Utility-optimal infra-marginal pari-mutuel bets: Monotone utilities

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Abstract. When the winning probabilities for a horse race with pari-mutuel payoffs are known, the bettor has to decide how to bet on individual horses. The problem with limited exposure is split into two problems: one with free exposure and one with constrained exposure. For the former, we prove that rational bettors will bet on under-bet horses and that the optimal set of horses depends only on the objective winning probabilities and is independent of the utility function. For the later, we prove that rational bettors will also bet on under-bet horses.

Keywords. Optimal parimutuel bets, Convex optimization, Criteria for active and non-active one-sided constraints, Optimal set of horses, Explicit solutions.

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1 Introduction

When a bettor knows the true winning probabilities for a horse race with pari-mutuel payoffs (a race in which the winnings are proportional to the bets), the bettor needs to decide how much to bet on each horse. In a groundbreaking paper [1], this decision problem was reduced to the maximization of the expected value of the logarithm of the growth factor, using the logarithmic utility function. The author provided explicit solutions for two-horse races and for races with many horses under the assumption of no track-take. Explicit solutions for many horses

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with track-take were given in [2], and later a simpler solution was provided in [3]. In their work, [4] modified the Kelly criterion for the case of a bettor who assumes a Beta prior distribution for the probability that the bettor selects the winning horse.

Studies [5] and [6], using simulations of a two-horse model, concluded that unrestricted betting causes significant oscillations in the bettor's wealth. To mitigate this issue, reference [7] suggested that the bettor should place bets on the same horses as if the total exposure F were unrestricted, but then proportionally reduce all bets to comply with the restriction.

Studies [1–7] share common limitations: they are restricted to logarithmic utility functions and focus on betting without any restrictions on the total fraction bet on multiple horses. To the best of the authors' knowledge, this paper is the first to extend the analysis to general utility functions and to consider constraints on total betting exposure.

This paper addresses the aforementioned limitations by considering general utility functions that are increasing and concave, while imposing constraints on total exposure F. We demonstrate how to determine the optimal bets under these conditions. We assume that the bettor's betting and risk preferences are described by a utility function $U(g)$ with the properties given in the assumption below.

Assumption 1 (Properties of the Utility Functions) *The utility function $U(g)$ is defined on $(0,+\infty)$ (and possibly on a larger set, such as $(-\infty,+\infty)$). The function U is differentiable, $\frac{dU}{dg} > 0$, $0 \in \overline{Range\left(\frac{dU}{dg}\right)}$, and $\frac{d^2U}{dg^2} < 0$, hence U is strictly concave down. Under this assumption, $\frac{dU}{dg}$ is decreasing, and therefore invertible.*

The average utility of the growth factor is given by

$$G(f_1, f_2, f_3, \ldots, f_H) = \sum_{h=1}^{H} p_h \cdot U(g_h(f_h)) = \sum_{h=1}^{H} p_h \cdot U\left(1 - F + D \cdot \frac{f_h}{\beta_h}\right). \quad (1)$$

The probabilistic interpretation when $U(g) = \ln(g)$ is provided in [1] and [3]. Since U is a concave down (convex) function, $G(f_1, f_2, f_3, \ldots, f_H)$ must also be concave down, as it is a convex combination of concave down functions, and the set of the admissible fractions $(f_1, f_2, f_3, \ldots, f_H)$ determined by constraints (2), (3), and (4), is a convex set with a vertex at $(0, 0, 0, \ldots, 0)$.

In this paper, which is, to the best of our knowledge, the first in this field to admit general utility functions, we solve the problem with limited exposure (Problem 1) for a bettor whose betting and risk preferences are described by a concave-down utility function. Since this problem is inconvenient to deal with, we analyze two other problems, Problem 2 with free exposure and Problem 3 with constrained exposure. For the problem with free exposure, we obtain the following results: Rational bettors do not always bet, they do not bet on all the horses, they choose their optimal horses independently of their utility, and they bet only a limited amount of money (see Sections 5.3 and 5.2).

For the problem with constrained exposure, we show that the set of optimal horses that rational bettors bet on can be determined before the solution is known, and that that set is independent of the utility function when the risk aversion is sufficiently high (see Theorem 1). However, the set of optimal horses depends on the exposure. In some cases the rational bettor bets on all the horses (see Proposition 4). For the problem with free exposure, the solution is reduced to solving a single equation (equation (26)) for the unknown exposure. For the problem with constrained exposure, the solution is reduced to solving a similar equation (equation (42)) but with a different optimal set of horses.

In addition, in Section 6.2 we show that in the case of fractional betting, when one bets less than what is recommended by the solution of the problem with free exposure, it is optimal to decrease the size of the set of horses to bet on. The last section contains two examples for which explicit solutions are known: constant relative risk aversion (CRRA) utility and constant absolute risk aversion (CARA) utility. Logarithmic utility, which is used in [3] and [2], is a special case of CRRA utility. Sections 5.4, 6.3, and 7 contain outlines of algorithms that will be useful to programmers. The results of this paper can also be used to analyze exacta, trifecta, and perfecta betting. For a review of the literature on trifecta betting, see [8].

The framework developed in this paper could be applied to other zero-sum games in which a decision-maker (e.g., an investor, a businessperson, or a policy maker) has to choose how much to invest in each of multiple projects that are available to them and knows the odds of success of each project. Future research could further enhance this framework by incorporating scenarios where the decision-maker does not know the true probabilities of success and must rely on estimations, thereby addressing the uncertainty inherent in real-world decision-making processes.

2 Pari-mutuel Betting and Notation

There are H horses in the race, indexed by $h \in \{1, 2, 3, \ldots, H\}$. The total amount of money bet on horse h is denoted by b_h, and $\beta_h = b_h/B$ is called the *subjective probability* that horse h will win, where $B = \sum_{h=1}^{H} b_h$. We assume that a large number of bettors bet on the horses, that $b_h > 0$ for all h, and that bets are infra-marginal (i.e., the amount of every bet placed on horse h by an individual bettor is small compared to b_h). The true, objective probability that horse h will win is p_h: When horse h wins, the track management takes the amount $tt \cdot B$ (the value tt is usually between 0.1 and 0.2), and in races with pari-mutuel payouts the rest, namely $B \cdot (1 - tt)$, is divided among the bettors who bet on the winning horse, in proportion to the amounts of the bets placed. The dividend rate is $D = (1 - tt) \in (0, 1]$.

The informed bettor has to decide how much to bet on each horse. In this paper, we assume that such a bettor knows the values of the winning probabilities

p_h. Recent studies into the estimation of p_h include [9]. In his ground-breaking paper, [1], Kelly assumed that the bettor has preferences characterized by the logarithmic utility and was able to reduce this decision problem to the maximisation of the expected value of the logarithm of the growth factor. Kelly's idea is as follows: The bettor bets a fraction f_h of his wealth on horse h, and we assume that there is no short-selling of bets, hence

$$f_h \geq 0, \tag{2}$$

and the total fraction of his money that he bets (i.e., his exposure) is

$$F = \sum_{h=1}^{H} f_h. \tag{3}$$

We show that for under-bet horses (defined in Remark 1) $f_h > 0$ and for over-bet horses $f_h = 0$. For under-bet horses we say that constraint (2) is not active, while for over-bet horses we say that it is active. We assume that the bettor restricts his bets so that

$$F \leq \Phi, \tag{4}$$

where $\Phi \in (0, 1]$ is a constant, as he doesn't borrow money and doesn't sell bets short. Simulations (see [5], [6]) suggest that high levels of exposure produce large fluctuations in the total wealth of the bettor. The standard antidote to this is the fractional Kelly criterion (see [7]) where the exposure is controlled directly by a stronger version of (4),

$$F = \Phi, \tag{5}$$

where $\Phi \in (0, 1]$ is a constant. Kelly assumed that bettors' bets are infra-marginal (negligible) in comparison to the total money bet on each horse, hence, assuming that horse h wins, we have that D/β_h is the revenue rate, $D \cdot \frac{p_h}{\beta_h}$ is the expected revenue rate per dollar bet, $D \cdot \frac{f_h}{\beta_h}$ is the bettor's revenue, $D \cdot \frac{f_h}{\beta_h} - F$ is the profit, and

$$g_h(f_h) = 1 - F + D \cdot \frac{f_h}{\beta_h}$$

is the growth factor of the bettor's wealth.

The expression $(1 - F)$ appears many times in the paper, hence we give it a name. Since 1 represents the total wealth, and F is the *betting expense* or the *exposure*, what is left $(1 - F)$ will be called *savings* or the *reserve*.

For a bettor, it is important to know the solution of the following convex optimization problem.

Problem 1 (Limited Exposure) *Given a function U that satisfies Assumption 1, find the fractions $(f_1, f_2, f_3, \ldots, f_H)$ that maximize the value of the concave down function in (1) on the convex set of admissible fractions $(f_1, f_2, f_3, \ldots, f_H)$ determined by constraints (2), (3), and (4).*

According to [10] and [11], the solution of this type of problem, maximisation of a convex function over a compact closed set, always exists and is unique.

3 Outline of the Paper

In Section 4, the problem with limited exposure (Problem 1) introduced in the previous section is split into two problems—Problem 2 (the problem with free exposure) and Problem 3 (the problem with constrained exposure)—as these problems are easier to deal with than Problem 1. Problems 2 and 3 are then converted into the corresponding Lagrangian problems (Problems 4 and 5, respectively). The corresponding systems of Lagrangian equations—(7) and (8) for free exposure, and (9) for constrained exposure—are difficult to solve. For this reason, preliminary analysis is conducted in Section 4.1. For the case of unrestricted (free) exposure, the criterion that determines when it is optimal to place bets is established in Proposition 1. That criterion indicates that rational bettors prefer to bet on under-bet horses.

In Section 5, where Problem 4 (the problem with free exposure) is solved, we begin with Lemma 3, which establishes when it is optimal to bet on at least one horse. Proposition 2 then establishes that for rational bettors it is optimal to bet only on under-bet horses. Under-bet horses are those for which constraint (2) is not active, hence determination of the optimal set of horses is equivalent to the determination of the set of non-active constraints. This allows us to split the Lagrangian equations into two forms: one for under-bet horses (equation (20)) and one for over-bet horses (equation (21)). These equations contain a total of $2H + 2$ unknowns: λ, the λ_h, F, and the f_h. Thus, we need $2H + 2$ equations. There are H equations corresponding to each of the equations (8) and (6). The two additional equations are constraint (3) and equation (7), the equations that determine F and λ, respectively. After splitting the horses into the under-bet ones and the over-bet ones, equation (7) is converted into equation (23), which one can use to determine λ. Equation (20) can easily be solved for f_h; the solution is given by the formulas in (22). These formulas are substituted into constraint (7), and then after the elimination of λ with the help of equation (24), we obtain equation (26), which contains only one of the initial $2H + 2$ unknowns (F). However, it contains one additional unknown (P), the optimal set of horses. It is proven in Theorem 1 that the optimal set (equivalently, the set of non-active constraints $f_h > 0$) depends only on the probability distributions p_h and β_h and is independent of the utility. A very simple criterion (inequalities (31)) is given for the determination of the optimal set. After the determination of the optimal set, it is proven that one can always find a unique solution (the exposure F) of equation (26).

The solution of Problem 5 is constructed in Section 6. Similarly to what was done in earlier sections, the Lagrangian equations are split into equations for under-bet and over-bet horses, and again we obtain an equation (equation (42)) with two unknowns: the optimal set of under-bet horses, P (equivalently, the set of non-active constraints $f_h > 0$) and the Lagrange multiplier λ. The optimal set of horses is determined in Theorem 3.

The structure of the last section (Section 8) follows the structure of the theoretical sections (Sections 5 and 6).

4 Lagrangian Equations

This section presents preparatory material essential for the subsequent analysis. A key insight is that one should place bets only on underbet horses, specifically those with high (unreduced) expected revenues $\frac{p_h}{\beta_h}$; see Remarks 1 and 2. Additionally, a necessary condition for betting is that the highest expected revenue, after accounting for the track take, satisfies the inequality

$$D \cdot \max_h \frac{p_h}{\beta_h} > 1.$$

The optimization problem corresponding to the bettor's objective of maximizing wealth growth with limited exposure is as follows:

Problem 1 (Limited Exposure, $0 \leq F \leq \Phi \leq 1$):

Given a function U that satisfies Assumption 1, find the fractions $(f_1, f_2, f_3, \ldots, f_H)$ that maximize the value of the concave down function in (5) on the convex set of admissible fractions $(f_1, f_2, f_3, \ldots, f_H)$ determined by constraints (1), (2), and (3).

For technical reasons, it is convenient to divide the problem above into two distinct subproblems: **Problem 2** (Free Exposure $0 \leq F \leq 1$) and **Problem 3** (Constrained Exposure $F = \Phi$).

Problem 2 (Free Exposure) *Given a function U that satisfies Assumption 1, find the fractions $(f_1, f_2, f_3, \ldots, f_H)$ that maximize the value of the concave down function (1) on the convex set of admissible fractions $(f_1, f_2, f_3, \ldots, f_H)$ determined by constraints (2), (3), and* $g_h = \left(1 - F + D \cdot \dfrac{f_h}{\beta_h}\right) \in Dom\,(U)$.

