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Abstract. There exist necessary and sufficient conditions on the generating functions of the FGM family, in order to obtain various dependence properties.
We present multivariate generalizations of this class studying symmetry and dependence concepts, measuring the dependence among the components of each class and providing several examples.

Keywords: copula, density function, FGM copulas, dependence, symmetry.

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1 Introduction

Analyzing the dependence between the components $X_1, \ldots, X_n$ of a random vector X is subject to various lines of statistical research. For this purpose, copula functions (or simply copulas) have been introduced by Sklar (1959) which allow for a separation between the marginal distributions and the dependence structure. Moreover, construction principles for copulas based on certain functions ("generator functions") have gained in importance. For example, Archimedean copulas are constructed by (a possibly rather complicated) composition of a specific generator function and its corresponding pseudo inverse. In contrast to that, Amblard and Girard (2002) discuss a very simple construction principle of copulas on the basis of certain generator functions and a "dependence parameter"θ. Specific generalized Farlie - Gumbel (or Sarmanov) copulas are generated by a single function (so-called generator or generator function) defined on the unit interval. An alternative approach to generalize the FGM family of copulas is to consider the semi-parametric family of symmetric copulas. This family is generated by a univariate function, determining the symmetry (radial symmetry, joint symmetry) and dependence property (quadrant dependence, total positivity) of copulas.
A multivariate data set, which exhibit complex patterns of dependence, particularly in the tails, can be modeled using a cascade of lower-dimensional copulas. Moreover, these copulas allow for a direct characterization of symmetry properties, ordering properties and association measures. Recently, Amblard and Girard (2004) also state a semiparametric estimation method for the underlying generator function. However, the parameter θ is not identified in the semiparametric context.

2 Definitions and properties

First we restrict ourselves to the bivariate case. Loosely speaking, a $2-$copula is a two-dimensional distribution function defined on the unit square with uniformly distributed marginals. More formally, a two-dimensional copula is a function $C : [0,1] \times [0,1] \to [0,1]$ which satisfies the following properties:

1. C is 2-increasing, i.e. for $0 \leq u_1 \leq v_1 \leq 1$ and $0 \leq u_2 \leq v_2 \leq 1$ holds:

$$C(v_1, v_2) - C(v_1, u_2) - C(u_1, v_2) + C(u_1, u_2) \geq 0.$$

2. For all $u, v \in [0,1] : C(u,0) = C(0,v) = 0$ and $C(u,1) = C(1,u) = u$.

Note that every copula is bounded below by $C^{min}(u,v) = \max\{u+v-1, 0\}$ and above by $C^{max}(u,v) = \min\{u,v\}$, the so-called Fréchet-Hoeffding bounds. Moreover the copula associated with the joint distribution of two independent uniform variables is given by $C^{\perp}(u,v) = uv$.
One of the most popular parametric family of copulas is the Farlie-Gumbel-Morgenstern (FGM) family defined when $\theta \in [-1,1]$ by

$$C_\theta^{FGM}(u,v) = uv + \theta u(1-u)v(1-v) \tag{1}$$

and studied in Farlie (1960), Gumbel (1960) and Morgenstern (1956).
An alternative approach to generalize the FGM family of copulas is to consider the semi-parametric family of symmetric copulas defined by

$$C^{SP}_{\theta,\phi}(u,v) = uv + \theta\phi(u)\phi(v), \tag{2}$$

with $\theta \in [-1,1]$ and ϕ is a function on $I = [0,1]$. It was first introduced in Rodríguez-Lallena (1992), and extensively studied in Amblard and Girard (2002, 2005).

2.1 Symmetry properties

Let $(a,b) \in \mathbb{R}^2$ and (X,Y) a random pair. We say that X is symmetric about a if the cumulative distribution functions of $(X-a)$ and $(a-X)$ are identical. The following definitions generalize this symmetry concept to the bivariate case:

- X and Y are exchangeable if (X,Y) and (Y,X) are identically distributed;
- (X,Y) is marginally symmetric about (a,b) if X and Y are symmetric about a and b respectively;
- (X,Y) is radially symmetric about (a,b) if $(X-a,Y-b)$ and $(a-X,b-Y)$ follow the same joint cumulative distribution function;
- (X,Y) is jointly symmetric about (a,b) if the pairs of random variables $(X-a,Y-b)$, $(a-X,b-Y)$, $(X-a,b-Y)$ and $(a-X,Y-b)$ have a common joint cumulative distribution function.

