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# The probability to reach an agreement as a foundation for axiomatic bargaining

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# The probability to reach an agreement as a foundation for axiomatic bargaining*


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**Abstract.** We revisit the Nash bargaining model and axiomatize a procedural solution that maximizes the probability of successful bargaining. This probability-based approach nests both the standard and the ordinal Nash solution, and yet need not assume that bargainers have preferences over lotteries or that choice sets are convex. We consider both mediator-assisted bargaining and standard unassisted bargaining. We solve a long-standing puzzle and offer a natural interpretation of the product operator underlying the Nash solution. We characterize other known solution concepts, including the egalitarian and the utilitarian solutions.



**Keywords:** cooperative bargaining, mediation, arbitration, benchmarking, copulas.

**JEL Classification Numbers:** C78, D81, D74.



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# Introduction

The axiomatic foundations laid out by Nash (1950) stand out as a cornerstone of two-person bargaining theory for simplicity and elegance. They have spawned a huge body of literature; see Thomson (1994). Along many glories, however, there is something amiss.

The Nash solution recommends to pick an outcome that maximizes the product of bargainers' utilities. Many economic models turn this precept into a shortcut for predicting how unstructured bargaining will be resolved, but hardly anybody use it in real situations. One reason for this disconnect is that the Nash solution "lacks a straightforward interpretation since the meaning of the product of two von Neumann–Morgenstern utility numbers is unclear" (Rubinstein, Safra and Thomson, 1992: 1172). It is difficult to advocate a solution that the bargainers cannot make sense of.

From a theoretical viewpoint, the Nash model assumes that the bargainers have expected utility preferences and abstracts away all the particulars except for their risk attitude. For example, if two risk-neutral agents bargain over a prize worth 1 euro to the first agent and $x$ euro to the second one, the Nash solution prescribes the same division regardless whether $x = 1$ or $x = 1000$. It is a challenge to argue that bargainers' risk aversion is the only defining issue when seeking a solution.

Motivated by these difficulties, Rubinstein, Safra and Thomson (1992: 1173) "switch from utility language to alternatives-preference language" and make three important contributions. First, they keep track of the physical alternatives and let bargainers have non-expected utility preferences. Second, they redirect the axiomatization of a solution from consistency across bargaining problems over different sets of alternatives towards consistency across different bargainers' preferences over a fixed set of alternatives. Third, their definition of an *ordinal-Nash solution* nests the standard Nash solution, but delivers a clear interpretation in terms of the competing risks faced by bargainers insisting to settle on different outcomes.

We move to a probability-based language and advance their work in all three directions. First, we let bargainers have only ordinal preferences over physical alternatives, dispensing with the (covert) requirement that they also rank lotteries among alternatives. Second, we introduce uncertainty over which alternatives bargainers are willing to accept; then, we formalize a benchmarking procedure that manages such uncertainty consistently with bargainers' ordinal preferences over the set of feasible alternatives. Third, we illustrate and axiomatize a solution that maximizes the probability that the bargainers reach an agreement; such solution encompasses the ordinal-Nash solution and provides a sound prescriptive advice in real situations. Finally, we extend our results to other major bargaining solutions and provide a unified interpretation for them.

A simple example is useful to illustrate our approach. Two agents bargain over a set $A$ of feasible alternatives, described in physical terms. Assume that $A$ is a nonempty, compact and convex subset of $\mathbb{R}^n$. Each bargainer $i = 1, 2$ has an ordinal continuous preference $\succsim_i$ over $A$.

The two agents hire a mediator to recommend a solution and help them reach an agreement. Agents' ordinal preferences are commonly known, but the mediator does not know what it takes for an agent to accept a proposal $a$ from $A$.

More formally, suppose that $i$ accepts a proposal $a$ if and only if $a \succsim_i t_i$, where $t_i$ in $A$ is $i$'s acceptance threshold or, for short, his *target*. The mediator has incomplete information about the bargainers' targets: she believes that each target is a random variable $T_i$, with a compact and convex support in $A$. Under her beliefs, she maps each proposal $a$ to a pair of individual acceptance probabilities $(p_1, p_2)$ in $[0, 1]^2$, where $p_i = P(a \succsim_i T_i)$. If the bargainers' targets are stochastically independent, the probability that both accept $a$ is the product $p_1 \cdot p_2$ of the individual acceptance probabilities.

The mediator can recommend any feasible alternative, but she cannot impose it: if she suggests $a$, it is left to the bargainers to accept it. Her goal is to find a proposal $a$ that maximises the probability that agents reach an agreement: if she believes that bargainers' targets are stochastically independent, she should advance a proposal $a$ that maximises the product $p_1 \cdot p_2$. In this example, we interpret the Nash solution as a rule for the mediator: maximise the probability that the bargainers reach an agreement, given that their targets are private information and independently distributed. From a prescriptive viewpoint, the mediator may use $p_1 \cdot p_2$ to *construct* a ranking over alternatives consistent with bargainers' ordinal preferences and she can provide a clear-cut argument for her recommendation.

The paper extends this example in two directions: a) we dispense with mediation and deal with unassisted bargaining; and b) we use the theory of copulas making stochastic independence a special case. Section 1 describes our approach and traces it back to Nash (1950). Section 2 highlights its key components for the special case of a mediator. Section 3 illustrates benchmarking as a procedure to align the incomplete information of an agent with the ordinal preferences of another bargainer. Section 4 provides the main results: assuming unassisted bargaining, we characterize the class of solutions that maximize the probability to reach an agreement, and the three solutions known as Nash, egalitarian and utilitarian as prominent special cases. Section 5 offers a commentary. Technicalities and proofs are relegated to the appendix.

