term structure models (see, for example, Kim and Wright (2005) and Adrian et al. (2013)), we decompose yields by considering the convexity term as distinct from the other two components.

2 Data and Estimation Method

2.1 Data

The model is estimated using monthly data on country-specific yields and yield volatilities and data on eurozone macroeconomic expectations starting in January 2000. Ten countries of the euro area are considered: Germany, France, Netherlands, Austria, Finland, Belgium, Italy, Spain, Portugal and Ireland. Moreover, we provide estimates for the Euro Area as a whole. We use yields with maturities between 2 and 10 years and yield variances, which are obtained by calculating the realized within-month variance of daily changes in yields. The data source is Bloomberg for the ten countries and the ECB website for the Euro Area.[1] For macro expectations, we use the average 1-, 2- and 5-year ahead forecasts of annual CPI growth and annual real GDP growth rates obtained from the ECB Survey of Professional Forecasters. As these data are available on a quarterly basis, we interpolate the series with a spline technique to derive monthly observations.

2.2 Estimation Method

The parameters of the state-space representation of the model are estimated by the quasi maximum likelihood method, with an approximate Kalman filter algorithm being used to calculate the values of the unobserved state variables. The use of approximate linear filtering

[1] As the ECB data start only in September 2004, for the period January 2000 to August 2004 the yield curve is calculated as the simple average of the yield curves of the ten countries considered.

is necessary in the cases in which the state vector has affine dynamics but is not Gaussian. In this scenario, an approximate transition equation can be obtained by exploiting the existence of an analytical expression of the first two conditional moments of the state vector (see, for example, Christoffersen et al. (2014)).

The estimation is performed separately for each country using the twelve country-specific series, i.e., 2- to 10-year yields and realized variance of yield changes at the 2-, 5-, and 10-year maturities, and the two global series, i.e., the eurozone inflation and real GDP growth forecasts. We obtain the corresponding observation equations by adding to each model-implied expression a normally distributed and homoskedastic error.

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