This problem is explored in Section 5, "Optimal Bets for Free Exposure." The most significant qualitative result is that the optimal set of horses to bet on is independent of the utility function. Consequently, all wealthy bettors who are indifferent to losses or fluctuations in wealth will bet on the same horses. This result is the primary consequence of Theorem 1. To determine the fractions f_h of wealth to bet, one must solve Equation (26) for the total exposure F. Subsequently, the optimal fractions f_h are given in Equation (25). A more precise summary of this process is provided in Section 5.4, "Summary and an Algorithm for the Problem with Free Exposure."

Problem 3 (Constrained Exposure) *Given a function U that satisfies Assumption 1, find the fractions $(f_1, f_2, f_3, \ldots, f_H)$ that maximize the concave function in (1) over the convex set of admissible fractions determined by constraints (2), (3), and (5).*

This problem is discussed in Section 6, "The Optimal Bets for Constrained Exposure." In this case, the optimal set of horses depends on both the utility function U and the chosen level of exposure Φ. Initially, the Lagrange multiplier is determined, after which the betting fractions are calculated. A precise summary of this process is provided in Theorem 4.

According to [10] and [11], the solution to such problems—which involve the maximization of a convex function over a compact, closed set—always exists and is unique.

If the solution of Problem 2 satisfies constraint (4), then it is also the solution of Problem 1. If it does not satisfy (4), then the solution of Problem 3 is the solution of Problem 1.

From the theory developed in [11], it follows that maximisation problems (such as Problems 2 and 3) are equivalent to the corresponding problems in Lagrangian form. For Problem 2, we construct the Lagrangian function

$$L\left(f_1,\ldots,\,\lambda_1,\ldots,\,\lambda\right)=\sum_{h=1}^{H}p_h\cdot U\left(g_h\left(f_h\right)\right)+\lambda\cdot\left(F-\sum_{h=1}^{H}f_h\right)+\sum_{h=1}^{H}\lambda_h\cdot f_h,$$

with constraints (2) and (3) and the KKT conditions

$$f_h\geq 0,\ \lambda_h\geq 0,\ \lambda_h\cdot f_h=0. \tag{6}$$

The corresponding Lagrangian equations are

$$0 \quad =\frac{\partial L}{\partial F}=-\sum_{h=1}^{H}p_h\cdot\frac{dU}{dg}\left(1-F+D\cdot\frac{f_h}{\beta_h}\right)+\lambda \tag{7}$$

$$0=\frac{\partial L}{\partial f_h}=p_h\cdot\frac{dU}{dg}\left(1-F+D\cdot\frac{f_h}{\beta_h}\right)\cdot\frac{D}{\beta_h}-\lambda+\lambda_h \tag{8}$$

This allows one to replace Problem 2 by the equivalent Lagrangian problem below.

Problem 4 (Free Exposure) *Given a function U that satisfies Assumption 1, find the solution $(f_1,f_2,\ldots,f_H,\ \lambda_1,\lambda_2,\ldots,\lambda_H,\lambda)$ of Lagrangian equations (7) and (8) that satisfies constraints (2), (3), and (6).*

For Problem 3, we construct the Lagrangian function

$$L\left(f_1,\ldots,\,\lambda_1,\ldots,\,\lambda\right)=\sum_{h=1}^{H}p_h\cdot U\left(g_h\left(f_h\right)\right)+\lambda\cdot\left(\Phi-\sum_{h=1}^{H}f_h\right)+\sum_{h=1}^{H}\lambda_h\cdot f_h,$$

where $g_h\left(f_h\right)=\left(1-\Phi+D\cdot\frac{f_h}{\beta_h}\right)$ and $\Phi\in(0,1]$ is a constant, with constraints (2) and (3) and the KKT conditions given in (6). The corresponding Lagrangian equations are

$$0=\frac{\partial L}{\partial f_h}=p_h\cdot\frac{dU}{dg}\left(1-\Phi+D\cdot\frac{f_h}{\beta_h}\right)\cdot\frac{D}{\beta_h}-\lambda+\lambda_h. \tag{9}$$

This allows one to replace Problem 3 by the equivalent Lagrangian problem below.

Problem 5 (Constrained Exposure) *Given a function U that satisfies Assumption 1, find the solution $(f_1, f_2, f_3, ..., f_H,\ \lambda_1, \lambda_2, \lambda_3, ..., \lambda_H,\ \lambda)$ of Lagrangian equations (9) that satisfies constraints (2), (3), (5), and (6).*

Sections 5 and 6 are devoted to the analysis of the two Lagrangian problems, Problems 4 and 5. The theory is illustrated with the construction of the explicit solutions for CRRA and CARA utilities in Section 8.

4.1 Important Inequalities

The results obtained in this section, in the form of necessary inequalities, permit us to simplify the analysis of the Lagrangian equations. We begin by establishing some basic inequalities that apply to both Lagrangian problems.

The bettor is interested only in bets that produce a better outcome than not betting. The latter produce $g_h(0) = 1$ for all h; therefore, the optimal fractions $(f_1, f_2, f_3, \ldots, f_H)$ should satisfy

$$G(f_1, \ldots, f_H) = \sum_{h=1}^{H} p_h \cdot U\left(1 - F + D \cdot \frac{f_h}{\beta_h}\right) > G(0, \ldots, 0) = U(1) \qquad (10)$$

Ultimately, the bettor wants to solve Problem 3 and find the bets that produce the maximum value of $G(f_1, f_2, f_3, \ldots, f_H)$ under constraints (2), (3), and (4).

It is possible that no selection of bets satisfies (10). The proposition below provides a simple necessary and sufficient condition for (10) to hold. If this condition is not satisfied, it is optimal not to bet on any horse.

Proposition 1. *Rational bettors will choose to bet, that is, (10) holds, if and only if*

$$D \cdot \max_h \left(\frac{p_h}{\beta_h}\right) > 1 \qquad (11)$$

(i.e., if and only if the maximum expected revenue rate exceeds 1).

Proof. A necessary and sufficient condition for (10) is existence of some selection of fractions $(f_1, f_2, f_3, \ldots, f_H)$ for which $G(f_1, f_2, f_3, \ldots, f_H) > U(1)$. A necessary and sufficient condition for this is existence of a direction $[\phi_1, \phi_2, \ldots, \phi_h, \ldots, \phi_H, \phi_F]$, where all the components are nonnegative and $\phi_F = \sum_{h=1}^{H} \phi_h$ corresponds to (3), for which the directional derivative $\frac{\partial G}{\partial [\ldots, \phi_h, \ldots, \phi_F]}(0) > 0$:

$$\begin{aligned} \frac{\partial G}{\partial [\ldots, \phi_h, \ldots, \phi_F]}(0) &= \frac{\partial G}{\partial f_h}(0) \cdot \phi_h + \frac{\partial G}{\partial F}(0) \cdot \phi_F \\ &= \frac{\partial G}{\partial f_h}(0) \cdot \phi_h + \frac{\partial G}{\partial F}(0) \cdot \sum_{h=1}^{H} \phi_h \\ &= \frac{dU}{dg}(1) \cdot \sum_{h=1}^{H} \left(D \cdot \frac{p_h}{\beta_h} - 1\right) \cdot \phi_h. \end{aligned}$$

We want to show that this expression is positive if $\phi_h \geq 0$ for all $h \in \{1, 2, 3, \ldots, H\}$. A necessary and sufficient condition for this is $\max_h \left(\frac{dU}{dg}(1) \cdot \left(D \cdot \frac{p_h}{\beta_h} - 1 \right) \right) > 0$. This condition is equivalent to (11) since $\frac{dU}{dg}(1) > 0$. □

This proposition asserts that it is unreasonable to bet if no horse has an expected revenue rate exceeding 1. Often it is optimal to bet on only some (i.e., not all) of the horses. In order to describe the optimal set of horses, P, we use a lemma from [2]. For the convenience of the reader, we quote it here without proof.

Lemma 1. *If* $[f_1, f_2, f_3, \ldots, f_H,\ \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_H,\ \lambda]$ *is the solution of Problem 4 or of Problem 5, then the following hold:*

(a) If $\lambda > 0$*, and if* $\frac{p_i}{\beta_i} > \frac{p_j}{\beta_j}$ *and* $f_j > 0$ *(horse* j *is in the optimal set), then also* $f_i > 0$ *(horse* i *is in the optimal set).*

(b) If $\frac{p_i}{\beta_i} \leq \frac{p_j}{\beta_j}$ *and* $f_j = 0$ *(horse* j *is not in the optimal set), then also* $f_i = 0$ *(horse* i *is not in the optimal set).*

Remark 1. Lemma 1 makes it convenient to reorder the horses so that the sequence of unreduced expected revenues $\frac{p_1}{\beta_1}, \frac{p_2}{\beta_2}, \frac{p_3}{\beta_3}, \ldots, \frac{p_H}{\beta_H}$ is non-increasing, and that $\frac{p_i}{\beta_i}$ is earlier in the sequence than $\frac{p_j}{\beta_j}$ if $\frac{p_i}{\beta_i} = \frac{p_j}{\beta_j}$ and $p_i > p_j$. The horses that a rational bettor would bet on are those with the larger values of p_i/β_i, hence they are the ones that will be called under-bet, while the horses with the smaller values of p_i/β_i will be called over-bet. From now on it is always assumed that the horses are ordered in this way.

The basis for determination of the optimal set P of horses in Theorems 1 and 3 (in Sections 5 and 6, respectively) is the following lemma:

Lemma 2. *If* $[f_1, f_2, f_3, \ldots, f_H,\ \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_H,\ \lambda]$ *is the solution of Problem 4 or of Problem 5, then*

$$h \in P \text{ if and only if } \lambda < \frac{dU}{dg}(1-F) \cdot \frac{p_h}{\beta_h} \cdot D, \tag{12}$$

$$h \notin P \text{ if and only if } \lambda \geq \frac{dU}{dg}(1-F) \cdot \frac{p_h}{\beta_h} \cdot D. \tag{13}$$

Proof. $h \in P$ implies that $f_h > 0$. In that case, KKT conditions (6) imply that $\lambda_h = 0$, hence Lagrangian equation (8) reduces to $\lambda = p_h \frac{dU}{dg}\left(1 - F + D\frac{f_h}{\beta_h}\right)\frac{D}{\beta_h}$. This implies (12), since $\frac{dU}{dg}$ is assumed to be a decreasing function. $h \notin P$ implies that $f_h = 0$. In that case, $\lambda_h \geq 0$ thanks to KKT conditions (6), hence Lagrangian equation (8) reduces to $\lambda \geq p_h \cdot \frac{dU}{dg}(1-F) \cdot \frac{D}{\beta_h}$, which is (13). □

Remark 2. When the horses are reordered as indicated in Remark 1, Lemma 1 and the fact that $\lambda > 0$ imply that the optimal set P can be described as follows: There is a nonnegative integer $\chi \le H$ such that the optimal set of horses (those with non-active constraints $f_h > 0$) is of the form $P = \{1, 2, 3, \ldots, \chi\}$ (P is empty if $\chi = 0$). We refer to horse χ as the cut-off horse. Lemma 2 above suggests that rational bettors will bet on under-bet horses. Lemma 2 enables us to construct the optimal set of horses. Since the Lagrange multiplier λ quantifies the sensitivity of the optimal utility value with respect to changes in the reserve rate $1 - F$, inequality (12) implies the following: one should bet only on horses for which the expected gain in utility exceeds this sensitivity to decreasing the reserve $1 - F$.

5 Optimal Bets for Free Exposure

We begin with the analysis of Problem 4, since it is simpler than Problem 5. With the help of Lemma 2, we prove that the optimal set of horses is independent of the utility and consists of those for which constraint (2) is not active. The optimal set of horses is determined in Theorem 1. When the optimal set of horses is known, the optimal exposure F should be determined by equation (26), and the solvability of this equation is analyzed in Proposition 3. The optimal fractions f_h are given by (25). If $F \le \Phi$, the solution of Problem 4 (the problem with free exposure) is also the solution of Problem 1 (the problem with limited exposure).

We begin with some helpful facts that simplify the analysis of Lagrangian equations (7) and (8). The following is an extension of inequalities proved in [2] for the case of the logarithmic utility.

Lemma 3. *If $[f_1, f_2, f_3, \ldots, f_H,\ \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_H,\ \lambda]$ is a solution of the problem with free exposure (Problem 4), then*

$$\lambda < \frac{dU}{dg}(1-F) \ \text{ if } f_h > 0 \text{ for some } h \in \{1,2,3,\ldots,H\}, \tag{14}$$

$$\lambda = \frac{dU}{dg}(1-F) = \frac{dU}{dg}(1) \ \text{ if } f_h = 0 \text{ for all } h \in \{1,2,3,\ldots,H\}. \tag{15}$$

If $D = 1$, then $\lambda_h = 0$ for all $h \in \{1, 2, 3, ..., H\}$. If $D \in [0, 1)$, then

$$\lambda > 0 \ \text{ if } \lambda_h > 0 \text{ for some } h \in \{1,2,3,...,H\}\,. \tag{16}$$

Proof. The inequality (14) with $\le$ instead of $<$ follows from (7) and (2), since $\frac{dU}{dg}$ is decreasing. If f_h is positive for at least one horse, the strict concavity of U implies that inequality (14) is strict. In order to derive (16), we multiply equations (8) by β_h, add them up, and obtain

$$\sum_{h=1}^{H} \beta_h \cdot \lambda = D \cdot \sum_{h=1}^{H} p_h \cdot \frac{dU}{dg}\left(1 - F + D \cdot \frac{f_h}{\beta_h}\right) + \sum_{h=1}^{H} \beta_h \cdot \lambda_h .$$

Then thanks to the second condition in (6), together with $\sum_{h=1}^{H} \beta_h = 1$ and (7), we have $(1 - D) \cdot \lambda = \sum_{h=1}^{H} \beta_h \cdot \lambda_h \geq 0$. If $D \in [0, 1)$, this implies that $\lambda > 0$ if $\lambda_h > 0$ for some $h \in \{1, 2, 3, ..., H\}$. If $D = 1$, then all $\lambda_h = 0$, and it is optimal to bet on all the horses. □

The inequalities in Lemma 3 allow us to prove that rational bettors prefer not to bet on all the horses.