# 1 The framework

The bargaining problem in Nash (1950: 155; emphasis added) considers a situation where two parties may cooperate to their mutual benefit, but "no action taken by one of the individuals *without the consent of the other* can affect the well-being of the other one." Thus, either agent is able to enforce a *default outcome*. Nash postulates "that the two individuals are highly rational, that each can accurately compare his desires for various things, that they are equal in bargaining skill, and that each has full knowledge of the tastes and preferences of the other." Then he argues that "a theoretical treatment of bargaining situations [abstracts] from the situation to form a mathematical model in terms of which to develop the theory." Our framework complies with these desiderata.

There is a set $A$ of available (physical) alternatives, including a default outcome $\delta$ that occurs if bargaining breaks down. We assume that $A$ is a subset of a topological space, endowed with the relative topology. Each bargainer $i = 1, 2$ has a continuous[1] preference relation $\succeq_i$ on $A$; thus, he has a most preferred element for each compact subset of $A$.

The quadruple $(A, \delta; \succeq_1, \succeq_2)$ is a *bargaining problem with ordinal preferences.* It is a modest but uncontroversial conclusion that any alternative $a^*$ associated with a solution to this problem should be:

*feasible:* $a^* \in A$;
*individually rational:* $a^* \succeq_i \delta$ for $i = 1, 2$;
*Pareto optimal:* there is no $a$ in $A$ such that $a \succeq_i a^*$ for $i$ and $a \succ_j a^*$ for $j = 3 - i$.

Individual rationality recognizes that no rational bargainer would consent to an outcome strictly worse than what he can unilaterally enforce. Pareto optimality prevents wasting any opportunity to benefit an agent without harming the other one.

Let $A^*$ be the set of feasible alternatives that are individually rational and Pareto optimal for the bargaining problem $(A, \delta; \succeq_1, \succeq_2)$. Clearly, $i$ and $j$ have opposing preferences on $A^*$. We assume that $A^*$ is compact so each bargainer $i$ has a *best choice* $M_i$ in $A^*$. Without loss of generality, let $M_i$ be unique and $M_1 \neq M_2$. Then $M_i \succ_i M_j$ for $j = 3 - i$ and $M_j$ is the *worst choice* in $A^*$ for $i$. All of the above is commonly known to the bargainers and to any third party—e.g., a mediator.

Any (non-empty) subset of $A^*$ is called an *ordinal solution.* When an ordinal solution can be rationalized as the set of maximal outcomes with respect to a complete ranking on $A$, we say that it is a *procedural solution.* This paper is concerned with procedural solutions.

Nash (1950) perfects the bargaining problem with ordinal preferences by adding the assumption that bargainers maximize expected utility.[2] It is common knowledge that each bargainer has expected utility preferences over the set of lotteries on $A$, with a Bernoulli index $u_i : A \to \mathbb{R}$ that is consistent with $\succeq_i$ on $A$ and is unique up to increasing affine transformations. For short, we say that the two bargainers have *EU-preferences* represented by the utility functions $u_1, u_2$.

The solution axiomatized by Nash concerns the smaller class of *bargaining problems with EU-preferences* $(A, \delta; u_1, u_2)$. The Nash solution is procedural, because it is rationalized by the ranking associated with the product

$$[u_1(a) - u_1(\delta)] \cdot [u_2(a) - u_2(\delta)]$$

[1] A preference relation is continuous on $A$ if, for any $a \in A$, the set $W(a) = \{b \in A : b \prec a\}$ of strictly worse alternatives and the set $B(a) = \{b \in A : b \succ a\}$ of strictly better alternatives are open.

[2] Nash (1950) offers no justification for this modelling choice, but Nash (1953: 128) argues for it with reference to agents' risk attitudes. Bleichrodt et al. (2016) suggest why at the time of writing the first paper Nash may have found his choice so compelling not to warrant a discussion.

This paper demonstrates a more general approach to perfect ordinal preferences and use it to derive procedural solutions. The key technical step in Nash's treatment relies on the expected utility assumption to map the problem $(A, \delta; u_1, u_2)$ into a pair $(S, d)$, where $S \subseteq \mathbb{R}^2$ and $d \in S$ are respectively the (convex hull of the) image of $A$ and the image of $\delta$. After the mapping from $(A, \delta; u_1, u_2)$ to $(S, d)$ is established, Nash's axioms concern sets or points in $\mathbb{R}^2$.

Porting the original bargaining problem to $\mathbb{R}^2$ makes for a simple and elegant axiomatization, but Nash's approach forcibly casts $(S, d)$ into a utility-based language; see Rubinstein et al. (1992). In the next section we cast an analogous porting in a probability-based language (LiCalzi, 1999) and axiomatize a procedural solution that nests the Nash solution. Our approach is consistent with, but does not require, the assumption that bargainers have expected utility preferences. Using a probability-based language delivers a straightforward interpretation and provides a rationale for the solution more natural than multiplying bargainers' utility functions.

## 2 Mediator-assisted bargaining

Let $(A, \delta; \succeq_1, \succeq_2)$ be a bargaining problem with ordinal preferences. Consider a third party—a mediator—who is hired by the two bargainers to recommend an alternative over which they can reach an agreement. The mediator can choose any alternative from $A$, but cannot impose it. After she irrevocably selects a proposal $a$, each bargainer decides—individually and simultaneously–whether to accept or refuse it. The mediation is successful if both accept the mediator's proposal; otherwise, the bargainers obtain the default outcome $\delta$.