The following theorem provides conditions on ϕ to ensure that the couple (X,Y) with associated copula C_θ is radially (or jointly) symmetric.

Theorem 1 **(i)** *If X and Y are identically distributed then X and Y are exchangeable. Besides, if (X,Y) is marginally symmetric about (a,b) then:*

(ii) *(X,Y) is radially symmetric about (a,b) if and only if either $\forall u \in I$, $\phi(u) = \phi(1-u)$ or $\forall u \in I$, $\phi(u) = -\phi(1-u)$;*

(iii) *(X,Y) is jointly symmetric about (a,b) if and only if $\forall u \in I$, $\phi(u) = -\phi(1-u)$.*

2.2 Concepts of dependence

In this section we note (X,Y) a random pair with joint cdf H, copula C and margins F and G. For the sake of simplicity, we assume that X and Y are exchangeable. Several concepts of dependence have been introduced and characterized in terms of copulas. X and Y are

- Positive Function Dependent (PFD) if for all integrable real-valued function g

$$\mathbb{E}_h[g(X)g(Y)] - \mathbb{E}_h[g(X)]\mathbb{E}_h[g(Y)] \geq 0,$$

where $\mathbb{E}_h$ is the expectation symbol relative to the density h.

- Positively Quadrant Dependent (PDQ) if $\mathbb{P}(X \leq x, Y \leq y) \geq \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y)$, for all $(x, y) \in \mathbb{R}^2$ or equivalently

$$\forall (u, v) \in I^2, \quad C(u, v) \geq uv. \tag{3}$$

- Left Tail Decreasing ($LTD(Y|X)$) if $\mathbb{P}(Y \leq y|X \leq x)$ is non-increasing in x for all y, or equivalently, see Theorem 5.2.5 in Nelsen (2006), $u \to C(u, v)/u$ is non-increasing for all $v \in I$.

- Right Tail Increasing ($RTI(Y|X)$) if $\mathbb{P}(Y > y|X > x)$ is non-decreasing in x for all y or, equivalently, $u \to (v - C(u, v))/(1 - u)$ is non-increasing for all $v \in I$.

- Stochastically Increasing ($SI(Y|X)$) if $\mathbb{P}(Y > y|X = x)$ is non-decreasing in x for all y .

- Left Corner Set Decreasing (LCSD) if $\mathbb{P}(X \leq x, Y \leq y|X \leq x', Y \leq y')$ is non-increasing in x' and y' for all x and y, or equivalently, see Corollary 5.2.17 in Nelsen (2006), C is a totally positive function of order 2 (TP_2), *i.e.* for all $(u_1, u_2, v_1, v_2) \in I^4$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$, one has

$$C(u_1, v_1)C(u_2, v_2) - C(u_1, v_2)C(u_2, v_1) \geq 0. \tag{4}$$

  This property is equivalent to Positively Likelihood Ratio Dependent (PLR), which is defined if and only if C is absolutely continuous and its density c satisfies (4), with C replaced by c.

- Right Corner Set Increasing (RCSI) if $\mathbb{P}(X > x, Y > y|X > x', Y > y')$ is non-decreasing in x' and y' for all x and y, or equivalently, the survival copula $\hat{C}$ associated to C is a totally positive function of order 2.

More broadly, one has the following definition:

Definition 1 *Let A and B be subsets of $[0, 1]$. A function C defined on $A \times B$ is said to be totally positive of order k, denoted TP_k, if for all m, $1 \leq m \leq k$ and all $u_1 < \ldots < u_m$, $v_1 < \ldots < v_m$ ($u_i \in A$, $v_j \in B$)*

$$C \begin{pmatrix} u_1, \ldots, u_m \\ v_1, \ldots, v_m \end{pmatrix} \equiv det \begin{bmatrix} C(u_1, v_1), \ldots, C(u_1, v_m) \\ C(u_m, v_1), \ldots, C(u_m, v_m) \end{bmatrix} \geq 0. \tag{5}$$

When the inequalities (5) are strict for $m = 1, \ldots, k$, C is called *strictly totally positive of order k (STP_k)*.
There are several obvious consequences of the definition.

1. If a and b are nonnegative functions defined, respectively, on A and B and if K is TP_k then $a(u)b(v)C(u, v)$ is TP_k.