Proposition 2. *If* $[f_1, f_2, f_3, \ldots, f_H,\ \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_H,\ \lambda]$ *is the optimal solution of the problem with free exposure (Problem 4), then* $f_h = 0$ *for some* $h \in \{1, 2, 3, ..., H\}$. *If, in addition,*

$$0 \leq \frac{dU}{dg} \left(1 - \Phi \cdot (1 - D)\right) = \frac{dU}{dg} \left(1 - \Phi + \Phi \cdot D\right), \tag{17}$$

then constraint (4) is satisfied.

Proof. If $f_h > 0$ for all $h \in \{1, 2, 3, ..., H\}$, then KKT conditions (6) imply that $\lambda_h = 0$ for all $h \in \{1, 2, 3, ..., H\}$, and this implies that $\lambda = 0$ which is impossible by Lemma 1, hence it is impossible that $f_h > 0$ for all $h \in \{1, 2, 3, ..., H\}$. □

5.1 Split Equations for Free Exposure

The derivative $\frac{dU}{dg}$ is assumed to be a positive, decreasing function on $(0, +\infty)$, and the closure of its range is assumed to include 0. Therefore, $\frac{dU}{dg}(1 - F) \geq \frac{dU}{dg}(1) > 0$ since $F \in (0, 1]$. Proposition 2 implies that it is optimal to bet only on horses belonging to the optimal set P and that P does not contain all the horses. Lemma 3 states that $\lambda > 0$, hence Lemma 2 implies that $P = \{h \mid h \leq \chi\}$ for some cut-off horse $\chi \in \{0, 1, 2, \ldots, H - 1\}$ (see Remark 2), where P is the set of under-bet horses for which constraint (2) is not active. In this section, we assume that the optimal set P of horses for Problem 4 is known, hence the KKT conditions (6) can be split into

$$f_h > 0,\ \lambda_h = 0, \quad h \in P, \tag{18}$$

$$f_h = 0,\ \lambda_h \geq 0, \quad h \notin P, \tag{19}$$

and the Lagrangian equations (8) for the f_h split into

$$\lambda = p_h \cdot \frac{dU}{dg} \left(1 - F + D \cdot \frac{f_h}{\beta_h}\right) \cdot \frac{D}{\beta_h}, \quad h \in P, \tag{20}$$

$$\lambda = D \cdot \frac{dU}{dg} (1 - F) \cdot \frac{p_h}{\beta_h} + \lambda_h, \quad h \notin P. \tag{21}$$

Equations (20) are for horses with non-active constraints (2), and these mean that bets on winning horses should be so chosen that the expected revenue of each horse should be equal the sensitivity to losses.

Equations (21) are for horses with active constraints (2) and these mean that that it is not reasonable to bet on horses with expected gains in utility smaller than the bettors' sensitivity to losses. Equation (20) can easily be solved for f_h:

$$f_h = \frac{\beta_h}{D} \cdot \left[\left(\frac{dU}{dg} \right)^{-1} \left(\frac{\lambda}{D} \cdot \frac{\beta_h}{p_h} \right) - 1 + F \right], \quad h \in P, \; f_h = 0, \quad h \notin P, \tag{22}$$

where $\left(\frac{dU}{dg} \right)^{-1}$ is the inverse of dU/dg.

The Lagrangian equation (7) for F can be rewritten as

$$\lambda = \sum_{h \in P} p_h \cdot \frac{dU}{dg} \left(1 - F + D \cdot \frac{f_h}{\beta_h} \right) + \frac{dU}{dg} (1 - F) \cdot \sum_{h \notin P} p_h. \tag{23}$$

Equation (23) can be transformed as follows: We multiply equations (20) by β_h, sum them over the optimal set of horses, and obtain

$$\lambda \cdot \sum_{h \in P} \beta_h = D \cdot \sum_{h \in P} p_h \cdot \frac{dU}{dg} \left(1 - F + D \cdot \frac{f_h}{\beta_h} \right) = D \cdot \left(\lambda - \frac{dU}{dg} (1 - F) \cdot \sum_{h \notin P} p_h \right),$$

hence (23) becomes

$$\lambda \cdot \left(D - \sum_{h \in P} \beta_h \right) = D \cdot \frac{dU}{dg} (1 - F) \cdot \sum_{h \notin P} p_h. \tag{24}$$

Now equations (22) take the form

$$f_h = \frac{\beta_h}{D} \cdot \left[\left(\frac{dU}{dg} \right)^{-1} \left(\frac{dU}{dg} (1 - F) \cdot \frac{\sum_{k \notin P} p_k}{D - \sum_{k \in P} \beta_k} \cdot \frac{\beta_h}{p_h} \right) - 1 + F \right], \; h \in P. \tag{25}$$

Remark 3. Formulas (25) hold only when

$$\left(\frac{dU}{dg} (1 - F) \cdot \frac{\sum_{k \notin P} p_k}{D - \sum_{k \in P} \beta_k} \cdot \frac{\beta_h}{p_h} \right) \in \mathit{Range} \left(\frac{dU}{dg} \right).$$

This is ensured by assuming that $0 \in \overline{\mathit{Range} \left(\frac{dU}{dg} \right)}$, since two later results (inequality (29) and the inequality on the right side in (34)) imply that

$$\frac{\sum_{k \notin P} p_k}{D - \sum_{k \in P} \beta_k} \cdot \frac{\beta_h}{p_h} \in (0, 1).$$

Formulas (25) still contain one unknown (F), defined by (3). In order to find an appropriate value for F, we add up equations (25) over the optimal set of horses and obtain the following equation for F:

$$0 = \Omega(F) \tag{26}$$

$$= \sum_{h\in P} \beta_h \left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{dU}{dg}(1-F) \frac{\sum_{k\notin P} p_k}{D - \sum_{k\in P} \beta_k} \cdot \frac{\beta_h}{p_h} \right) - 1 + F \right] - D{\cdot}F \tag{27}$$

This equation contains two unknowns, the exposure level F and the optimal set of horses P. Next, we will construct P, and then in Section 5.3 we will discuss the solvability of this equation.

5.2 Determination of the Set of Optimal Horses

In this section, we show that if the bettor is rational, the optimal set of horses is independent of the utility function U. We assume that horses are reordered as indicated in Remark 1, that is, the sequence of unreduced expected revenues $\frac{p_1}{\beta_1}, \frac{p_2}{\beta_2}, \frac{p_3}{\beta_3}, \ldots, \frac{p_H}{\beta_H}$ is non-increasing. Proposition 2 implies that for Problem 4 the optimal set of horses P does not contain all the horses, and according to Remark 2 there is a cut-off horse $\chi \in \{0, 1, 2, \ldots, H-1\}$ such that $P = \{1, 2, 3, \ldots, \chi\}$. This section is devoted to the determination of the cut-off horse χ. We begin by stating the following lemma:

Lemma 4. *If the optimal set P for Problem 4 is not empty, then*

$$D \cdot \sum_{h\in P} p_h > \sum_{h\in P} \beta_h. \tag{28}$$

Proof. By Proposition 2, P does not contain all the horses, hence Lemma 3 implies that $\lambda > 0$. Therefore, (24) implies that

$$D > \sum_{h\in P} \beta_h, \tag{29}$$

hence we can use (24) to obtain

$$\frac{\lambda}{D} = \frac{dU}{dg}(1-F) \cdot \frac{\sum_{k\notin P} p_k}{D - \sum_{k\in P} \beta_k}, \tag{30}$$

and now (14) implies that

$$\frac{dU}{dg}(1-F) \cdot \frac{1}{D} > \frac{\lambda}{D} = \frac{dU}{dg}(1-F) \cdot \frac{\sum_{k\notin P} p_k}{D - \sum_{k\in P} \beta_k}.$$

Also, $\dfrac{dU}{dg}(1-F) \geq \dfrac{dU}{dg}(1) > 0$, as stated earlier, hence $\dfrac{1}{D} > \dfrac{\sum_{k \notin P} p_k}{D - \sum_{k \in P} \beta_k} = \dfrac{1 - \sum_{k \in P} p_k}{D - \sum_{k \in P} \beta_k}$, which is equivalent to (28). It is important to note that this condition is independent of the utility function. Therefore, while rational bettors may bet different amounts, they all choose to bet on the same horses. □

With the help of inequality (4), we can prove the most important result of this section:

Theorem 1 (Set of profitable horses for Problem 4: free exposure). *For Problem 4 the cut-off horse* $\chi \in \{0, 1, 2, 3, \ldots, H-1\}$ *defined in Remark 2 is uniquely determined by the two inequalities*

$$T(\chi) < D \leq T(\chi + 1), \tag{31}$$

where the non-decreasing threshold function $T(k)$ *is given by* $T(0) = 0, T(H+1) = \infty$, *and*

$$T(k) = \sum_{i \leq k} \beta_i + \left(1 - \sum_{i \leq k} p_i\right) \cdot \frac{\beta_k}{p_k}, \quad k \in \{1, 2, 3, \ldots, H-1\} \tag{32}$$

The cut-off horse is independent of the utility function, hence we can determine the horses for which constraint (2) is not active, since $T(k)$ *is also independent of the bettor's utility. The cut-off horse is* $\chi > 0$, *that is, the optimal set of horses is not empty, if and only if*

$$D \cdot \frac{p_1}{\beta_1} > 1. \tag{33}$$

Proof. Since $P = \{h \mid h \leq \chi\}$ and the sequence $\dfrac{p_1}{\beta_1}, \ldots, \dfrac{p_H}{\beta_H}$ is non-increasing, condition (12) is equivalent to $\lambda < \dfrac{dU}{dg}(1-F) \cdot \dfrac{p_\chi}{\beta_\chi} \cdot D$, and condition (13) is equivalent to $\lambda \geq \dfrac{dU}{dg}(1-F) \cdot \dfrac{p_{\chi+1}}{\beta_{\chi+1}} \cdot D$. These two conditions for the cut-off horse, with the help of (30), may be written as

$$\frac{dU}{dg}(1-F) \cdot \frac{p_{\chi+1}}{\beta_{\chi+1}} \cdot D \leq D \cdot \frac{dU}{dg}(1-F) \cdot \frac{\sum_{k > \chi} p_k}{D - \sum_{k \leq \chi} \beta_k} < \frac{dU}{dg}(1-F) \cdot \frac{p_\chi}{\beta_\chi} \cdot D\,.$$

Since $D \cdot \dfrac{dU}{dg}(1-F) \neq 0$, these two conditions are equivalent to

$$\frac{p_{\chi+1}}{\beta_{\chi+1}} \leq \frac{\sum\limits_{k>\chi} p_k}{D - \sum\limits_{k\leq\chi} \beta_k} < \frac{p_\chi}{\beta_\chi}. \tag{34}$$

However, it is more convenient to rewrite the inequality on the right side in (34) as the inequality on the left side in (31). Similarly, the inequality on the left side in (34) can be rewritten as

$$D \leq \sum_{h\leq\chi} \beta_h + \frac{\beta_{\chi+1}}{p_{\chi+1}} \cdot \left(1 - \sum_{h>\chi} p_h\right). \tag{35}$$

$T(k)$ satisfies a helpful identity,

$$\begin{aligned} T(k+1) &= \sum_{i\leq k+1} \beta_i + \left(1 - \sum_{i\leq k+1} p_i\right) \cdot \frac{\beta_{k+1}}{p_{k+1}}, \\ &= \sum_{i\leq k} \beta_i + \left(1 - \sum_{i\leq k} p_i\right) \cdot \frac{\beta_{k+1}}{p_{k+1}}, \end{aligned} \tag{36}$$

hence (35) can be rewritten as the inequality on the right side in (31). Thus, we have proved that the two inequalities in (31) indeed characterize the cut-off horse. In order to finish the proof, we need to show the unique solvability of the two inequalities in (31). To prove uniqueness of the solution χ of the two inequalities in (31), it suffices to prove that $T(k)$ given by (32) is non-decreasing in k. Indeed, with the help of identity (36), we can write

$$T(k+1) - T(k) = \left(\frac{\beta_{k+1}}{p_{k+1}} - \frac{\beta_k}{p_k}\right) \cdot \left(1 - \sum_{h\leq k} p_h\right) \geq 0,$$

where the last inequality follows from the fact that $\dfrac{\beta_k}{p_k}$ is non-decreasing in k, thanks to Remark 1. Thus $T(k)$ given by (32) is non-decreasing. In order to prove solvability of the two inequalities in (31), it suffices to prove that $T(H) \geq D$, since $T(k)$ is non-decreasing in k and $T(1) < D$. Indeed,

$$T(H) = \left(1 - \sum_{i\leq H} p_i\right) \cdot \frac{\beta_H}{p_H} + \sum_{i\leq H} \beta_i = 1 \geq D,$$

since $D \in (0,1]$. Also,

$$T(1) = \left(1 - \sum_{i\leq 1} p_i\right) \cdot \frac{\beta_1}{p_1} + \sum_{i\leq 1} \beta_i = (1-p_1) \cdot \frac{\beta_1}{p_1} + \beta_1 = \frac{\beta_1}{p_1},$$

hence $T\left(1\right)=\dfrac{\beta_1}{p_1}<D$ is equivalent to (33). □

Remark 4. If the objective probabilities p_h are all equal to the corresponding subjective probabilities β_h, then $T\left(k\right)=1$ for all k, hence the cut-off horse is $\chi=0$, so the set of optimal horses is empty. If $p_1=1$ and $p_h=0$ for all the other horses, and $\beta_h>0$ for all horses, then $\chi=1$, that is, it is optimal to bet only on the winning horse. Thus, in horse races the track management and informed bettors fleece uninformed bettors.