The mediator's goal is to maximise the probability of a successful mediation. We expect the mediator to acquaint herself with bargainers' expectations, prevailing social norms, customary or legal precedents, and other contextual elements that bear relevance to the problem at hand. Eventually, she confronts her (subjective) uncertainty about what each bargainer $i$ is willing to accept and turns to questions such as "*how likely is bargainer $i$ to accept a given alternative $a$?*"

Having reached her best understanding of the bargaining problem under consideration, the mediator formulates an assessment for the probability $P_i(a)$ that an alternative $a$ will be accepted by bargainer $i$. Let $P_i : A \to [0, 1]$ denote the mediator's assessment for bargainer $i = 1, 2$. The assessment $P_i$ might be generated by a model of bargainer's behavior as in the example from the introduction, but this is not a requirement. We assume only that the mediator's assessment $P_i$ satisfies two minimal properties of consistency with her knowledge of the bargaining problem.

C.1 (*Monotonicity*) $P_i(a)$ represents $\succeq_i$ in $A$.

This states that the mediator ranks the probabilities of acceptance for two alternatives consistently with the bargainer's preferences: that is, $i$ likes $a$ better than $a'$ if and only if the mediator believes that $i$ is more likely to accept $a$ than $a'$.

C.2 (*Bargainer's rationality*) $P_i(\delta) = 0$ and $P_i(M_i) = 1$.

The first part states that the mediator believes that $i$ would refuse the default outcome for sure, because he could secure it without going through the hassle of bargaining. The second part states that she believes that $i$ would accept for sure his best choice $M_i$ from the set $A^*$: a rational bargainer $i$ knows that he cannot obtain more than $M_i$, because $j$ would veto it. If he were not willing to accept his best choice $M_i$ for sure, he would have refused to enter (or would have walked away from) the bargaining situation.

The next section defines a *benchmarking procedure* by which the mediator can construct a consistent assessment $P_i$ against a standard of reference. Theorem 2 provides sufficient conditions for the existence of two continuous assessments $P_1$ and $P_2$ that satisfy C.1–C.2. For expositional purposes, this section assumes that the pair $(P_1, P_2)$ of mediator's assessments is given.

The pair $(P_1, P_2)$ maps a bargaining problem with ordinal preferences $(A, \delta; \succeq_1, \succeq_2)$ into a subset $B \subseteq [0,1]^2$, where each element of $B$ is a pair of acceptance probabilities $(p_1, p_2)$; see Figure 1. The default outcome $\delta$ is mapped to the point $(P_1(\delta), P_2(\delta)) = (0,0)$: thus $B$ contains the origin. The best choices $M_1$ and $M_2$ are mapped to two points $(1, p_2)$ and $(p_1, 1)$, with $p_2 = P_2(M_1)$ and $p_1 = P_1(M_2)$: thus both the intersections of $B$ with the line $p_1 = 1$ and with the line $p_2 = 1$ are not empty. Finally, the image of the (compact) Pareto optimal set $A^*$ maps to a (compact) Pareto frontier $B^*$ in $[0,1]$, because $P_1$ and $P_2$ are continuous.

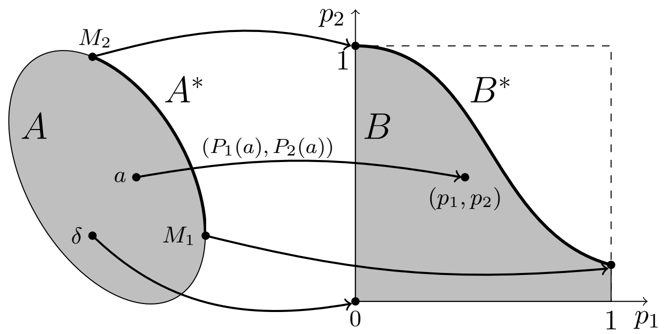


Figure 1: Mapping alternatives to acceptance probabilities.

In short, the mediator's assessments map the bargaining problem with ordinal preferences $(A, \delta; \succeq_1, \succeq_2)$ to a subset $B \subseteq [0,1]^2$ of pairs of acceptance probabilities $(p_1, p_2)$, which includes the origin, intersects $p_1 = 1$ and $p_2 = 1$, and has a compact Pareto frontier $B^*$.

The analogy with Nash's reformulation of a bargaining problem with EU-preferences as a pair $(S, d)$ should be apparent. The Nash construction uses the mapping $(u_1, u_2)$ associated with the two bargainer's utility functions and abstracts away the particulars except for their risk attitude. Our construction uses the mapping $(P_1, P_2)$ associated with the two mediator's assessments and abstracts away the particulars except for the mediator's beliefs. The Nash solution concerns bargaining problems with expected utility preferences. Our solution concerns bargaining problems with mediator's assessments. We show in Section 4 that the analogy runs much deeper and extends to the case of unassisted bargaining. The rest of this section provides a behavioral characterization for a procedural solution on $B$, defined as a (non-empty) selection from $B^*$ that can be rationalized by a complete ranking on $B$. It is understood that the selection from $B^*$ can be mapped back to a solution from $A^*$.

### A behavioral characterization

We assume that the mediator has a preference relation over all pairs of acceptance probabilities in $[0,1]^2$, consistent with her goal to maximize the probability of a successful mediation. Intuitively, one may imagine that she aggregates her (individual) probability assessments $P_1$ and $P_2$ into a single (joint) assessment for the probability of a successful mediation. The equivalent of the Nash solution is recovered when the two mediator's assessment are stochastically independent: then the probability that an alternative $a$ is accepted by both bargainers is the product of $P_1(a)$ and $P_2(a)$.

Denote by $\mathbf{p} = (p_1, p_2)$ an element in $[0,1]^2$. We view $[0,1]^2$ as a mixture space for the $\oplus$ operation, under the standard interpretation where $\alpha\mathbf{p} \oplus (1-\alpha)\mathbf{q}$ is a lottery that delivers $\mathbf{p}$ in $[0,1]^2$ with probability $\alpha$ in $[0,1]$ and $\mathbf{q}$ in $[0,1]^2$ with the complementary probability $1-\alpha$; see Herstein and Milnor (1953). Moreover, let $\mathbf{p} \vee \mathbf{q} = (\max(p_1, q_1), \max(p_2, q_2))$ and $\mathbf{p} \wedge \mathbf{q} = (\min(p_1, q_1), \min(p_2, q_2))$.