2. If g and h are defined on A and B, respectively, and monotone in the same direction, and if C is TP_k on $g(A) \times h(B)$, then $C(g(u), h(v))$ is TP_k on $A \times B$.

The following Corollary 5.2.6 in Nelsen [8] gives us the criteria for tail monotonicity in terms of the partial derivatives of C.

Corollary 1 *Let X and Y be continuous random variables with copula C. Then*

1. *LTD$(Y|X)$ if and only if for any v in $\mathbf{I}$, $\frac{\partial C(u,v)}{\partial u} \leq \frac{C(u,v)}{u}$ for almost all u;*
2. *LTD$(X|Y)$ if and only if for any u in $\mathbf{I}$, $\frac{\partial C(u,v)}{\partial v} \leq \frac{C(u,v)}{v}$ for almost all v;*
3. *$RTI(Y|X)$ if and only if for any v in $\mathbf{I}$, $\frac{\partial C(u,v)}{\partial u} \geq \frac{v-C(u,v)}{(1-u)}$ for almost all u;*
4. *$RTI(X|Y)$ if and only if for any u in $\mathbf{I}$, $\frac{\partial C(u,v)}{\partial v} \geq \frac{u-C(u,v)}{(1-v)}$ for almost all v.*

When X and Y are exchangeable, there are no reason to distinguish $SI(Y|X)$ and $SI(X|Y)$, which will be both noted SI. Similarly, we will denote LTD the equivalent properties $LTD(Y|X)$ and $LTD(X|Y)$, and RTI, $RTI(Y|X)$ or $RTI(X|Y)$. The following theorem in [1] is devoted to the study of properties of positive dependence of any pair (X, Y) associated with the copula C_θ defined by (2). Similar results can be established for the corresponding concepts of negative dependence.

Theorem 2 *Let $\theta > 0$ and (X, Y) a random pair with copula C_θ.*

- *X and Y are PFD.*
- *X and Y are PQD if and only if either $\forall u \in I, \phi(u) \geq 0$ or $\forall u \in I, \phi(u) \leq 0$.*
- *X and Y are LTD if and only if $\phi(u)/u$ is monotone.*
- *X and Y are RTI if and only if $\phi(u)/(u-1)$ is monotone.*
- *X and Y are $LCSD$ if and only if they are LTD.*
- *X and Y are $RCSI$ if and only if they are RTI.*
- *X and Y are SI if and only if $\phi(u)$ is either concave or convex.*
- *X and Y have the $TP2$ density property if and only if they are SI.*

3 The general case

Many of the dependence properties encountered in earlier sections have natural extensions to the multivariate case. In three or more dimensions, rather than quadrants we have "orthants," and the generalization of quadrant dependence is known as *orthant dependence*. First of all we recall the definition of n-copula due to A. Sklar in 1959: an *n-copula* is the restriction to the unit cube $[0, 1]^n$ of a multivariate cumulative distribution function, whose marginals are uniform on $[0, 1]$.
More precisely, an n-copula is a function $C : [0, 1]^n \to [0, 1]$ that satisfies:

(a) $C(\mathbf{u}) = 0$ if $u_i = 0$ for any $i = 1, \ldots, n$, that is C is *grounded*;

(b) $C(\mathbf{u}) = u_i$ if all coordinates of $\mathbf{u}$ are 1 except u_i, that is C has *uniform one-dimensional marginals*;

(c) C is *n-increasing*, i.e. $V_C(B) \geq 0$ for any n-box $B = [u_1, v_1] \times [u_2, v_2] \times \ldots \times [u_n, v_n] \subseteq [0,1]^n$ with $u_i \leq v_i$, $i = 1, 2, \ldots, n$, where the C-volume of the n-box B is given by

$$V_C(B) = \sum \epsilon(z_1, \ldots, z_n) \cdot C(z_1, \ldots, z_n) \geq 0, \tag{6}$$

with

$$\epsilon(z_1, \ldots, z_n) = \begin{cases} 1 & \text{if } z_i = u_i \text{ for an even number of i's,} \\ -1 & \text{if } z_i = u_i \text{ for an odd number of i's} \end{cases}$$

and the sum in (6) is extended to all vertices of B.

Conditions (a) and (b) are known as *boundary conditions*, whereas condition (c) is known as *monotonicity.*
Now we are going to examine the role played by n-copulas in the study of multivariate dependence.