Remark 5. The cut-off horse χ for Problem 4 can also be characterized by the value of k for which the function $NL(k)$, defined as

$$NL(k)=D\cdot\frac{\sum_{j>k}p_j}{\sum_{j>k}\beta_j-tt}, \tag{37}$$

attains its global minimum on the set $\{0,1,2,3,...,H\}$ at $k=\chi$. This means that $NL\left(\chi-1\right)>NL\left(\chi\right)\leq NL\left(\chi+1\right)$; that is $NL(k)$ decreases until $k=\chi$ and then increases thereafter, as demonstrated in [1]. Alternatively, the value of χ can be determined using an algorithm described in [2] and [3].

The value $NL(k)$ can be interpreted as the expected rate of forfeited gains when betting on the first k horses with the highest expected return rates $D\cdot\frac{p_j}{\beta_j}$. Therefore, the inequalities in (31) are equivalent to the statement that a rational bettor selects the horses to bet on so as to minimize the expected forfeited loss $NL(k)$.

Remark 6. When $D=1$ and the sequence $\dfrac{p_1}{\beta_1},\dfrac{p_2}{\beta_2},\dfrac{p_3}{\beta_3},\ldots,\dfrac{p_H}{\beta_H}$ is non-increasing, the inequalities in (31) imply that the optimal set of horses contains either all horses except horse H.

5.3 The Optimal Bets

When the cut-off horse χ, defined in Remark 2, is determined by solving the two inequalities in (31) and $\chi=0$, it is optimal not to bet at all. When $\chi>0$, it is optimal to bet on at least one horse. However, the formulas (25) for the optimal fractions contain one more unknown (F), which has to be found using equation (26); we discuss its solvability below.

Proposition 3. *Suppose P is not empty. If $\Omega(F^*)<0$ for some $F^*>0$, then equation (26) has a unique solution $F>0$. If, in addition, $\Omega(\Phi)\leq 0$, then $F\leq\Phi$, hence the solution of the free exposure problem is also the solution of the limited exposure problem because it satisfies constraint (4).*

Proof. Both $\dfrac{dU}{dg}$ and $\left(\dfrac{dU}{dg}\right)^{-1}$ are assumed to be decreasing functions. Since $F\geq 0$ in any solution F of Problem 4, strict downward concavity of U implies that $\frac{dU}{dg}(1-F) \geq \frac{dU}{dg}(1) > 0$. Next, we rewrite formula (27) as follows:

$$\Omega(F) = \sum_{h\in P}\beta_h \cdot \left[\left(\frac{dU}{dg}\right)^{-1}\left(\frac{dU}{dg}(1-F)\cdot\frac{\sum_{k\notin P} p_k}{D-\sum_{k\in P}\beta_k}\cdot\frac{\beta_h}{p_h}\right)-1\right] - \left(D-\sum_{h\in P}\beta_h\right)\cdot F.$$

Thanks to this and inequality (28), the argument of $\left(\frac{dU}{dg}\right)^{-1}$ in the formula above is an increasing function of F, hence the expression in the square brackets in this formula is a decreasing function of F, and so is the term $-\left(D-\sum_{h\leq\chi}\beta_h\right)\cdot F$, thanks to inequality (29). Therefore, $\Omega(F)$ given by (27) is a decreasing function of F. This implies uniqueness of the solution F of equation (26) if one exists. If $\Omega(F^*) < 0$ for some $F^* > 0$, then to prove the existence of a solution of (26), it suffices (by the intermediate value theorem for continuous functions) to prove that

$$\Omega(0) = \sum_{h\leq\chi}\beta_h \cdot \left[\left(\frac{dU}{dg}\right)^{-1}\left(\frac{dU}{dg}(1)\cdot\frac{\sum_{k>\chi} p_k}{D-\sum_{k\leq\chi}\beta_k}\cdot\frac{\beta_h}{p_h}\right)-1\right] > 0. \tag{38}$$

The inequality on the right side in (34) implies that

$$\frac{\sum_{k\notin P} p_k}{D-\sum_{k\in P}\beta_k}\cdot\frac{\beta_h}{p_h} < 1,$$

hence

$$\frac{dU}{dg}(1)\cdot\frac{\sum_{k\notin P} p_k}{D-\sum_{k\in P}\beta_k}\cdot\frac{\beta_h}{p_h} < \frac{dU}{dg}(1).$$

Therefore,

$$\left(\frac{dU}{dg}\right)^{-1}\left(\frac{dU}{dg}(1)\cdot\frac{1-\sum_{k\leq\chi} p_k}{D-\sum_{k\leq\chi}\beta_k}\cdot\frac{\beta_h}{p_h}\right) > \left(\frac{dU}{dg}\right)^{-1}\left(\frac{dU}{dg}(1)\right) = 1,$$

since $\left(\frac{dU}{dg}\right)^{-1}$ is a decreasing function. This shows that the expression in the square brackets in (38) is positive, hence (38) is proved, which means that (27) is solvable. Since $\Omega(F) = 0$ and Ω is a decreasing function, $F \leq \Phi$ if $\Omega(\Phi) \leq 0$. □

Remark 7. It is easy to prove that the following inequality is true:

$$\Omega(F) < const - \left(D - \sum_{h \in P} \beta_h\right) \cdot F.$$

This inequality seems to imply that $\Omega(F^*) < 0$ for some $F^* > 0$, especially when $Dom(\Omega(F))$ contains large numbers. However, $(0, \infty)$ is not always a subset of $Dom(\Omega(F))$; for CRRA utilities, for example, $Dom(\Omega(F)) = (0, 1)$. In all the examples in Section 8, the solution of (26) always exists and is positive, and sometimes this solution is greater than 1. However, if a positive solution of (26) does not exist, this implies that the objective function (1) is unbounded on a possibly unbounded convex set described in the formulation of Problem 2. In such a case the solution of Problem 1 happens to be the solution of Problem 3.

5.4 Summary and an Algorithm for the Problem with Free Exposure

The results obtained thus far for Problem 4 can be summarized as follows:

Theorem 2. *The cut-off horse χ defined in Remark 2 for Problem 4 is the solution of the two inequalities in (31). If the two inequalities in (31) have no solution with an optimal set P that is not empty, then $\chi = 0$ and it is optimal not to bet at all, that is, $f_h = 0$ for all horses h. If $\chi > 0$, the optimal set of horses is $P = \{1, 2, 3, \ldots, \chi\}$. The optimal fractions f_h are given by (25), where F is the solution of (26) and $\Omega(F)$ is given by (27). If $\Omega(\Phi) \leq 0$, then $F \leq \Phi$, hence the solution of Problem 2 (the problem with free exposure) is also the solution of Problem 1 (the problem with limited exposure). If $F > \Phi$, the solution of Problem 1 happens to be the solution of Problem 3 (the problem with restricted exposure).*

A practical approach to solving the problem with free exposure—either Problem 4 or Problem 2—is presented in the following pseudocode:

Algorithm 1 (Pseudocode For Free Exposure)

1. *Check whether inequality (33) holds; that is, verify whether $D \cdot \frac{p_1}{\beta_1} > 1$ is satisfied. If it is not satisfied, the optimal solution is $f_h = 0$ for all h, implying that it is rational not to bet at all and to stop. Otherwise, proceed to the next step.*
2. *Construct the threshold function $T(\chi)$ as defined in equation (32):*
   $T(k) = \sum_{i \leq k} \beta_i + \left(1 - \sum_{i \leq k} p_i\right) \cdot \frac{\beta_k}{p_k}, \quad k \in \{1, 2, 3, \ldots, H-1\}$
3. *Find the cut-off horse $\chi \in \{1, 2, 3, \ldots, H-1\}$ that satisfies the two inequalities in (31): $T(\chi) < D \leq T(\chi + 1)$. There is only one such horse χ, since $T(k)$ is non-decreasing in k. Bets are placed only on horses with indices $h \leq \chi$; that is, set the optimal fractions $f_h = 0$ for all $h > \chi$.*

4. *Construct the function $\Omega(F)$ as defined in equation (27):*
$$\Omega\left(F\right)=\sum_{h\in P}\beta_h\left[\left(\frac{dU}{dg}\right)^{-1}\left(\frac{dU}{dg}\left(1-F\right)\frac{\sum_{k\notin P}p_k}{D-\sum_{k\in P}\beta_k}\cdot\frac{\beta_h}{p_h}\right)-1+F\right]-D\cdot F$$
5. *Solve the equation $\Omega(F)=0$ for F.*
6. *If equation $\Omega\left(F\right)=0$ has no solution in the interval $(0,1)$, then stop. In this case, it is optimal to borrow money to place very large bets.*
7. *Compute the optimal fractions f_h for $h\leq\chi$ using equation (25):*
$$f_h=\frac{\beta_h}{D}\cdot\left[\left(\frac{dU}{dg}\right)^{-1}\left(\frac{dU}{dg}\left(1-F\right)\cdot\frac{\sum_{k\notin P}p_k}{D-\sum_{k\in P}\beta_k}\cdot\frac{\beta_h}{p_h}\right)-1+F\right],\ h\in P,$$
and set $f_h=0$ for all $h>\chi$. Then stop.

If $F\leq\Phi$, that is, constraint (4) is satisfied, then this solution is also the solution of Problem 1 (the problem with limited exposure).

6 The Optimal Bets for Constrained Exposure

In this section, we describe a method for finding the optimal set of horses for the problem with constrained exposure. We begin with splitting Lagrangian equations (9).

6.1 Split Equations for Constrained Exposure

In this section, we assume that the optimal set of horses (those for which constraint (2) is not active) for Problem 5 is known, and we split the Lagrangian equations accordingly. In this way we discover the necessary and sufficient conditions (39) and (40) for the optimal set of horses. These conditions are used later (in Section 6.2) to determine the optimal set.

The KKT conditions (6) split into (18) and (19), exactly as was done when solving Problem 4, and the Lagrangian equations (9) for f_h split into

$$\lambda=p_h\cdot\frac{dU}{dg}\left(1-\Phi+D\cdot\frac{f_h}{\beta_h}\right)\cdot\frac{D}{\beta_h},\quad h\in P,\tag{39}$$
$$\lambda=D\cdot\frac{dU}{dg}\left(1-\Phi\right)\cdot\frac{p_h}{\beta_h}+\lambda_h,\quad h\notin P.\tag{40}$$

In these equations, Φ is the constant from constraint (5), and the Lagrange multiplier λ is an additional unknown. For every subset P of the set of all horses and every value of λ, one can calculate the corresponding fractions f_h for all horses h:

$$f_h=\frac{\beta_h}{D}\cdot\left[\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\lambda}{D}\cdot\frac{\beta_h}{p_h}\right)-1+\Phi\right],\quad h\in P,\ f_h=0,\quad h\notin P.\tag{41}$$

Note that P and λ have to be chosen so that KKT conditions (18) and (19) are satisfied.

Constraints (3) and (5) imply that

$$F = \sum_{h \in P} f_h = \sum_{h \in P} \frac{\beta_h}{D} \cdot \left[\left(\frac{dU}{dg} \right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{\lambda}{D} \right) - 1 + \Phi \right],$$

which is more convenient to use when written as follows:

$$\Omega\left(P, \lambda, \Phi\right) = 0, \tag{42}$$

$$\Omega\left(P, \lambda, \Phi\right) = \sum_{h \in P} \beta_h \cdot \left[\left(\frac{dU}{dg} \right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{\lambda}{D} \right) - 1 + \Phi \right] - D \cdot \Phi, \tag{43}$$

and this can be used to find the Lagrange multiplier λ. Since $\left(\frac{dU}{dg} \right)^{-1}$ is a decreasing function, the solution λ is unique whenever it exists, and it exists whenever $Range \left(\frac{dU}{dg} \right)^{-1}$ is sufficiently large, for instance when $Dom \left(\frac{dU}{dg} \right) \supset (0, \infty)$ and a solution must exist, since Problem 5 is about maximisation of a convex function over a convex set and always has a unique solution.

Since λ is the solution of equation (42), and $\left(\frac{dU}{dg} \right)^{-1}$ is a decreasing function, $\lambda > 0$ is equivalent to

$$\Omega\left(\chi, 0, \Phi\right) = \sum_{h \in P} \beta_h \cdot \left[\left(\frac{dU}{dg} \right)^{-1} (0) - 1 + \Phi \right] - D \cdot \Phi > 0,$$

and this is an additional inequality for $\sum_{h \in P} \beta_h$.