We assume that the mediator's preference relation $\succsim$ over $[0,1]^2$ satisfies four properties.

A.1 (Regularity) $\succsim$ is a complete preorder, continuous and mixture independent.

This implies that there exists a real-valued function $V : [0,1]^2 \to [0,1]$, unique up to positive affine transformations, that represents $\succsim$ and is linear with respect to $\oplus$; that is, $V(\alpha\mathbf{p} \oplus (1-\alpha)\mathbf{q}) = \alpha V(\mathbf{p}) + (1-\alpha)V(\mathbf{q})$, for any $\alpha$ in $[0,1]$ and any $\mathbf{p}, \mathbf{q}$ in $[0,1]^2$. See Theorem 8.4 in Fishburn (1970).

A.2 (Non-triviality) $(1,1) \succ (0,0)$.

This states that the mediator strictly prefers a proposal that is accepted for sure by both bargainers to another proposal that is refused for sure by both bargainers. Its purpose is to rule out the trivial case where the mediator is always indifferent.

A.3 (Disagreement indifference) for any $p, q$ in $[0,1]$, $(p, 0) \sim (0, q)$.

This is named after Assumption DI in Border and Segal (1997), who study a preference relation over utility-based solutions. In our framework, this property states that the mediator is indifferent about which bargainer refuses a proposal for sure.

$A.4$ (Consistency) For any $p, q \in [0, 1]$

$$p(1, q) \oplus (1-p)(0, q) \sim (p, q) \quad \text{and} \quad p(q, 1) \oplus (1-p)(q, 0) \sim (q, p).$$

This is a separability property. Suppose that bargainer $i$ is known to accept a proposal with probability $q$. Then the mediator is indifferent between proposing an alternative that bargainer $j = 3 - i$ accepts with probability $p$, or facing a lottery where with probability $p$ bargainer $j$ accepts for sure and with probability $1 - p$ rejects for sure.

Our first result implies that A.1-2-3-4 characterize the mediator's ranking over pairs of acceptance probabilities by the product rule. Its proof is postponed to Section 4, where the result is stated as Theorem 6.

**Theorem 1.** *The preference relation $\succsim$ on $[0, 1]^2$ satisfies A.1-2-3-4 if and only if it is represented by the product $p_1 \cdot p_2$.*

This result uncovers an appealing interpretation for the product operator underlying the Nash solution for cooperative bargaining. Given her probability assessments $P_1, P_2$ on $A$, the mediator's ranking over pairs of acceptance probabilities in $[0, 1]^2$ induces a preference relation over alternatives in $A$ represented by the function $V(a) = P_1(a) \cdot P_2(a)$. Thus, she recommends an alternative $a$ that maximizes the product $P_1(a) \cdot P_2(a)$. If the bargaining problem is framed with respect to acceptance probabilities (instead of utilities), the product operator corresponds to the assumption that the contribution of the individual probabilities to the joint acceptance probability satisfies stochastic independence.

Our result is a behavioral characterization. However, it fits also with the normative view expounded in Border and Segal (1997: 5), who interpret "axioms as characteristics of the arbitrator that both bargainers can accept". Similarly, it is consistent with the practitioners' view as argued by Subramanian (2010: 109): "the implications for negotiation strategy change dramatically when we move away from the assumption that dealmakers will accept deals that are just better than their [default outcome] to the more realistic and nuanced assumption that the likelihood the other side will say yes increases with the incentives to do so."

# 3 The benchmarking procedure

Let $(A, \delta; \succeq_1, \succeq_2)$ be a bargaining problem with ordinal preferences. We axiomatize a benchmarking procedure by which an agent $k \neq i$ (e.g., the mediator) formulates beliefs consistent with $i$'s ordinal preferences, using a standard of reference to calibrate her assessment $P_{k(i)}(a)$ for the probability that $i$ accepts $a$ as bargaining outcome. The benchmarking procedure generates a unique assessment for the acceptance probabilities of bargainer $i$, as perceived by agent $k$.

The *standard of reference* is the set $\mathcal{L}_i$ of simple lotteries over $\{M_i, \delta\}$. A lottery $L_p = pM_i \oplus (1-p)\delta$ in $\mathcal{L}_i$ yields $M_i$ with probability $p$ in $[0,1]$ and $\delta$ with probability $1-p$. Suppose that $k$ plays $L_p$ to select a recommendation for $i$: with probability $p$ she proposes $M_i$ and $i$ accepts for sure, and with probability $1-p$ she proposes $\delta$ and $i$ refuses for sure. Let $\hat{P}_{k(i)} : \mathcal{L} \to [0,1]$ be $k$'s assessment for a lottery $L_p$; then $\hat{P}_{k(i)}(L_p) = p$ for any $p$ because the outcome of $L_p$ is accepted with (objective) probability $p$.

The benchmarking procedure extends $\hat{P}_i$ on $\mathcal{L}_i$ to an *assessment* $P_{k(i)}$ on $A \cup \mathcal{L}_i$ by comparing each alternative $a$ against the lotteries in $\mathcal{L}_i$: if agent $k$ feels that $a$ is as likely to be accepted by $i$ as $L_p$, then she sets $P_{k(i)}(a) = p$.