Definition 2 *Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be an n-dimensional random vector.*

1. $\mathbf{X}$ *is positively lower orthant dependent (PLOD) if for all* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *in* $\mathbf{R}^n$,

$$P[\mathbf{X} \leq \mathbf{x}] \geq \prod_{i=1}^{n} P[X_i \leq x_i]; \tag{7}$$

2. $\mathbf{X}$ *is positively upper orthant dependent (PUOD) if for all* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *in* $\mathbf{R}^n$,

$$P[\mathbf{X} > \mathbf{x}] \geq \prod_{i=1}^{n} P[X_i > x_i]; \tag{8}$$

3. $\mathbf{X}$ *is positively orthant dependent (POD) if for all* $\mathbf{x}$ *in* $\mathbf{R}^n$*, both* (7) *and* (8) *hold.*

Negative lower orthant dependence (NLOD), negative upper orthant dependence (PUOD) and negative orthant dependence (NOD) are defined analogously, by reversing the sense of the inequalities in (7) and (8).
For $n = 2$, (7) and (8) are equivalent to (3).
The following definitions are from Brindley and Thompson (1972), Harris (1970), Joe (1997).

Definition 3 *Let $X = (X_1, X_2, \ldots, X_n)$ be an n-dimensional random vector and let the sets A and B partition of {1,2,...,n}.*

1. *LTD*$(\mathbf{X}_B|\mathbf{X}_A)$ *if* $P[\mathbf{X}_B \leq \mathbf{x}_B|\mathbf{X}_A \leq \mathbf{x}_A]$ *is nonincreasing in* $\mathbf{x}_A$ *for all* $\mathbf{x}_B$*;*
2. *RTI*$(\mathbf{X}_B|\mathbf{X}_A)$ *if* $P[\mathbf{X}_B > \mathbf{x}_B|\mathbf{X}_A > \mathbf{x}_A]$ *is nondecreasing in* $\mathbf{x}_A$ *for all* $\mathbf{x}_B$*;*

3. *$SI(\mathbf{X}_B|\mathbf{X}_A)$ if $P[\mathbf{X}_B > \mathbf{x}_B|\mathbf{X}_A = \mathbf{x}_A]$ is nondecreasing in $\mathbf{x}_A$ for all $\mathbf{x}_B$;*

4. *$LCSD(\mathbf{X})$ if $P[\mathbf{X} \le \mathbf{x}|\mathbf{X} \le \mathbf{x}']$ is nonincreasing in $\mathbf{x}'$ for all $\mathbf{x}$;*

5. *$RCSI(\mathbf{X})$ if $P[\mathbf{X} > \mathbf{x}|\mathbf{X} > \mathbf{x}']$ is nondecreasing in $\mathbf{x}'$ for all $\mathbf{x}$.*

We recall that for $\mathbf{x} \in \mathbb{R}^n$ a phrase such as "nondecreasing in $\mathbf{x}$" means nondecreasing in each component $x_i,\ i = 1, 2, \ldots, n$.

In the bivariate case, the corner set monotonicity properties were expressible in terms of total positivity (Corollary 5.2.16 in [8]). The same is true in the multivariate case with the following generalization of total positivity: a function f from $\overline{\mathbf{R}}^n$ to $\mathbf{R}$ is *multivariate totally positive of order two* (MTP_2) if

$$f(\mathbf{x} \vee \mathbf{y})f(\mathbf{x} \wedge \mathbf{y}) \ge f(\mathbf{x})f(\mathbf{y}) \tag{9}$$

for all $\mathbf{x}, \mathbf{y} \in \overline{\mathbf{R}}^n$, where

$$\mathbf{x} \vee \mathbf{y} = (\max(x_1, y_1), \max(x_2, y_2), \ldots, \max(x_n, y_n)),$$

$$\mathbf{x} \wedge \mathbf{y} = (\min(x_1, y_1), \min(x_2, y_2), \ldots, \min(x_n, y_n)).$$

Lastly, $\mathbf{X}$ is positively likelihood ratio dependent if its joint n-dimensional density h is MTP_2.