Under constraint (5), necessary and sufficient conditions (12) and (13) for the optimal set of horses take the form

$$\frac{\beta_h}{p_h} \cdot \frac{\lambda}{D} < \frac{dU}{dg} \left(1 - \Phi\right), \quad h \in P, \tag{44}$$

$$\frac{dU}{dg} \left(1 - \Phi\right) \leq \frac{\beta_h}{p_h} \cdot \frac{\lambda}{D}, \quad h \notin P. \tag{45}$$

These inequalities make it reasonable to introduce thresholds $\lambda_k(\Phi)$ for $k = 1, ..., H$ as follows:

$$\lambda_k\left(\Phi\right) = D \cdot \frac{p_k}{\beta_k} \cdot \frac{dU}{dg} \left(1 - \Phi\right) \tag{46}$$

Next, we use the thresholds $\lambda_k(\Phi)$ and the function $\Omega\left(P, \lambda, \Phi\right)$ defined in (43) to determine the optimal set of horses.

6.2 The Optimal Set of Horses for Constrained Exposure

Remark 2 implies that the bettor's optimal set of horses is of the form $P = \{h \mid h \leq \chi\}$. The threshold $\lambda_k(\Phi)$ defined by (46) is non-increasing in k, since $\frac{p_k}{\beta_k}$ is non-increasing, thanks to Remark 1. We substitute $\lambda = \lambda_\chi(\Phi)$ and $P = \{h \mid h \leq \chi\}$ into (43) and write the result as

$$\begin{aligned} \Omega\left(\chi, \lambda_\chi(\Phi), \Phi\right) &= \sum_{h \leq \chi} \beta_h \cdot \left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{\lambda_\chi(\Phi)}{D}\right) - 1 + \Phi \right] - D \cdot \Phi, \\ &= \sum_{h \leq \chi} \beta_h \cdot \left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{p_\chi}{\beta_\chi} \cdot \frac{dU}{dg}(1 - \Phi)\right) - 1 + \Phi \right] - D \cdot \Phi \,. \end{aligned} \tag{47}$$

We will use $\Omega\left(\chi, \lambda_\chi(\Phi), \Phi\right)$ to determine the optimal set of horses.

Remark 8. The term in the summation in (47) that corresponds to $h = \chi$ in Ω is always zero, since $\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\beta_\chi}{p_\chi} \cdot \frac{p_\chi}{\beta_\chi} \cdot \frac{dU}{dg}(1 - \Phi)\right) - 1 + \Phi = (1 - \Phi) - 1 + \Phi = 0$, hence

$$\Omega\left(\chi, \lambda_\chi(\Phi), \Phi\right) = \sum_{h \leq (\chi - 1)} \beta_h \cdot \left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{p_\chi}{\beta_\chi} \cdot \frac{dU}{dg}(1 - \Phi)\right) - 1 + \Phi \right] - D \cdot \Phi.$$

Therefore, $\Omega\left(\chi, \lambda_\chi(\Phi), \Phi\right) = \Omega\left(\chi - 1, \lambda_\chi(\Phi), \Phi\right)$. This will be used in the proofs of the next two results.

In order to determine the optimal set of horses, we need the following technical lemma:

Lemma 5. *$\Omega\left(\chi, \lambda_\chi(\Phi), \Phi\right)$, given by 47, is a non-constant, non-decreasing function of χ; there is a unique solution χ of the two inequalities*

$$\Omega\left(\chi, \lambda_\chi(\Phi), \Phi\right) < 0, \tag{48}$$

$$\Omega\left(\chi + 1, \lambda_{\chi+1}(\Phi), \Phi\right) \geq 0, \tag{49}$$

and $\chi \geq 1$. If $\Omega\left(H, \lambda_H(\Phi), \Phi\right) \geq 0$, then $\chi < H$. If $\Omega\left(H, \lambda_H(\Phi), \Phi\right) < 0$, we set $\chi = H$.

Proof. First, with the help of the trick mentioned in Remark 8, we show that $\Omega\left(1, \lambda_1(\Phi), \Phi\right) = \beta_1 \cdot \left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{dU}{dg}(1 - \Phi)\right) - 1 + \Phi \right] - D \cdot \Phi = -D \cdot \Phi < 0$. Therefore, $\chi = 1$ always satisfies (48). Next, we show that Ω is a non-decreasing function of χ. Indeed,

$$\begin{aligned}
&\Omega\left(\chi+1, \lambda_{\chi+1}\left(\Phi\right), \Phi\right) - \Omega\left(\chi, \lambda_{\chi}\left(\Phi\right), \Phi\right) \\
&\qquad = \sum_{h \leq \chi+1} \beta_h \cdot \left(\frac{dU}{dg}\right)^{-1} \left[\left(\frac{\beta_h}{p_h} \cdot \frac{\lambda_{\chi+1}}{D}\right) - 1 + \Phi\right] \\
&\qquad - \sum_{h \leq \chi} \beta_h \cdot \left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{\lambda_{\chi}}{D}\right) - 1 + \Phi\right] \\
&\qquad = \beta_{\chi+1} \cdot \left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{\beta_{\chi+1}}{p_{\chi+1}} \cdot \frac{\lambda_{\chi+1}}{D}\right) - 1 + \Phi\right] \\
&+ \sum_{h \leq \chi} \beta_h \cdot \left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{\lambda_{\chi+1}}{D}\right) - \left(\frac{dU}{dg}\right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{\lambda_{\chi}}{D}\right)\right].
\end{aligned}$$

Thanks to the definition of the thresholds in (46) and the trick mentioned in Remark 8, the first term in the sum above is zero. Moreover, the second term is non-negative: Since λ_χ is non-increasing in χ and $\left(\frac{dU}{dg}\right)^{-1}$ is a decreasing function (as the inverse of a decreasing function dU/dg), we have

$$\left(\frac{dU}{dg}\right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{\lambda_{\chi+1}}{D}\right) \geq \left(\frac{dU}{dg}\right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{\lambda_{\chi}}{D}\right),$$

hence

$$\Omega\left(\chi+1, \lambda_{\chi+1}\left(\Phi\right), \Phi\right) \geq \Omega\left(\chi, \lambda_{\chi}\left(\Phi\right), \Phi\right).$$

This monotonicity implies that if the solution of the two inequalities exists, it must be unique. Since $\Omega\left(1, \lambda_1\left(\Phi\right), \Phi\right) = -D \cdot \Phi < 0$, as proven earlier, we have that $\chi < H$ if $\Omega\left(H, \lambda_H\left(\Phi\right), \Phi\right) \geq 0$. If $\Omega\left(H, \lambda_H\left(\Phi\right), \Phi\right) < 0$, then we set $\chi = H$. □

Now we are ready to describe the optimal set of horses.

Theorem 3 (Set of profitable horses for Problem 5: constrained exposure). *The horse χ determined in Lemma 5 for Problem 5 is the cut-off horse χ defined in Remark 2. More precisely: If the cut-off horse χ satisfies (48) and (49) and we set $P = \{h \mid h \leq \chi\}$ in equation (42) and solve it for λ, then for this χ and λ, inequalities (44) and (45) are satisfied.*

Proof. Let us set $P = \{h \mid h \leq \chi\}$ in (43). Then $\Omega\left(\chi, \lambda_\chi\left(\Phi\right), \Phi\right) < 0$, since χ satisfies (48), where λ_χ is defined by (46) with $k = \chi$. With the help of the trick mentioned in Remark 8 and the identity

$$\left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{\beta_{\chi+1}}{p_{\chi+1}} \cdot \frac{p_{\chi+1}}{\beta_{\chi+1}} \cdot \frac{dU}{dg}\left(1-\Phi\right)\right) - 1 + \Phi\right] = \left[\left(1-\Phi\right) - 1 + \Phi\right] = 0,$$

we obtain

$$\begin{aligned}
&\Omega\left(\chi,\lambda_{\chi+1}\left(\Phi\right),\Phi\right)\\
&\quad=\Omega\left(\chi,\lambda_{\chi+1}\left(\Phi\right),\Phi\right)+\beta_{\chi+1}\cdot\left[\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\beta_{\chi+1}}{p_{\chi+1}}\cdot\frac{p_{\chi+1}}{\beta_{\chi+1}}\cdot\frac{dU}{dg}\left(1-\Phi\right)\right)-1+\Phi\right]\\
&\quad=\Omega\left(\chi+1,\lambda_{\chi+1}\left(\Phi\right),\Phi\right)\geq 0,
\end{aligned}$$

since χ satisfies (49). The two inequalities

$$\Omega\left(\chi,\lambda_{\chi}\left(\Phi\right),\Phi\right)<0\leq\Omega\left(\chi,\lambda_{\chi+1}\left(\Phi\right),\Phi\right)$$

imply that $\lambda_{\chi+1}\leq\lambda<\lambda_{\chi}$, where λ is the solution of equation (42) with $P=\{h\mid h\leq\chi\}$, since $\left(\frac{dU}{dg}\right)^{-1}$ is a decreasing function. Also, $\frac{p_h}{\beta_h}$ is non-increasing in h, by Remark 1; therefore, $\frac{p_\chi}{\beta_\chi}\leq\frac{p_h}{\beta_h}$ and hence $\frac{\beta_h}{p_h}\cdot\frac{p_\chi}{\beta_\chi}\leq 1$ if $h\leq\chi$. This and $\lambda<\lambda_\chi$ imply that

$$\frac{dU}{dg}\left(1-\Phi+D\cdot\frac{f_h}{\beta_h}\right)=\frac{\beta_h}{p_h}\cdot\frac{\lambda}{D}<\frac{\beta_h}{p_h}\cdot\frac{\lambda_\chi}{D}=\frac{\beta_h}{p_h}\cdot\frac{p_\chi}{\beta_\chi}\cdot\frac{dU}{dg}\left(1-\Phi\right)\leq\frac{dU}{dg}\left(1-\Phi\right)$$

for all $h\leq\chi$, which is (44). This implies that $f_h>0$ for all $h\leq\chi$. The inequality $\lambda_{\chi+1}\leq\lambda$, together with (46) for $k=\chi+1$, implies that $D\cdot\frac{p_{\chi+1}}{\beta_{\chi+1}}\cdot\frac{dU}{dg}\left(1-\Phi\right)=\lambda_{\chi+1}\leq\lambda$, hence for all $h>\chi$

$$\frac{dU}{dg}\left(1-\Phi\right)\leq\frac{\beta_h}{p_h}\cdot\frac{p_{\chi+1}}{\beta_{\chi+1}}\cdot\frac{dU}{dg}\left(1-\Phi\right)=\frac{\beta_h}{p_h}\cdot\frac{\lambda_{\chi+1}}{D}\leq\frac{\beta_h}{p_h}\cdot\frac{\lambda}{D},$$

since $\frac{\beta_h}{p_h}\cdot\frac{p_{\chi+1}}{\beta_{\chi+1}}\geq 1$ for all $h>\chi$, as $\frac{p_h}{\beta_h}$ is non-increasing in h, by Remark 1. Thus (45) is proved for all $h>\chi$, which implies that $\lambda_h\geq 0$ for all $h>\chi$, which is the second KKT condition in (6). Therefore, the set $P=\{h\mid h\leq\chi\}$ described in this proof is indeed the optimal set of horses. □

It is interesting to determine the levels of exposure for which one should bet on all the horses and the levels of exposure for which one should bet on only some of them.

Proposition 4. *If*

$$\Omega\left(H,\lambda_H\left(1\right),1\right)<0,\tag{50}$$

where $\Omega\left(\chi,\lambda_\chi\left(\Phi\right),\Phi\right)$ is given by 47, then there exists a solution $\Phi_H\in(0,1)$ of the equation

$$\Omega\left(H,\lambda_H\left(\Phi\right),\Phi\right)=0\tag{51}$$

such that the set of optimal horses includes all the horses if $\Phi>\Phi_H$. If $\beta_h=p_h$ for all h, or these are close (in the sense of the Kullback–Leibler metric or any other metric), then $\Omega\left(H,\lambda_H\left(\Phi\right),\Phi\right)<0$ for all Φ, hence the set of optimal horses includes all the horses for all $\Phi\in(0,1]$.