Formally speaking, let $\succeq_i$ be the preference relation of bargainer $i$ on the set of alternatives $A$ and $\succeq_{k(i)}$ be a preorder on $A \cup \mathcal{L}_i$. We interpret an element in $A \cup \mathcal{L}_i$ as a proposal from agent $k$ to $i$: she can pick an alternative $a$ or (unbeknown to the bargainer) let Chance decide between $M_i$ and $\delta$. The preorder $\succeq_{k(i)}$ represents agent $k$'s ranking over her proposals, where $x \succeq_{k(i)} y$ if and only if agent $k$ believes that bargainer $i$ is no less likely to accept $x$ than $y$.

The two consistency requirements C.1–C.2 for $P_i$ from Section 2 correspond to two assumptions on the mediator's ranking $\succeq_{k(i)}$:

B.1 (*Monotonicity*) the preorder $\succeq_{k(i)}$ represents $\succeq_i$ on $A$.

B.2 (*Bargainer's rationality*) $\quad \delta \sim_{k(i)} L_0$ and $M_i \sim_{k(i)} L_1$.

The benchmarking procedure extends the assessment $\hat{P}_{k(i)}$ on $\mathcal{L}_i$ to an assessment $P_{k(i)}$ on $A \cup \mathcal{L}_i$ that satisfies B.1–B.2. The procedure is well-defined and yields a unique extension under two additional conditions. The first one is that shifting probability from $\delta$ to $M_i$ increases $k$'s confidence that $i$ accepts the proposal.

B.3 (*Stochastic dominance*) if $p > q$, then $L_p \succ_{k(i)} L_q$.

The last condition is technical. The subset $A$ has the relative topology. The set $\mathcal{L}_i$ has the natural topology generated by the metric $d(L_p, L_q) = |p-q|$. We endow the set $A \cup \mathcal{L}_i$ with the *disjoint union topology*: a set $\mathcal{O} \subset A \cup \mathcal{L}_i$ is open if and only if $\mathcal{O} \cap A$ and $\mathcal{O} \cap \mathcal{L}_i$ are open in $A$ and in $\mathcal{L}_i$, respectively.

B.4 (*Continuity*) the preorder $\succeq_{k(i)}$ is continuous in the disjoint union topology.

Our second result characterizes the benchmarking procedure. Its proof is in the Appendix.

**Theorem 2.** *Suppose that the preorder $\succeq_{k(i)}$ satisfies B.2-3-4. Then the benchmarking procedure uniquely extends the assessment $\hat{P}_{k(i)}$ on $\mathcal{L}_i$ to a continuous assessment $P_{k(i)}$ on $A \cup \mathcal{L}_i$ with $P_{k(i)}(\delta) = 0$ and $P_{k(i)}(M_i) = 1$, that represents $\succeq_{k(i)}$. Moreover, the restriction of $P_{k(i)}$ to $A$ is continuous and, if B.1 holds, also represents $\succeq_i$.*

### 3.1 Benchmarking and EU-preferences

As it turns out, the EU-preferences in Nash (1950, 1953) are a special case of the benchmarking procedure, when the set of feasible proposals includes all randomizations over the available (physical) alternatives in $A$. Under expected utility, the preferences of bargainer $i$ are defined over the set $\mathcal{L}(A)$ of lotteries on $A$. The choice space is no longer $A$ but $\mathcal{L}(A)$, and an EU-preference is represented by the expected value of a Bernoulli index $u_i : A \to \mathbb{R}$, consistent with $\succeq_i$ on $A$ and unique up to increasing affine transformations.

Suppose that an agent $k \neq i$ formulates beliefs $P_{k(i)}$ on $\mathcal{L}(A)$ consistent with $i$'s preferences on such set. Note that $i$'s preferences and $k$'s benchmarking concern $\mathcal{L}(A) \supseteq A \cup \mathcal{L}_i$: the agent $k \neq i$ derives beliefs for $i$'s probability to accept a (randomized) proposal from the larger set $\mathcal{L}(A)$, but her task is made easier because $k$ knows $i$'s preferences over all lotteries on $A$. Applying B.1 and B.2 on the domain $\mathcal{L}(A)$, it follows immediately that $k$ sets $P_{k(i)}(\alpha) = Eu_i(\alpha)$ for any (possibly, degenerate) lottery $\alpha \in \mathcal{L}(A)$, where $u_i$ is normalized so that $u_i(\delta) = 0$ and $u_i(M_i) = 1$.

This observation ties nicely the Nash approach with mediator-assisted bargaining. Suppose that the mediator $k \neq i$ faces a bargaining problem with EU-preferences and knows $i$'s and $j$'s preferences on $\mathcal{L}(A)$. If her ranking respects B.1 (Monotonicity) on the domain $\mathcal{L}(A)$ of $i$'s preferences and B.2 (Bargainer's rationality), then she sets $P_{k(i)}(\alpha) = Eu_i(\alpha)$; hence, her assessment for $i$'s acceptance probability of a (randomized) proposal $\alpha$ coincides with its expected utility for bargainer $i$. Then, given a bargaining problem with EU-preferences, Theorem 1 states that the mediator ranks alternatives as if she is maximizing the Nash product of the expected utilities for the two bargainers.

The target-based example in the introduction is another special case of the benchmarking procedure. Suppose that the mediator $k \neq i$ believes that each bargainer $i$ accepts any proposal that meets $i$'s target $T_i$, but knows only the probability distributions for $T_1$ and $T_2$. Then she would set $P_{k(i)}(a) = P(a \succeq_i T_i)$; see Castagnoli and LiCalzi (1996). If the mediator believes that $T_1$ and $T_2$ are stochastically independent and have continuous distributions, Theorem 1 applies again. More generally, Theorem 3 below deals with the case when independence does not hold. The following Section 4 dispenses with the mediator and assumes that each agent $j$ applies benchmarking to his counterparty $i$.