A first one-parameter multivariate extension of the class of copulas given by (1) is

$$C_\theta(\mathbf{u}) = \prod_{i=1}^{n} u_i + \theta \prod_{i=1}^{n} \phi_i(u_i), \qquad \mathbf{u} \in [0,1]^n, \tag{10}$$

where $\theta \in \mathbf{R}$ and ϕ_i, $1 \le i \le n$, are n non-zero absolutely continuous functions such that $\phi_i(0) = \phi_i(1) = 0$. Note that all the $k-$dimensional margins, $2 \le k < n$, are $\prod^k$. The density function of (10) is

$$c_\theta(\mathbf{u}) = 1 + \theta \prod_{i=1}^{n} \phi_i'(u_i), \tag{11}$$

whose parameter θ has the admissible range

$$-1/\sup_{\mathbf{u} \in D^+}(\prod_{i=1}^{n})\phi_i'(u_i) \le \theta \le -1/\inf_{\mathbf{u} \in D^-}(\prod_{i=1}^{n})\phi_i'(u_i),$$

where $D^- = \{\mathbf{u} \in [0,1]^n : \prod_{i=1}^n \phi_i'(u_i) < 0\}$ and $D^+ = \{\mathbf{u} \in [0,1]^n : \prod_{i=1}^n \phi_i'(u_i) > 0\}$.

The survival function and the survival n-copula associated with the n-copula C_θ are given by $\overline{C}_\theta(\mathbf{u}) = \prod_{i=1}^n (1-u_i) + (-1)^n \theta \prod_{i=1}^n \phi_i(u_i)$ and $\hat{C}_\theta(\mathbf{u}) = \prod_{i=1}^n u_i + (-1)^n \theta \prod_{i=1}^n \phi_i(1-u_i)$, respectively, for every $\mathbf{u} \in [0,1]^n$.

Let C_θ be the corresponding family of n-copulas given by (10). Then, C_θ is positively ordered if and only if $\prod_{i=1}^n \phi_i(u_i) \ge 0$ for all $\mathbf{u}$ in $[0,1]^n$.

Let $C_{\theta_1}(\mathbf{u}) = \prod_{i=1}^n u_i + \theta_1 \prod_{i=1}^n \phi_i(u_i)$ and $C_{\theta_2}(\mathbf{u}) = \prod_{i=1}^n u_i + \theta_2 \prod_{i=1}^n \gamma_i(u_i)$ be two

$n-$copulas. Then, C_{θ_1} is more PLOD (respectively, PUOD) than C_{θ_2} if and only if $\theta_1 \prod_{i=1}^{n} \phi_i(u_i) \geq \theta_2 \prod_{i=1}^{n} \gamma_i(u_i)$ (respectively, $(-1)^n \theta_1 \prod_{i=1}^{n} \phi_i(1-u_i) \geq (-1)^n \theta_2 \prod_{i=1}^{n} \gamma_i(1-u_i)$).
Much of the theory of bivariate dependence presents considerable difficulty when one attempts to generalize it to more than two dimensions. We want to extend in this paper to more than two random variables, $X_1, \ldots, X_n$ the problem of dependence.
The following theorem is from Dolati and Úbeda-Flores (2006) [4].

Theorem 3 *Let* $\mathbf{X}$ *be an* n*-dimensional random vector whose associated* n*-copula* C_θ *is defined by* (10) *and such that the functions* ϕ_i, $i = 1, \ldots, n$ *and* θ *are non-negative. Let* $\mathbf{X}_A$ *and* $\mathbf{X}_B$ *be two subsets of* $\mathbf{X}$ *as in the preceding definition. Then:*

(i) $LTD(\mathbf{X}_B|\mathbf{X}_A)$ *if and only if* $\phi_i(u) \geq u\phi_i'(u)$ *for all* $u \in [0,1]$ *and for every* $i \in A$;

(ii) $RTI(\mathbf{X}_B|\mathbf{X}_A)$ *if and only if* $\phi_i(u) \geq (u-1)\phi_i'(u)$ *for all* $u \in [0,1]$ *and for every* $i \in A$;

(iii) $SI(\mathbf{X}_B|\mathbf{X}_A)$ *if and only if* $(-1)^n \phi_i''(u) \prod_{h \in A - \{i\}} \phi_h'(u_h) \geq 0$ *for every* $i \in A$, *and* $u,\ u_h \in [0,1]$.

3.1 Other properties

Now we want to study the previous properties extended to n dimensions, using the copula approach, in particular with regard to the family given by (10). So, we prove the following theorem.