Proof. If follows from Lemma 5 and Theorem 3 that the set of optimal horses includes all the horses if and only if $\Omega(H, \lambda_H(\Phi), \Phi) < 0$, that is, if and only if

$$\Omega(H, \lambda_H(\Phi), \Phi) = \sum_{h=1}^{H} \beta_h \cdot \left[\left(\frac{dU}{dg} \right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{p_H}{\beta_H} \cdot \frac{dU}{dg}(1-\Phi) \right) - 1 + \Phi \right] - D \cdot \Phi < 0.$$

If $\beta_h = p_h$ for all h, then

$$\Omega(H, \lambda_H(\Phi), \Phi) = \sum_{h=1}^{H} \beta_h \cdot \left[\left(\frac{dU}{dg} \right)^{-1} \left(\frac{dU}{dg}(1-\Phi) \right) - 1 + \Phi \right] - D \cdot \Phi = -D \cdot \Phi < 0,$$

hence $\Omega(H, \lambda_H(\Phi), \Phi) < 0$ for all $\Phi \in (0, 1]$ if the two probability distributions β_h and p_h are close. In order to deal with the case where β_h and p_h are not close, we note that $\Omega(H, \lambda_H(\Phi), \Phi)$ is a continuous function of Φ, hence in order to prove the existence of a solution $\Phi_H \in (0, 1)$ of (51) it suffices to show that $\Omega(H, \lambda_H(0), 0) > 0$ and $\Omega(H, \lambda_H(1), 1) < 0$, the latter being (50). We need to prove that $\Omega(H, \lambda_H(0), 0) > 0$, where

$$\Omega(H, \lambda_H(0), 0) = \sum_{h=1}^{H} \beta_h \cdot \left[\left(\frac{dU}{dg} \right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{p_H}{\beta_H} \cdot \frac{dU}{dg}(1) \right) - 1 \right].$$

$\frac{p_h}{\beta_h} \geq \frac{p_H}{\beta_H}$ for all h, hence $\frac{\beta_h}{p_h} \cdot \frac{p_H}{\beta_H} \leq 1$ for all h, and for at least one h this is a strict inequality since β_h and p_h are not close. Since $\frac{\beta_h}{p_h} \cdot \frac{p_H}{\beta_H} \cdot \frac{dU}{dg}(1) \leq \frac{dU}{dg}(1)$ and $\left(\frac{dU}{dg} \right)^{-1}$ is a decreasing function, we obtain

$$\left(\frac{dU}{dg} \right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{p_1}{\beta_1} \cdot \frac{dU}{dg}(1) \right) \geq \left(\frac{dU}{dg} \right)^{-1} \left(\frac{dU}{dg}(1) \right) = 1$$

for all h, and for some h this inequality is strict, hence $\Omega(H, \lambda_H(0), 0) > 0$. Thus $\Omega(H, \lambda_H(0), 0) > 0$ and $\Omega(H, \lambda_H(1), 1) < 0$, hence there exists a solution (and possibly many solutions) $\Phi_H \in (0, 1)$ of (51). We denote Φ_H to be the largest of these solutions. Now if $\Phi > \Phi_H$, then $\Omega(H, \lambda_H(\Phi), \Phi) < 0$, hence the optimal set of horses contains all the horses, thanks to Lemma 5 and Theorem 3. □

If the cut-off horse χ and the Lagrange multiplier λ are determined as described in Theorem 3, the optimal fractions f_h are given by (41).

If the bettor is interested in the solution of Problem 1 (the problem with limited exposure), then it would be useful to know whether the solution of Problem 5 (the problem with constrained exposure) could be the solution of Problem 1 (the problem with limited exposure) without making an effort to solve equation (42). The criterion for its being a solution of both problems is given in the proposition below.

Proposition 5. *The solution of Problem 5 (the problem with constrained exposure) is also the solution of Problem 1 (the problem with limited exposure) if*

$$\sum_{h\in P}\beta_h\cdot\left[\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\beta_h}{p_h}\cdot\frac{dU}{dg}\left(1-\Phi\right)\cdot\frac{1-\sum_{k\in P}p_k}{D-\sum_{k\in P}\beta_k}\right)-1+\Phi\right]-D\cdot\Phi\geq 0. \quad (52)$$

Proof. Let us denote the solution of Problem 5, $[f_1,\ldots,f_H,\,\lambda_1,\ldots,\lambda_H,\,\lambda]$, by $[f(\Phi),\lambda(\Phi)]$. This is also a solution of Problem 1 if and only if $\frac{d}{d\Phi}G\left(f\left(\Phi\right),\Phi\right)\geq 0$, where function G is described in (1). By the envelope theorem, $\frac{d}{d\Phi}G\left(f\left(\Phi\right),\Phi\right)=\frac{\partial G}{\partial\Phi}+\lambda\left(\Phi\right)$, hence $\frac{d}{d\Phi}G\left(f(\Phi),\Phi\right)\geq 0$, is equivalent to

$$\begin{aligned}0\leq\frac{\partial G}{\partial\Phi}+\lambda&=-\sum_{h=1}^{H}p_h\cdot\frac{dU}{dg}\left(1-\Phi+D\cdot\frac{f_h}{\beta_h}\right)+\lambda\\&=-\sum_{h\in P}p_h\cdot\frac{dU}{dg}\left(1-\Phi+D\cdot\frac{f_h}{\beta_h}\right)-\sum_{h\notin P}p_h\cdot\frac{dU}{dg}\left(1-F\right)+\lambda.\end{aligned}$$

The Lagrangian equations (39) imply that $p_h\cdot\frac{dU}{dg}\left(1-\Phi+D\cdot\frac{f_h}{\beta_h}\right)=\frac{\lambda}{D}\cdot\beta_h$, hence we obtain from the above inequality $\frac{dU}{dg}(1-\Phi)\cdot\sum_{h\notin P}p_h\leq\frac{\lambda}{D}\cdot\left(D-\sum_{h\in P}\beta_h\right)$. Since $\frac{dU}{dg}\left(1-\Phi\right)$ and λ have the same sign thanks to Lemma 2, it must be the case that $\left(D-\sum_{h\in P}\beta_h\right)>0$, hence we obtain

$$\frac{\lambda}{D}\geq\frac{dU}{dg}\left(1-\Phi\right)\cdot\frac{1-\sum_{h\in P}p_h}{D-\sum_{h\in P}\beta_h}. \quad (53)$$

This condition can be checked before λ is known, in the following manner: $\Omega\left(P,\lambda,\Phi\right)$, which is defined in (43), is a decreasing function of $\frac{\lambda}{D}$, hence (53) is equivalent to $\Omega\left(P,\lambda,\Phi\right)\geq 0$ when the right-hand side of (53) is substituted for λ/D in (43). This is equivalent to (52). □

Variability of the Optimal Set of Horses Using simulations to study the model with two horses, [5] and [6] concluded that the bets indicated by the Kelly criterion produce large oscillations in the wealth of the bettor. To alleviate this problem, it was suggested that the bettor bet only a fraction of the bets that correspond to the optimal values of f_h from the formulas in Section 5.3. This was called the 'fractional Kelly criterion,' and it was recommended that all the f_h be decreased proportionally in order to decrease the exposure F, without discussing the appropriate changes in the optimal set of horses. Here we discuss dependency of the cut-off horse χ on Φ. The next result implies that when the sum of the f_h is less than F, where F is the optimal exposure from Proposition 3, it is optimal to decrease f_h for the cut-off horse χ. In order to prove this, we introduce the following definition:

Definition 1. *For all $k \in \{1, 2, 3, \ldots, H\}$, the k-th Φ-stop is the solution Φ_k of the equation*

$$\begin{aligned} 0 &= \Omega\left(k, \lambda_k\left(\Phi_k\right), \Phi_k\right) && (54)\\ &= \sum_{h \le k} \beta_h \cdot \left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{p_k}{\beta_k} \cdot \frac{dU}{dg} \left(1 - \Phi_k\right) \right) - 1 + \Phi_k \right] - D \cdot \Phi_k && (55) \end{aligned}$$

Proposition 6. *If (29) holds, Φ_k is uniquely determined.*

Proof. First, by Lemma 5, $\Omega\left(k, \lambda_k\left(\Phi_k\right), \Phi_k\right)$ is a non-decreasing function of Φ_k, at least as long as k is so small that $\sum_{h \le k} \beta_h < D$. Next,

$$\begin{aligned} \Omega\left(k, \lambda_k(0), 0\right) &= \sum_{h \le k} \beta_h \cdot \left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{p_k}{\beta_k} \cdot \frac{dU}{dg} (1 - 0) \right) - 1 + 0 \right] - D \cdot 0 \\ &= \sum_{h \le k} \beta_h \cdot \left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{p_k}{\beta_k} \cdot \frac{dU}{dg} (1) \right) - 1 \right] \ge 0, \end{aligned}$$

since $\frac{dU}{dg}(1) > 0$; $\frac{\beta_h}{p_h} \cdot \frac{p_k}{\beta_k} \le 1$ for all $h \le k$ thanks to Remark 1; and $\frac{dU}{dg}$ is assumed to be a decreasing function. Finally, $\Omega\left(k, \lambda_k(\Phi), \Phi\right)$ is negative for large Φ; therefore, equation (54) always has a unique positive solution Φ_k. □

In order to prove that the cut-off horse χ defined in Remark 2 decreases together with Φ, we prove the following proposition:

Proposition 7. *The function Φ_k: $k \in \{1, 2, 3, \ldots, H-1\} \to \Phi_k(\chi) \in R$ from Definition 1 is non-decreasing in k, at least as long as k is so small that $\sum_{h \le k} \beta_h < D$, that is, when (29) holds. For small Φ, the cut-off horse χ defined in Remark 2 can be characterized by the two inequalities*

$$\Phi_\chi \le \Phi, \tag{56}$$

$$\Phi < \Phi_{\chi+1}. \tag{57}$$

Proof. First, we prove that $\varPhi_k$ is non-decreasing in k. When one subtracts the equation
$\varOmega\left(\chi,\lambda_\chi\left(\varPhi_\chi\right),\varPhi_\chi\right)=0$ from $\varOmega\left(\chi+1,\lambda_{\chi+1}\left(\varPhi_{\chi+1}\right),\varPhi_{\chi+1}\right)=0$, one obtains

$$\begin{aligned}
0 &= \varOmega\left(\chi+1,\lambda_{\chi+1}\left(\varPhi_{\chi+1}\right),\varPhi_{\chi+1}\right)-\varOmega\left(\chi,\lambda_\chi\left(\varPhi_\chi\right),\varPhi_\chi\right)\\
&= \sum_{h\le\chi}\beta_h\cdot\left[\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\beta_h}{p_h}\cdot\frac{p_{\chi+1}}{\beta_{\chi+1}}\cdot\frac{dU}{dg}\left(1-\varPhi_{\chi+1}\right)\right)\right.\\
&\qquad\left.-\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\beta_h}{p_h}\cdot\frac{p_\chi}{\beta_\chi}\cdot\frac{dU}{dg}\left(1-\varPhi_\chi\right)\right)\right]\\
&\qquad-\left(D-\sum_{h\le\chi}\beta_h\right)\cdot\left(\varPhi_{\chi+1}-\varPhi_\chi\right)\\
&= \sum_{h\le\chi}\beta_h\cdot\left[\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\beta_h}{p_h}\cdot\frac{p_{\chi+1}}{\beta_{\chi+1}}\cdot\frac{dU}{dg}\left(1-\varPhi_{\chi+1}\right)\right)\right.\\
&\qquad-\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\beta_h}{p_h}\cdot\frac{p_{\chi+1}}{\beta_{\chi+1}}\cdot\frac{dU}{dg}\left(1-\varPhi_\chi\right)\right)\\
&\qquad+\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\beta_h}{p_h}\cdot\frac{p_{\chi+1}}{\beta_{\chi+1}}\cdot\frac{dU}{dg}\left(1-\varPhi_\chi\right)\right)\\
&\qquad\left.-\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\beta_h}{p_h}\cdot\frac{p_\chi}{\beta_\chi}\cdot\frac{dU}{dg}\left(1-\varPhi_\chi\right)\right)\right]\\
&\qquad-\left(D-\sum_{h\le\chi}\beta_h\right)\cdot\left(\varPhi_{\chi+1}-\varPhi_\chi\right),
\end{aligned}$$

hence

$$\begin{aligned}
\sum_{h\le\chi}\beta_h &\cdot\left[\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\beta_h}{p_h}\cdot\frac{p_{\chi+1}}{\beta_{\chi+1}}\cdot\frac{dU}{dg}\left(1-\varPhi_\chi\right)\right)\right.\\
&\qquad\left.-\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\beta_h}{p_h}\cdot\frac{p_\chi}{\beta_\chi}\cdot\frac{dU}{dg}\left(1-\varPhi_\chi\right)\right)\right]\\
&= -\sum_{h\le\chi}\beta_h\cdot\left[\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\beta_h}{p_h}\cdot\frac{p_{\chi+1}}{\beta_{\chi+1}}\cdot\frac{dU}{dg}\left(1-\varPhi_{\chi+1}\right)\right)\right.\\
&\qquad\left.-\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\beta_h}{p_h}\cdot\frac{p_{\chi+1}}{\beta_{\chi+1}}\cdot\frac{dU}{dg}\left(1-\varPhi_\chi\right)\right)\right]\\
&\qquad+\left(D-\sum_{h\le\chi}\beta_h\right)\cdot\left(\varPhi_{\chi+1}-\varPhi_\chi\right).
\end{aligned}$$

Since it is assumed that $D > \sum_{h \leq \chi} \beta_h$ and $-\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\beta_h}{p_h} \cdot \frac{p_{\chi+1}}{\beta_{\chi+1}} \cdot \frac{dU}{dg}\left(1 - \Phi_{\chi+1}\right)\right)$ is an increasing function of $\Phi_{\chi+1}$, the right-hand side of the equation above is also an increasing function of $\Phi_{\chi+1}$, and it is zero if $\Phi_{\chi+1} = \Phi_\chi$. Each term in the summation on the left-hand side is independent of $\Phi_{\chi+1}$ and is non-negative, since $\left(\frac{dU}{dg}\right)^{-1}$ is a decreasing function and $\frac{p_h}{\beta_h} \geq \frac{p_{\chi+1}}{\beta_{\chi+1}}$ for all $h \leq \chi$ thanks to Remark 1. Therefore, $\Phi_{\chi+1} \geq \Phi_\chi$, that is Φ_k is non-decreasing in k. Next, for k that is so small that (29) holds, and holding k constant, the function $\Omega\left(k, \lambda_k\left(\Phi_k\right), \Phi_k\right)$ is decreasing as a function of Φ_k. We use this to prove that (56) and (57) imply (48) and (49). Indeed, since $\Omega\left(k, \lambda_k\left(\Phi_k\right), \Phi_k\right)$ is decreasing as a function of k and $\Phi_\chi < \Phi$, we have that $\Omega\left(\chi, \lambda_\chi\left(\Phi\right), \Phi\right) < \Omega\left(\chi, \lambda_\chi\left(\Phi_\chi\right), \Phi_\chi\right) = 0$, which is (48). Similarly, $\Phi \leq \Phi_{\chi+1}$ implies that

$$\Omega\left(\chi, \lambda_{\chi+1}\left(\Phi\right), \Phi\right) \geq \Omega\left(\chi, \lambda_{\chi+1}\left(\Phi_{\chi+1}\right), \Phi_{\chi+1}\right) = \Omega\left(\chi+1, \lambda_{\chi+1}\left(\Phi_{\chi+1}\right), \Phi_{\chi+1}\right) = 0,$$

which is (49). By Theorem 3, inequalities (48) and (49) characterize the cut-off horse. □

Since Φ_k is non-decreasing in k by Proposition **??**, inequality (56) implies that the cut-off horse decreases when Φ decreases. In Section 8 it is shown that for CRRA utilities, the conclusions of this section are valid without the restriction in (29).