## 4 Unassisted bargaining

Consider a bargaining problem $(A, \delta; \succeq_1, \succeq_2)$ with ordinal preferences. Unassisted bargaining concerns the situation when there is no mediator. It is useful to conceive unassisted bargaining in two stages. In the first stage, the two bargainers deliberate over a set of guidelines to pinpoint a *shared ranking* over the feasible alternatives. This ranking substantiates normative issues and defines a common ground based on mutually acceptable principles, leading to a procedural solution $a$ that is submitted to each bargainer's individual consideration for approval. In the second stage, each bargainer decides whether to accept or refuse $a$. The bargaining is successful if both parties accept $a$; otherwise, the bargainers obtain the default outcome $\delta$.

Similarly to the mediator's case, each bargainer $i$ takes into account all relevant information and formulates an assessment for the probability $P_{i(j)}(a)$ that an alternative $a$ will be accepted by the other bargainer $j = 3 - i$; that is, the individual probability assessment $P_{i(j)}$ describes $i$'s beliefs about $j$'s propensity to accept an alternative $a$. We assume that $P_{i(j)}$ and $P_{j(i)}$ are commonly known and that each is derived using the benchmarking procedure in Section 3; see the discussion of Mechanism I in Harsanyi (1962: 33–34). In particular, $P_{i(j)}$ is consistent with $\succeq_j$ and uses the set $\mathcal{L}_j$ of simple lotteries on $\{M_j, \delta\}$ as standard of reference; moreover, $P_{i(j)}$ is continuous on $A \cup \mathcal{L}_j$ with $P_{i(j)}(\delta) = 0$ and $P_{i(j)}(M_j) = 1$. This set of assumptions for unassisted bargaining is weaker than Nash's (1950) model, as argued in Section 3.1.

This section characterises procedural solutions for unassisted bargaining via a natural generalization of the setup with a mediator in Section 2. In mediator-assisted bargaining, there is one mediator benchmarking her beliefs about the bargainers; in unassisted bargaining, each bargainer benchmarks his beliefs about his counterparty and such beliefs are commonly known. The latter generalizes the former when $P_{1(2)} = P_2$ and $P_{2(1)} = P_1$.

There are two key steps in our construction. The first step is to let the pair $(P_{2(1)}, P_{1(2)})$ of bargainers' assessments map the bargaining problem $(A, \delta; \succeq_1, \succeq_2)$ into a subset $B \subseteq [0,1]^2$ of acceptance probabilities. Each element in $B$ represents a pair $(p_1, p_2)$ of acceptance probabilities, where $p_i = P_{j(i)}(a)$ represents $j$'s assessment about $i$'s propensity to accept $a$ if this is submitted for individual consideration in the second stage. As in Section 2, the set $B$ includes the origin, intersects $p_1 = 1$ and $p_2 = 1$, and has a compact Pareto frontier $B^*$.

The second step is to deal with the bargainers' shared ranking instead of the mediator's ranking. The axioms on the shared ranking $\succeq$ subsume the normative or prescriptive issues faced in the first stage of bargaining. They can also be interpreted as behaviorally observable choices, where $a \succeq a'$ means that both bargainers agree to rank $a$ above $a'$ on general principles when seeking a procedural solution.

Next, we derive characterizations of the shared ranking $\succeq$ under which the bargainers strive to maximize the probability of successful bargaining, similarly to the mediator's effort of maximizing the probability of a successful mediation.

### General characterisations

We assume that the bargainers' shared ranking $\succeq$ over $[0,1]^2$ satisfies five properties. The first three properties are exact analogs of those presented for the mediator-assisted bargaining. We state them again for completeness and to review their interpretation for unassisted bargaining. (Appendix A.2 shows that, under the first one, the other four properties are logically independent.)

A.1 (Regularity) $\succeq$ is a complete preorder, continuous and mixture independent.

This implies that the bargainers' shared ranking over pairs of probabilities is representable by a real-valued function $V : [0,1]^2 \to [0,1]$, unique up to positive affine transformations and linear with respect to $\oplus$. Nash (1950, p. 157) takes for granted a similar assumption under EU-preferences: he defines a "two-person anticipation as a combination of two one-person anticipations" and states that the two-person anticipation "will have the same linearity property".

A.2 (Non-triviality) $(1,1) \succ (0,0)$.

This states that the bargainers' ranking strictly favors an alternative that each of them believes that the counterparty accepts for sure over another alternative that each believes that the counterparty refuses for sure. Intuitively, they agree that the first proposal has a strictly higher probability of success.

A.3 (Disagreement indifference) for any $p, q$ in $[0,1]$, $(p,0) \sim (0,q)$.

This means that the bargainers rank equally all alternatives where at least one party is believed to refuse for sure. Intuitively, they agree that the identity of the refuser is irrelevant.

A.4$^w$ (Weak consistency) for any $p$ in $[0,1]$,

$$p(1,1) \oplus (1-p)(0,1) \sim (p,1) \quad \text{and} \quad p(1,1) \oplus (1-p)(1,0) \sim (1,p).$$

This is a weakening of A.4 (Consistency) in Section 2 and states the following. Assume that bargainer $i = 1,2$ is believed to accept for sure. Then a lottery where with probability $p$ the other party $j = 3-i$ accepts for sure and with probability $(1-p)$ refuses for sure is indifferent to a proposal where $j$ accepts with probability $p$. Intuitively, the first lottery has an "objective" probability $p$ of success, while the second proposal has a "subjective" probability with the same value $p$. The bargainers agree to be indifferent over these two modalities.