Theorem 4 *Let* $\mathbf{X}$ *be an* n*-dimensional random vector whose associated* n*-copula* C_θ *is defined by* (10) *and such that the functions* ϕ_i, $i = 1, \ldots, n$ *and* θ *are non-negative. Let* $\mathbf{X}_A$ *and* $\mathbf{X}_B$ *be two subsets of* $\mathbf{X}$ *as in the preceding theorem. Then:*

(i) $\mathbf{X}$ *is* PFD *if* n *is even;*

(ii) $\mathbf{X}$ *is* $PLOD$;

(iii) $\mathbf{X}$ *is* MTP_2 *if* $\mathbf{X}_A$ *and* $\mathbf{X}_B$ *are* LTD;

(iv) $\mathbf{X}$ *is* $RCSI$ *if* $\mathbf{X}_A$ *and* $\mathbf{X}_B$ *are* RTI;

(v) $\mathbf{X}_A$ *and* $\mathbf{X}_B$ *are* SI *if and only if* $\mathbf{X}$ *has the* MTP_2 *density property.*

Proof.

(i) Let g be an integrable real-valued function on I. The density distribution c_θ of the cumulative distribution C_θ is given by (11). Routine calculations yield

$$\mathbf{E}_{c_\theta}[g(X_1)\ldots g(X_n)] - \mathbf{E}_{c_\theta}[g(X_1)]\ldots\mathbf{E}_{c_\theta}[g(X_n)] = \theta\left[\int_0^1 g(t)\phi_i'(t)dt\right]^n \geq 0,$$

since $\theta \geq 0$ and n is even.

(ii) The vector $\mathbf{X}$ is $PLOD$ if and only if the uniform I-margins vector $\mathbf{U}$ with distribution C_θ is $PLOD$. For $\mathbf{U}$, condition (7) simply rewrites $C(u_1,\dots,u_n) \geq u_1 \dots u_n$, that is $\theta \prod_{i=1}^n \phi_i(u_i) \geq 0$, $\forall u_i \in I$ and the conclusion follows.

(iii) Let the partition of $\{1,2,\dots,n\}$ be in two subsets A and B, such that $\max(u_i,v_i) = u_i$ and $\max(u_j,v_j) = v_j$, $\forall i \in A$ and $\forall j \in B$ respectively. So,

$$C_\theta(\mathbf{u} \vee \mathbf{v}) = C_\theta(\dots,u_i,\dots,v_j,\dots) = \prod_{\substack{i\in A\\ j\in B}} u_i v_j + \theta \prod_{\substack{i\in A\\ j\in B}} \phi_i(u_i)\phi_j(v_j) \qquad \mathbf{u},\mathbf{v} \in [0,1]^n,$$

and

$$C_\theta(\mathbf{u} \wedge \mathbf{v}) = C_\theta(\dots,u_i,\dots,v_j,\dots) = \prod_{\substack{i\in A^C\\ j\in B^C}} u_i v_j + \theta \prod_{\substack{i\in A^C\\ j\in B^C}} \phi_i(u_i)\phi_j(v_j) \qquad \mathbf{u},\mathbf{v} \in [0,1]^n.$$

We observe that $A^C = B$ and $A \cup A^C = \{1,\dots,n\}$. Therefore

$$\begin{aligned}
&C_\theta(\mathbf{u} \vee \mathbf{v})C_\theta(\mathbf{u} \wedge \mathbf{v}) - C_\theta(\mathbf{u})C_\theta(\mathbf{v}) = \\
&= \Big(\prod_{\substack{i\in A\\ j\in B}} u_i v_j + \theta \prod_{\substack{i\in A\\ j\in B}} \phi_i(u_i)\phi_j(v_j)\Big)\Big(\prod_{\substack{i\in A^C\\ j\in B^C}} u_i v_j + \theta \prod_{\substack{i\in A^C\\ j\in B^C}} \phi_i(u_i)\phi_j(v_j)\Big) - \\
&- \Big(\prod_{i=1}^n u_i + \theta \prod_{i=1}^n \phi_i(u_i)\Big)\Big(\prod_{i=1}^n v_i + \theta \prod_{i=1}^n \phi_i(v_i)\Big) = \\
&= \Big(\prod_{i=1}^n u_i v_i + \theta^2 \prod_{i=1}^n \phi_i(u_i)\phi_i(v_i) + \theta \prod_{\substack{i\in A\\ j\in B}} u_i v_j \prod_{\substack{i\in A^C\\ j\in B^C}} \phi_i(u_i)\phi_j(v_j) + \\
&+ \theta \prod_{\substack{i\in A^C\\ j\in B^C}} u_i v_j \prod_{\substack{i\in A\\ j\in B}} \phi_i(u_i)\phi_j(v_j)\Big) - \Big(\prod_{i=1}^n u_i v_i + \theta^2 \prod_{i=1}^n \phi_i(u_i)\phi_i(v_i) + \\
&+ \theta \prod_{i=1}^n u_i \phi_i(v_i) + \theta \prod_{i=1}^n v_i \phi_i(u_i)\Big).
\end{aligned}$$