6.3 Summary and an Algorithm for the Problem with Constrained Exposure

The results obtained thus far for Problem 5 can be summarized as follows:

Theorem 4. *The cut-off horse χ, defined in Remark 2 and applied to Problem 5, is the solution of inequalities (48) and (49). The optimal set of horses is $P = \{h \mid h \leq \chi\}$. Following the determination of the set P, the Lagrange multiplier λ has to be determined using equation (42), and then the optimal fractions f_h are given by formulas (41). If (52) holds, the solution of Problem 5 (the problem with constrained exposure) is also the solution of Problem 1 (the problem with limited exposure). If (52) is violated, the solution of Problem 1 happens to be the solution of Problem 2 (the problem with free exposure).*

A practical approach to solving the problem with constrained exposure—either Problem 5 or Problem 3—is presented in the following pseudocode.

Algorithm 2 (Pseudocode for Constrained Exposure)

1. *Construct the thresholds $\lambda_k(\Phi)$ for $k = 1, 2, 3, \ldots, H$ as recommended in equation (46):* $\lambda_k\left(\Phi\right) = D \cdot \frac{p_k}{\beta_k} \cdot \frac{dU}{dg}\left(1 - \Phi\right)$.
2. *Construct the function $\Omega\left(\chi, \lambda_\chi\left(\Phi\right), \Phi\right)$, as defined in equation (47):*
$$\Omega\left(\chi, \lambda_\chi\left(\Phi\right), \Phi\right) = \sum_{h \leq \chi} \beta_h \cdot \left[\left(\frac{dU}{dg}\right)^{-1}\left(\frac{\beta_h}{p_h} \cdot \frac{p_\chi}{\beta_\chi} \cdot \frac{dU}{dg}\left(1 - \Phi\right)\right) - 1 + \Phi\right] - D \cdot \Phi\ .$$

3. *Find the cut-off horse* χ *that satisfies the inequalities (48) and (49):* $\Omega(\chi, \lambda_\chi(\Phi), \Phi) < 0$ *and* $\Omega(\chi+1, \lambda_{\chi+1}(\Phi), \Phi) \geq 0$. *This determines the set of horses to bet on.*
4. *Set the optimal set of horses to* $P = \{h \mid h \leq \chi\}$. *Then, construct the function (43):*
   $\Omega(P, \lambda, \Phi) = \sum_{h \in P} \beta_h \cdot \left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{\lambda}{D}\right) - 1 + \Phi \right] - D \cdot \Phi$
   Then, solve equation (42), that is, $\Omega(P, \lambda, \Phi) = 0$, *for the Lagrange multiplier* λ. *An approximate solution can be obtained using a convex combination of* $\lambda_\chi(\Phi)$ *and* $\lambda_{\chi+1}(\Phi)$.
5. *Compute the optimal fractions* f_h *using equation (41):*
   $f_h = \frac{\beta_h}{D} \cdot \left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{\lambda}{D} \cdot \frac{\beta_h}{p_h}\right) - 1 + \Phi \right]$.

If (52) holds, then the solution of Problem 5 (the problem with constrained exposure) is also the solution of Problem 1 (the problem with limited exposure).

7 Algorithm for the Problem with Limited Exposure

For the sake of completeness, here we provide an algorithm for solving Problem 1 (the problem with limited exposure):

Problem 6 (Limited Exposure) *Given a function U that satisfies Assumption 1, find the fractions* $[f_1, f_2, f_3, \ldots, f_H]$ *that maximize the value of the concave down function (1) on the convex set of admissible fractions* $(f_1, f_2, f_3, \ldots, f_H)$ *determined by constraints* $f_h \geq 0$ *and* $\sum_{h=1}^{H} f_h \leq \Phi$.

There are two possible approaches: Begin by solving Problem 4 (the problem with free exposure), or begin by solving Problem 5 (the problem with constrained exposure).

If one begins by solving Problem 4, one would proceed as follows: Execute steps 1–6 of Algorithm 1, and then check the condition on

$$\Omega(\Phi) = \sum_{h \leq \chi} \beta_h \cdot \left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{dU}{dg}(1-\Phi) \cdot \frac{\sum_{k>\chi} p_k}{D - \sum_{k \leq \chi} \beta_k} \cdot \frac{\beta_h}{p_h} \right) - 1 + \Phi \right] - D \cdot \Phi.$$

from Proposition 3. If $\Omega(\Phi) \leq 0$, then execute the remaining steps of Algorithm 1. If $\Omega(\Phi) > 0$, then execute Algorithm 2.

If one begins by solving Problem 5, one would proceed as follows: Execute steps 1–3 of Algorithm 2, and then check condition (52):

$$\sum_{h \leq \chi} \beta_h \cdot \left[\left(\frac{dU}{dg}\right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{dU}{dg}(1-\Phi) \cdot \frac{\sum_{k>\chi} p_k}{D - \sum_{k \leq \chi} \beta_k} \right) - 1 + \Phi \right] - D \cdot \Phi \geq 0.$$

If it is satisfied, execute the remaining steps of the Algorithm 2; otherwise, execute Algorithm 1.

8 Examples

In this section, we illustrate the theory developed in Sections 4–6 and show that equation (26) in Step 5 of Algorithm 1 (for solving Problem 4) and equation (42) in Step 4 of Algorithm 2 (for solving Problem 5) are solvable for both CRRA and CARA utilities.

8.1 The CRRA Utility

CRRA utility functions are of the form $U(g) = \frac{g^{1-a} - 1}{1 - a}$ with $a > 0$, $Dom(g) = (0, \infty)$, and $\frac{dU}{dg}(g) = g^{-a} = u$, and the inverse of $\frac{dU}{dg}$ is $g = \left(\frac{dU}{dg}\right)^{-1}(u) = u^{-1/a}$. $U(g)$ satisfies Assumption 1. The logarithmic utility $\ln(g)$ is a special case of CRRA utility in the limit as $a \to 1$, and all the formulas of this section convert to corresponding formulas from [3] after passing to the limit $a \to 1$. Hyperbolic utility corresponds to $a = 2$: $U(g) = \frac{g^{1-2} - 1}{1 - 2} = \frac{g^{-1} - 1}{-1} = 1 - g^{-1}$. Since $\frac{dU}{dg}(g) = g^{-a} > 0$ for all $a > 0$, the optimal set of horses is of the form $\{h \mid h \leq \chi\}$, as noted in Remark 2.

The Optimal Bets for Free Exposure The cut-off horse χ has to be determined as described in Step 3 of Algorithm 1. Next, the exposure level F has to be determined using equation (26), which takes the following form:

$$0 = (1 - F) \cdot \sum_{h \leq \chi} \beta_h \cdot \left[\left(\frac{\sum_{k > \chi} p_k}{D - \sum_{k \leq \chi} \beta_k} \cdot \frac{\beta_h}{p_h} \right)^{-1/a} - 1 \right] - D \cdot F$$

The solution is $F = S/(S + D)$, where $S = \sum_{h \leq \chi} S_h$ and

$$S_h = \beta_h \cdot \left[\left(\frac{\sum_{k > \chi} p_k}{D - \sum_{k \leq \chi} \beta_k} \cdot \frac{\beta_h}{p_h} \right)^{-1/a} - 1 \right] \geq 0,$$

hence $F \in (0,1)$ since $D > 0$ and

$$\left(\frac{D - \sum\limits_{k\le\chi} \beta_k}{\sum\limits_{k>\chi} p_k} \cdot \frac{p_h}{\beta_h} \right)^{1/a} > 1,$$

for all horses h in the optimal set of horses thanks to the inequality on the right side in (34). The formulas (25) for the fractions that correspond to horses from the optimal set are $f_h = S_h / (S + D) = F \cdot S_h / S$. If F satisfies constraint (4), the resulting solution is also the solution of Problem 1; otherwise, the solution obtained in the next section is the solution of Problem 1.

The Optimal Bets for Constrained Exposure The cut-off horse χ can be determined as described in Theorem 3 with the help of the thresholds $\lambda_k(\Phi)$. However, when analytical solutions are available, it is better to use the Φ-stops Φ_k, as described in Proposition **??**. These are the solutions of equations (54):

$$0 = (1 - \Phi_k) \cdot \sum_{h\le k} \beta_h \cdot \left[\left(\frac{\beta_h}{p_h} \cdot \frac{p_k}{\beta_k} \right)^{-1/a} - 1 \right] + D \cdot (1 - \Phi_k) - D.$$

The solution is $\Phi_k = S_k / (S_k + D) < 1$, where $S_k = \sum\limits_{h\le k} \beta_h \cdot \left[\left(\frac{\beta_h}{p_h} \cdot \frac{p_k}{\beta_k} \right)^{-1/a} - 1 \right] \ge 0$. Since $\frac{p_k}{\beta_k}$ is non-increasing in k, we have that $\left(\frac{p_h}{\beta_h} \cdot \frac{\beta_k}{p_k} \right) \ge 1$ for all $h \le k$, hence $S_k \ge 0$ and $S_k > 0$ for $k > 1$. Therefore, each $\Phi_k \in [0,1)$, and the sequence S_k is non-decreasing in k:

$$\begin{aligned} S_{k+1} - S_k &= \sum_{h\le k+1} \beta_h \cdot \left[\left(\frac{p_h}{\beta_h} \cdot \frac{\beta_{k+1}}{p_{k+1}} \right)^{1/a} - 1 \right] - \sum_{h\le k} \beta_h \cdot \left[\left(\frac{p_h}{\beta_h} \cdot \frac{\beta_k}{p_k} \right)^{1/a} - 1 \right] \\ &= \sum_{h\le k} \beta_h \cdot \left[\left(\frac{p_h}{\beta_h} \cdot \frac{\beta_{k+1}}{p_{k+1}} \right)^{1/a} - \left(\frac{p_h}{\beta_h} \cdot \frac{\beta_k}{p_k} \right)^{1/a} \right] \\ &= \sum_{h\le k} \beta_h \cdot \left(\frac{p_h}{\beta_h} \right)^{1/a} \cdot \left[\left(\frac{\beta_{k+1}}{p_{k+1}} \right)^{1/a} - \left(\frac{\beta_k}{p_k} \right)^{1/a} \right] \ge 0, \end{aligned}$$

since $\frac{\beta_{k+1}}{p_{k+1}}$ is non-decreasing in k, and hence Φ_k is non-decreasing in k.

Now the cut-off horse χ is the solution to the two inequalities $\Phi_\chi \le \Phi < \Phi_{\chi+1}$ (inequalities (56) and (57)), and when $\Phi \ge \Phi_H = S_H / (S_H + D) < 1$ it is optimal to bet on all the horses. The Lagrange multiplier λ can be determined using equation (42):

$$0 = \sum_{h\le\chi} \beta_h \cdot \left[\left(\frac{dU}{dg} \right)^{-1} \left(\frac{\beta_h}{p_h} \cdot \frac{\lambda}{D} \right) - 1 + \Phi \right] - D \cdot \Phi,$$

and the solution is

$$\left(\frac{\lambda}{D}\right)^{-1/a} = \frac{\sum_{h\leq\chi}\beta_h + \left(D - \sum_{h\leq\chi}\beta_h\right)\cdot\Phi}{\sum_{h\leq\chi}\beta_h\cdot\left(\frac{p_h}{\beta_h}\right)^{1/a}} = \frac{(1-\Phi)\cdot\sum_{h\leq\chi}\beta_h + D\cdot\Phi}{\sum_{h\leq\chi}\beta_h\cdot\left(\frac{p_h}{\beta_h}\right)^{1/a}}.$$

Now the optimal fractions f_h for $h \leq \chi$ can be found using (41):

$$f_h = \frac{\beta_h}{D}\cdot\left[\frac{(1-\Phi)\cdot\sum_{h\leq\chi}\beta_h + D\cdot\Phi}{\sum_{h\leq\chi}\beta_h\cdot\left(\frac{p_h}{\beta_h}\right)^{1/a}}\cdot\left(\frac{\beta_h}{p_h}\right)^{-1/a} - 1 + \Phi\right] > 0,$$

By Proposition 5, the resulting solution of Problem 5 is also the solution of Problem 1 if condition (52) holds. This can be reduced to $\Phi \leq S/(S+D)$, where

$$S = \sum_{h\leq\chi}\beta_h\cdot\left[\left(\frac{p_h}{\beta_h}\cdot\frac{D-\sum_{k\leq\chi}\beta_k}{\sum_{k>\chi}p_k}\right)^{1/a} - 1\right].$$

8.2 The CARA Utility

CARA utility functions are of the form $U(g) = -\frac{e^{-a\cdot x}-1}{a}$. The inverse of $\frac{dU}{dg}$ is $g = \left(\frac{dU}{dg}\right)^{-1}(u) = -\frac{\ln(u)}{a}$, and CARA utilities satisfy Assumption 1 whenever $a \neq 0$. The linear utility is a special case of CARA, equivalent to $a = 0$. Linear utility functions satisfy Assumption 1 only if $a > 0$, hence in this case all of the theory presented in Sections 5 and 6 is applicable. Since $\frac{dU}{dg}(g) = e^{-a\cdot x} > 0$, the optimal set of horses is of the form $\{h \mid h \leq \chi\}$, as noted in Remark 2.