A.5$^w$ (Weak complementarity) for any $\mathbf{p}, \mathbf{q}$ in $[0,1]^2$,

$$(1/2)\,(\mathbf{p} \vee \mathbf{q}) \oplus (1/2)\,(\mathbf{p} \wedge \mathbf{q}) \succeq (1/2)\mathbf{p} \oplus (1/2)\mathbf{q}.$$

This is an analog of Axiom S in Francetich (2013). It states that a fifty-fifty lottery over two pairs of acceptance probabilities $\mathbf{p}$ and $\mathbf{q}$ does not rank above a fifty-fifty lottery over their extremes (under the component–wise ordering). The interpretation is the following. Suppose $p_1 \geq q_1$ and $q_2 \geq p_2$. When the individual acceptance probabilities improve from $(q_1, p_2)$ to $(p_1, p_2)$, the increase in the probability of success for a proposal cannot be greater than when they change from $(q_1, q_2)$ to $(p_1, q_2)$. Whatever advantage is gained when one bargainer raises his belief that the counterparty accepts by $p_1 - q_1$, this is made stronger when the corresponding probability assessed by the other bargainer also increases. Intuitively, the bargainers agree that their individual acceptance probabilities are (weakly) complementary with respect to the probability of success. For a particularly sharp illustration, let $p_1 = q_2 = 1$ and $p_2 = q_1 = 0$: clearly, joint acceptance occurs only at $(1, 1)$, and a fifty-fifty lottery between $(1, 1)$ and $(0, 0)$ is strictly better than a fifty-fifty lottery between $(1, 0)$ and $(0, 1)$.

Our third result gives a behavioural characterisation for the bargainers' shared ranking. Such ranking takes as input the two bargainers' individual probability assessments and delivers as output their shared evaluation for the probability of successful bargaining, via a unique copula function that merges the two assessments into a joint probability. Appendix A.1 recalls a few basic notions about copulas.

**Theorem 3.** *The shared ranking $\succeq$ satisfies A.1-2-3-$4^w$-$5^w$ if and only if there exists a unique copula $C : [0, 1]^2 \to [0, 1]$ that represents $\succeq$, in the sense that*

$$\mathbf{p} \succeq \mathbf{q} \quad \textit{if and only if} \quad C(\mathbf{p}) \geq C(\mathbf{q}).$$

This result has a straightforward interpretation. A pair $(p_1, p_2)$ of acceptance probabilities represents the beliefs held by each bargainer about his counterparty accepting the underlying alternative. If the shared bargainers' ranking satisfies A.1-2-3-$4^w$-5, they agree on a (unique) copula $C$ to compute the joint probability $C(p_1, p_2)$ of successful bargaining from the acceptance probabilities. An alternative $a$ mapping to a pair of acceptance probabilities $(P_{2(1)}(a), P_{1(2)}(a)) = (p_1, p_2)$ in $B$ is ranked by its probability of success $C(p_1, p_2)$.

Some technical comments about Theorem 3 are in order. First, it characterizes a ranking $\succeq$ that has a maximal element (and thus a procedural solution exists), because any copula is Lipschitz continuous and $B^*$ is compact; note that $B$ is not required to be convex or even connected. Second, the procedural solution may not be unique: in general, the set of maximal elements is an equivalence class of pairs of individual acceptance probabilities. Third, the theorem puts no restriction on the dependence structure between $P_{2(1)}$ and $P_{1(2)}$ because the copula is arbitrary. Fourth, the copula is linear with respect to $\oplus$, in the sense that $C(\alpha\mathbf{p} \oplus (1 - \alpha)\mathbf{q}) = \alpha C(\mathbf{p}) + (1 - \alpha)C(\mathbf{q})$; the bargainers agree to evaluate the probability of success for a lottery over $\mathbf{p}$ and $\mathbf{q}$ by assessing first their respective probabilities of success $C(\mathbf{p})$ and $C(\mathbf{q})$, and then mixing those with the same "objective" weights of the lottery.

There are stronger versions of Theorem 3. We present two of these, concerning respectively (a) the case when the shared ranking $\succeq$ respects the (strong) Pareto ordering; and (b) the case when $\succeq$ satisfies an elementary notion of fairness. These two properties are independent and may hold concurrently. Proofs are omitted, because they are trivial modifications of the proof for Theorem 3.

Because any copula $C$ is (weakly) increasing in each argument, the shared ranking $\succeq$ in Theorem 3 satisfies the (weak) Pareto ordering: if $p_1 \geq q_1$ and $p_2 \geq q_2$, then $(p_1, p_2) \succeq (q_1, q_2)$.

But it might violate the (strong) Pareto ordering for which $(p_1 - q_1)(p_2 - q_2) > 0$ implies $(p_1, p_2) \succ (q_1, q_2)$. A mild strengthening of A.5$^w$ rules out this possibility. Let $\mathbf{p} \bowtie \mathbf{q}$ denote that $\mathbf{p}$ and $\mathbf{q}$ are not comparable in the component–wise ordering; that is, $(p_1 - q_1)(p_2 - q_2) < 0$.

A.5 (Complementarity) for any $\mathbf{p} \bowtie \mathbf{q}$ in $[0, 1]^2$,

$$(1/2)\,(\mathbf{p} \vee \mathbf{q}) \oplus (1/2)\,(\mathbf{p} \wedge \mathbf{q}) \succ (1/2)\mathbf{p} \oplus (1/2)\mathbf{q}.$$

In combination with A.1-2-3-4$^w$, this property rules out "thick" indifference curves and implies that $\succeq$ is consistent with the (strong) Pareto ordering. This follows from the next result, because any strictly supermodular copula is strictly increasing in each argument.

**Theorem 4.** *The shared ranking $\succeq$ satisfies A.1-2-3-4$^w$-5 if and only if there exists a unique strictly supermodular copula $C : [0, 1]^2 \to [0, 1]$ that represents $\succeq$.*

The second variant incorporates the property that the shared ranking $\succeq$ should not be affected by the bargainers' identity.

A.6 (Anonymity) for any $p, q$ in $[0, 1]$, $(p, q) \sim (q, p)$.