So, by rearranging the expression, we have

$$\begin{aligned}
&C_\theta(\mathbf{u} \vee \mathbf{v})C_\theta(\mathbf{u} \wedge \mathbf{v}) - C_\theta(\mathbf{u})C_\theta(\mathbf{v}) = \\
&= \theta \prod_{i=1}^n u_i v_i \Big(\prod_{\substack{i\in A\\ j\in B}} \frac{\phi_i(u_i)\phi_j(v_j)}{u_i v_j} + \prod_{\substack{i\in A^C\\ j\in B^C}} \frac{\phi_i(u_i)\phi_j(v_j)}{u_i v_j} - \prod_{i=1}^n \frac{\phi_i(u_i)}{u_i} - \prod_{i=1}^n \frac{\phi_i(v_i)}{v_i}\Big) = \\
&= \theta \prod_{i=1}^n u_i v_i \left[\prod_{i\in A} \frac{\phi_i(u_i)}{u_i} - \prod_{i\in A} \frac{\phi_i(v_i)}{v_i}\right]\left[\prod_{j\in B} \frac{\phi_j(v_j)}{v_j} - \prod_{j\in B} \frac{\phi_j(u_j)}{u_j}\right].
\end{aligned}$$

Now $\dfrac{\phi_i(u)}{u}$ is derivable because the ratio of two derivable functions and we have

$$\frac{d}{du}\Big(\prod_{i\in A}\frac{\phi_i(u)}{u}\Big)=\Big(\frac{\phi_i'(u)u-\phi_i(u)}{u^2}\Big)\prod_{h\in A\setminus\{i\}}\frac{\phi_h(u_h)}{u_h}\leq 0, \qquad \forall u\in[0,1]$$

for the hypothesis of LTD. The same happens to the other factor. So we have two monotonically decreasing functions and, as a consequence, MTP_2 property, that is our thesis.

(iv) It is similar to (iii). In fact $\mathbf{X}$ is $RCSI$ if and only if the survival copula associated to C, $\hat{C}_\theta(\mathbf{u})=\prod_{i=1}^n u_i+(-1)^n\theta\prod_{i=1}^n\phi_i(1-u_i)$ is MTP_2. So we have

$$\hat{C}_\theta(\mathbf{u}\vee\mathbf{v})\hat{C}_\theta(\mathbf{u}\wedge\mathbf{v})-\hat{C}_\theta(\mathbf{u})\hat{C}_\theta(\mathbf{v})=$$

$$=(-1)^n\theta\prod_{i=1}^n u_iv_i\left[\prod_{i\in A}\frac{\phi_i(1-u_i)}{u_i}-\prod_{i\in A}\frac{\phi_i(1-v_i)}{v_i}\right]\left[\prod_{j\in B}\frac{\phi_j(1-v_j)}{v_j}-\prod_{j\in B}\frac{\phi_j(1-u_j)}{u_j}\right].$$

Now we do the same thought as in the previous case:

$$\Big(\frac{\phi_i(1-u)}{u}\Big)'=\frac{-u\phi_i'(1-u)-\phi_i(1-u)}{u^2}.$$

We use RTI property, by putting $u'=1-u$ and in fact we have

$$-u\phi_i'(1-u)-\phi_i(1-u)=(u'-1)\phi_i'(u')-\phi_i(u')\leq 0, \qquad \forall u'\in[0,1]$$

and so we have MTP_2 property again.