The Optimal Bets for Free Exposure The cut-off horse χ has to be determined as described in Step 3 of Algorithm 1. The formulas (25) for the fractions corresponding to horses in the optimal set produce

$$f_h = \frac{\beta_h}{a\cdot D}\cdot\ln\left(\frac{D-\sum_{k\leq\chi}\beta_k}{\sum_{k>\chi}p_k}\cdot\frac{p_h}{\beta_h}\right) > 0$$

and hence

$$F = \frac{1}{a \cdot D} \cdot \sum_{h \le \chi} \beta_h \cdot \ln \left(\frac{D - \sum_{k \le \chi} \beta_k}{\sum_{k > \chi} p_k} \cdot \frac{p_h}{\beta_h} \right).$$

If $F \le \Phi$, the resulting solution of Problem 4 is also the solution of Problem 1.

The Optimal Bets for Constrained Exposure For a CARA utility, $\frac{dU}{dg}(g) = \exp(-a \cdot x) > 0$, hence by Remark 2 and Lemma 2 the optimal set of horses is of the form $P = \{h \mid h \le \chi\}$. The cut-off horse χ can be determined as described in Theorem 3 with the help of the thresholds $\lambda_k(\Phi)$. However, when analytical solutions are available, it is better to use the Φ-stops Φ_k, as described in Proposition **??**. These are the solutions of equations (54):

$$0 = -\frac{1}{a} \cdot \sum_{h \le k} \beta_h \cdot \ln \left(\frac{\beta_h}{p_h} \cdot \frac{p_k}{\beta_k} \cdot \exp\left(-a \cdot (1 - \Phi_k)\right) \right) - \sum_{h \le k} \beta_h \cdot (1 - \Phi_k) - D \cdot \Phi_k$$

In this case the solution is given by $\Phi_k = \frac{1}{D \cdot a} \cdot \sum_{h \le k} \beta_h \cdot \ln \left(\frac{p_h}{\beta_h} \cdot \frac{\beta_k}{p_k} \right)$. Since $\frac{p_k}{\beta_k}$ is non-increasing in k, we have that $\left(\frac{p_h}{\beta_h} \cdot \frac{\beta_k}{p_k} \right) \ge 1$ for all $h \le k$, and $\Phi_k > 0$, hence Φ_k is increasing in k. The only relevant Φ-stops are those with $\Phi_k \le 1$. One can easily show that $\Phi_1 = 0$ and $\Phi_2 = \frac{1}{D \cdot a} \cdot \beta_2 \cdot \ln \left(\frac{p_1}{\beta_1} \cdot \frac{\beta_2}{p_2} \right)$, hence $\Phi_2 > 1$ for small a. This implies that solutions of Problem 5 converge to solutions for a linear utility, and it is optimal to bet everything on the best horse.

Now the cut-off χ horse is determined using $\Phi_\chi \le \Phi < \Phi_{\chi+1}$ (inequalities (56) and (57)), and then the Lagrange multiplier λ can be determined using equation (42):

$$0 = \sum_{h \le \chi} \beta_h \cdot \left[-\frac{1}{a} \cdot \ln \left(\frac{\beta_h}{p_h} \cdot \frac{\lambda}{D} \right) - 1 + \Phi \right] - D \cdot \Phi$$

The solution is given by

$$\frac{1}{a} \cdot \ln \left(\frac{\lambda}{D} \right) + 1 - \phi = -\frac{\frac{1}{a} \cdot \sum_{k \le \chi} \beta_k \cdot \ln \left(\frac{\beta_k}{p_k} \right) + D \cdot \Phi}{\sum_{k \le \chi} \beta_k}.$$

Now the optimal fractions f_h can be found using (41):

$$f_h = \frac{\beta_h}{D \cdot a} \cdot \left[-ln \left(\frac{\beta_h}{p_h} \right) + \frac{\sum_{k \le \chi} \beta_k \cdot \ln \left(\frac{\beta_k}{p_k} \right) + D \cdot a \cdot \Phi}{\sum_{k \le \chi} \beta_k} \right] > 0.$$

By Proposition 5, the solution of Problem 5 is also the solution of Problem 1 as long as (52) holds. This can be reduced to

$$\Phi \leq \frac{1}{a \cdot D} \cdot \sum_{h \leq \chi} \beta_h \cdot \ln \left(\frac{D - \sum_{k \leq \chi} \beta_k}{\sum_{k > \chi} p_k} \cdot \frac{p_h}{\beta_h} \right).$$

This is always satisfied when a is small, hence if one is interested in the solution of Problem 1 for CARA utilities with small a, it is better to begin by solving Problem 5.

8.3 An Example: Following the Leaders

In this section, we consider a horse race scenario involving two types of bettors: noise bettors and proportional bettors. Noise bettors are unsophisticated and unaware of the horses' win probabilities; hence, they do not optimize their bets. They distribute their bets evenly among all the horses: $\beta_h^\nu = \beta^\nu = \frac{1}{N}$, where N is the number of horses. Proportional bettors, on the other hand, are informed bettors who know the win probabilities p_h of horses. They allocate their bets proportionally to these probabilities: $\beta_h^c = p_h$. We assume that the proportional bettors constitute a fraction $w \in (0, 1)$ of all bettors, while the noise bettors make up the remaining fraction $z = 1 - w$. Therefore, the resulting distribution of belief probabilities is: $\beta_h = w \cdot \beta_h^c + z \cdot \beta^\nu$.

Additionally, we introduce a small group of sophisticated casual bettors who do not know win probabilities p_h, but are capable of optimizing their bets. These bettors choose to follow the strategy of "doing what the big players are doing." We assume that their bets are negligible in size and do not significantly affect the total amounts bet on each horse.

The sophisticated bettors observe the amount bet on the horse with the lowest total bets and assume that this amount represents the bets placed by noise bettors, who distribute their bets uniformly. By subtracting the amount bet on the lowest horse from the total bets on each horse, they can estimate the amounts bet by the proportional bettors. Therefore, the sophisticated bettors approximate the win probabilities as: $p_h = \beta_h^c$, effectively mimicking the behavior of the proportional bettors. To apply the theory developed in this paper, they need to order the horses according to decreasing expected revenue rates:

$$\frac{p_h}{\beta_h} = \frac{\beta_h^c}{w \cdot \beta_h^c + z \cdot \beta^\nu} \geq \frac{p_{h+1}}{\beta_{h+1}} = \frac{\beta_{h+1}^c}{w \cdot \beta_{h+1}^c + z \cdot \beta^\nu}.$$

These inequalities are equivalent to: $\beta_h^c \geq \beta_{h+1}^c$. Thus, we assume that horses are relabeled and ordered such that $\beta_1^c \geq \beta_2^c \geq \ldots \geq \beta_N^c$

The requirement $D \cdot \frac{p_1}{\beta_1} > 1$ is equivalent to:

$$D \cdot \frac{\beta_1^c}{w \cdot \beta_1^c + z \cdot \beta^\nu} > 1 \iff D \cdot \beta_1^c > w \cdot \beta_1^c + (1-w) \cdot \beta^\nu = \beta_1^c - (1-w) \cdot \beta_1^c + (1-w) \cdot \beta^\nu$$

$$\iff (1-w)\left(1-\frac{\beta^\nu}{\beta_1^c}\right) = (1-w)\left(1-\frac{1}{N\cdot\beta_1^c}\right) > (1-D) = tt,$$

This condition is easily satisfied, especially when $w < D$. The cut-off horse χ for free exposure can be found by identifying the smallest χ satisfying the inequalities:

$$\frac{p_\chi}{\beta_\chi} > NL(\chi) \geq \frac{p_{\chi+1}}{\beta_{\chi+1}}.$$

The average logarithm of the growth factors for the sophisticated uninformed bettors is:

$$G(f^{opt}) = \ln(D) + \sum_{h\in S} p_h \cdot \ln\left(\frac{p_h}{b_h}\right) + \sum_{h\notin S} p_h \cdot \ln\left(\frac{\sum_{h\notin S} p_h}{D - \sum_{h\in S} b_h}\right).$$

where S is the set of horses on which they choose to bet. If they bet as proportional bettors do, then their average logarithm of the growth factors is:

$$G(f^p) = \sum_h p_h \cdot \ln\left(\frac{D\cdot p_h}{b_h}\right) = \ln(D) + \sum_h p_h \cdot \ln\left(\frac{\beta_h^c}{w\cdot\beta_h^c + z\cdot\beta^\nu}\right).$$

For noise bettors, the average logarithm of the growth factors is:

$$G(f^\nu) = \ln(D) + \sum_h p_h \cdot \ln\left(\frac{1/N}{w\cdot\beta_h^c + z\cdot\beta^\nu}\right).$$

The difference between the average logarithms of the growth factors of the proportional informed bettors and the noise bettors is:

$$G(f^p) - G(f^\nu) = \sum_h p_h \cdot \ln\left(\frac{\beta_h^c}{1/N}\right) = \ln(N) + \sum_h p_h \cdot \ln(p_h) \geq 0. \quad (58)$$

This difference represents the "sophistication premium" gained by the proportional bettors over the noise bettors.

Similarly, the difference between the sophisticated uninformed bettors and the proportional bettors is:

$$G(f^{opt}) - G(f^p) = \sum_{h\notin S} p_h \cdot \ln\left(\frac{\sum_{h\notin S} p_h}{D - \sum_{h\in S} b_h}\Big/\frac{p_h}{b_h}\right) > 0,$$

since the sequence $\frac{p_h}{b_h}$ is non-increasing, and the criterion for the cut-off horse is:

$$\frac{p_\chi}{b_\chi} > \frac{\sum_{h\geq\chi} p_h}{D - \sum_{h<\chi} b_h} \geq \frac{p_{\chi+1}}{b_{\chi+1}}.$$

8.3.1 Numerical Example: Consider a numerical example where the dividend rate ($D = 0.85$). There are seven horses with win probabilities p_h, subjective probabilities β_h, and expected revenue rates $e_h = \frac{p_h}{\beta_h}$ as given in Table 1:

Table 1: Numerical Example Parameters

Horse (h)	1	2	3	4	5	6	7
Win Probability (p_h)	0.0032	0.0032	0.0032	0.0162	0.2270	0.1620	0.5840
Belief Probability (β_h)	0.0250	0.0375	0.0625	0.1250	0.2500	0.3130	0.1880
Expected Revenue Rate (e_h)	0.1299	0.0866	0.0520	0.1298	0.9090	0.5190	3.1170
New Horse Number After Reordering by e_h	4	6	7	5	2	3	1

The horses are reordered according to decreasing expected revenue rates e_h, with the new horse numbers shown in the last row of Table 1. In Table 2, we present the reordered data and additional calculations.

Table 2: Numerical Example Results

Row	**1**	**2**	**3**	**4**	**5**	**6**	**7**
New Horse Number (h)	1	2	3	4	5	6	7
Win Probability (p_h)	0.5844	0.2273	0.1623	0.0032	0.0160	0.0030	0.0030
Belief Probability (β_h)	0.1875	0.2500	0.3125	0.0250	0.1250	0.0380	0.0630
Expected Revenue Rate (er_h)	3.1168	0.9092	0.5194	0.1299	0.1300	0.0870	0.0520
$NL(h)$	0.6273	0.4564	0.2600	0.3033	-0.1304	-0.0374	-0.00019
To Bet or Not?	bet	bet	bet	no	no	no	no
Optimal Fraction to Bet (f_h) **for Logarithmic Utility**	0.5356	0.1623	0.0811	-0.0032	-0.0163	-0.0065	-0.0130
Optimal Fractions for CRRA Utility with $\alpha = 0.5$	0.8535	0.0895	0.0298	-0.0006	-0.0029		

The smallest positive entry in $NL(h)$ row determines the cut-off horse, which in this case is ($\chi = 3$). Thus, ($NL(\chi) = 0.26$) and the optimal set of horses is ($\{1, 2, 3\}$). The bottom two rows of Table 2 show the optimal fractions to bet on each horse for the logarithmic utility ($\alpha = 1$) and for a Constant Relative Risk Aversion (CRRA) utility function with ($\alpha = 0.5$). The total fraction of wealth bet for the logarithmic utility is ($F = 0.779$), and for ($\alpha = 0.5$), the total fraction is ($F = 0.972886$).

The differences between the logarithms of the growth factors are as follows:

- Between the informed bettors and the noise bettors:
  $G(f^p) - G(f^\nu) = 0.1510.$
- Between the sophisticated bettors and the informed bettors:
  $G(f^{\text{opt}}) - G(f^p) = 0.1042.$
- Between the sophisticated bettors and the noise bettors:
  $G(f^{\text{opt}}) - G(f^\nu) = 0.2552.$

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