(v) X has the MTP_2 density property if and only if the density of the copula verifies

$$c_\theta(\mathbf{u}\vee\mathbf{v})c_\theta(\mathbf{u}\wedge\mathbf{v})-c_\theta(\mathbf{u})c_\theta(\mathbf{v})\geq 0, \tag{12}$$

which rewrites solving the calculations like in the point (iii)

$$c_\theta(\mathbf{u}\vee\mathbf{v})c_\theta(\mathbf{u}\wedge\mathbf{v})-c_\theta(\mathbf{u})c_\theta(\mathbf{v})=\Big(1+\theta\prod_{\substack{i\in A\\ j\in B}}\phi_i'(u_i)\phi_j'(v_j)\Big)\Big(1+\theta\prod_{\substack{i\in A^C\\ j\in B^C}}\phi_i'(u_i)\phi_j'(v_j)\Big)-$$

$$-\Big(1+\theta\prod_{i=1}^n\phi_i'(u_i)\Big)\Big(1+\theta\prod_{i=1}^n\phi_i'(v_i)\Big)=\Big(1+\theta^2\prod_{i=1}^n\phi_i'(u_i)\phi_i'(v_i)+\theta\prod_{\substack{i\in A^C\\ j\in B^C}}\phi_i'(u_i)\phi_j'(v_j)+$$

$$+\theta\prod_{\substack{i\in A\\ j\in B}}\phi_i'(u_i)\phi_j'(v_j)\Big)-\Big(1+\theta^2\prod_{i=1}^n\phi_i'(u_i)\phi_i'(v_i)+\theta\prod_{i=1}^n\phi_i'(v_i)+\theta\prod_{i=1}^n\phi_i'(u_i)\Big)=$$

$$=\theta\Big(\prod_{\substack{i\in A\\ j\in B}}\phi_i'(u_i)\phi_j'(v_j)+\prod_{\substack{i\in A^C\\ j\in B^C}}\phi_i'(u_i)\phi_j'(v_j)-\prod_{i=1}^n\phi_i'(v_i)-\prod_{i=1}^n\phi_i'(u_i)\Big)=$$

$$=\theta\left[\prod_{i\in A}\phi_i'(u_i)-\prod_{i\in A}\phi_i'(v_i)\right]\left[\prod_{j\in B}\phi_j'(v_j)-\prod_{j\in B}\phi_j'(u_j)\right].$$

Now,

$$\frac{d}{du}\Big(\prod_{i\in A}\phi_i'(u_i)\Big) = \pm\,\phi_i''(u)\prod_{h\in A\setminus\{i\}}\phi_h'(u_h) \geq 0$$

for our hypothesis. The same happens to the other factor and so we have proved our thesis.
Conversely, assume that (12) holds. So, the function $\prod_{i\in A}\phi_i'$ is either increasing or decreasing and then $\mathbf{X}_A$ and $\mathbf{X}_B$ are SI.

Example We can consider the example 2.2 proposed by Dolati and Úbeda-Flores in [4]. Let $f_i(u) = u^b(1-u)^a$, $1 \leq i \leq 3$, with $a, b \geq 1$. Then, for all $(u_1, u_2, u_3) \in [0,1]^3$, the function

$$C_\theta(u_1, u_2, u_3) = u_1u_2u_3 + \theta u_1^b(1-u_1)^a u_2^b(1-u_2)^a u_3^b(1-u_3)^a$$

is a 3-copula. In particular, if $a = b = 1$, we have a one-parametric trivariate extension of the FGM family with $\theta \in [-1,1]$. Suppose $\theta > 0$, then, from theorem 2.1 in [4] we have that C_θ is LTD if and only if $b = 1$, C_θ is RTI if and only if $a = 1$, and C_θ is SI if and only if $a = b = 1$.
As a consequence from the theorem 4, we can also conclude that C_θ is MTP_2 if $b = 1$. If $a = 1$ C_θ is $RCSI$ and it has the MTP_2 density property if and only if $a = b = 1$. Moreover C_θ is $PLOD$, but it is not PFD.

4 Concluding remarks

In this work we have studied a multivariate generalization of one-parametric family of copulas. In particular we have analyzed concepts of dependence with regard to the example (2) in [4]. In fact we have continued that analysis, by extending the links between these concepts and exploring ways in which copulas can be used in the study of dependence between random variables. However, the case with $\phi_i(u_i)$ and θ negative and with more parameters are open problems. Moreover, the study of dependence properties for other classes of one-parametric n-copulas that generalize (1) can be considered in a further work.

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