This states that the shared ranking for any pair $(p, q)$ of individual acceptance probabilities is invariant to their permutation, and hence is anonymous. The following result is immediate.

**Theorem 5.** *The shared ranking $\succeq$ satisfies A.1-2-3-4$^w$-5$^w$-6 if and only if there exists a unique symmetric copula $C : [0, 1]^2 \to [0, 1]$ that represents $\succeq$.*

Theorem 3 and its two variants are characterisations of the bargainers' shared ranking as the aggregation (via a suitable copula) of their individual assessments into a joint probability of successful bargaining. Their generality leaves unspecified the dependence structure modelled by the copula. Our next three results specialize the dependence structure and recover characterizations for the analogs of the three well-known bargaining solutions: Nash, egalitarian, and (relative) utilitarian. More precisely, the Nash solution is associated with the stochastic independence of the bargainers' assessments; the other two solutions correspond to the extreme cases of maximal positive dependence and maximal negative dependence, respectively.

## The Nash solution

Theorem 1 in Section 2 states a characterization of the Nash solution for the case of mediator-assisted bargaining. We state it for unassisted bargaining and make explicit that the product copula is associated with stochastic independence. Recall A.4 (Consistency) from Section 2.

*A*.4 (Consistency) For any $p, q \in [0, 1]$

$$p(1, q) \oplus (1 - p)(0, q) \sim (p, q) \quad \text{and} \quad p(q, 1) \oplus (1 - p)(q, 0) \sim (q, p).$$

This property strenghtens A.$4^w$ (Weak consistency) and, in combination with A.1-2-3, implies A.5; see Lemma A.4 in the appendix. The following characterization encompasses Theorem 1.

**Theorem 6.** *The shared ranking $\succeq$ on $[0,1]^2$ satisfies A.1-2-3-4 if and only if it is represented by the copula $\Pi(p,q) = p \cdot q$.*

The Nash solution recommends to maximise the probability of successful bargaining when the two bargainers' assessments about their counterparties' acceptance probabilities are stochastically independent. Given that each bargainer formulates his own individual assessment about the other party, this seems a very natural requirement that explains how the ubiquity of the Nash solution is inextricably linked to the product of two values.

In particular, consider the standard Nash model based on a bargaining problem with EU-preferences that are commonly known to the bargainers. When each agent has beliefs that satisfy B.1 and B.2 as in Section 3.1, $i$'s assessment for the probability that $j$ accepts a (randomized) proposal $\alpha$ equates $j$'s expected utility. Then Theorem 6 reinterprets the product of expected utilities advocated by the Nash solution as computing the probability of successful bargaining from two independent individual assessments about the probability that the counterparty agrees to a proposal.

The analogy between expected utilities and probabilities makes it possible to shuffle them. When he claimed the equivalence between Zeuthen's and Nash's theories of bargaining, Harsanyi (1956: 149) derived Zeuthen's model by postulating that "each party can estimate correctly the probability that the other party will definitely reject a certain offer." After the equivalence had been acknowledged, Harsanyi (1962: 29) subsumed Zeuthen's and Nash's theories under the assumption that both parties "know [...] each other's attitudes towards risk".

### The egalitarian solution

In the utility–based language consistent with Nash's approach, the egalitarian solution (Kalai, 1977) recommends the point on the Pareto frontier at which utility gains from the disagreement point $d$ are equal. Given a bargaining problem with EU-preferences, the egalitarian solution is procedural because it is rationalized by the ranking associated with the function $\min\{(u_1 - d_1), (u_2 - d_2)\}$.

In the more general probability-based language, the egalitarian solution for a bargaining problem with ordinal preferences arises when the shared ranking $\succeq$ aggregates the two bargainers' probability assessments through the copula $M(p,q) = \min(p,q)$. This copula obtains when we replace A.4 (Consistency) with the following property.

A.7 (Meet indifference) for any $p, q$ in $[0,1]$, $(p, p \wedge q) \sim (p \wedge q, q)$.

This is an analog of Meet preservation in Voorneveld (2014). It states that the shared ranking is indifferent when pairs of acceptance probabilities have the same minimum value.

Intuitively, the evaluation of a pair $(p, q)$ is consistent with prioritizing the smallest value between $p$ and $q$. Clearly, A.7 implies A.3 (Disagreement indifference) and A.6 (Anonymity). The next characterization is immediate.

**Theorem 7.** *The shared ranking $\succeq$ satisfies A.1-2-4$^w$-5-7 if and only if it is represented by the copula $M(p, q) = \min(p, q)$.*

The copula $M(p, q) = \min(p, q)$ is known as the Fréchet upper bound, associated with the strongest possibile positive dependence between two marginal distributions. Thus, the egalitarian solution recommends to maximise the probability of successful bargaining when the two bargainer's assessments about their counterparties' acceptance probabilities are maximally positively dependent. Intuitively, when this occurs, the shared ranking views beliefs about bargainers' propensities to accept a deal as *perfect complements*: the probability of successful bargaining equals the smallest individual probability assessment.

### The utilitarian solution

There exist alternative formulations of the utilitarian solution for the Nash model. They share the general principle that the solution recommends an alternative that maximises the sum of utilities (or utility increments over the disagreement point). We follow Arrow (1963) and consider relative utilitarianism, based on the sum of cardinal utility functions with range normalized to the interval $[0, 1]$; see Dhillon and Mertens (1999).

When using a probability-based language, the normalisation is a natural step before mapping the utilitarian precept into the recommendation of maximising the sum of individual acceptance probabilities. We show that this recommendation is consistent with maximizing the probability of successful bargaining, when the individual probability assessments are maximally negatively dependent. Consider the following property.

A.8 (Average indifference) for any $p, q$ in $[0, 1]$, $(p, q) \sim (\frac{p+q}{2}, \frac{p+q}{2})$.

This states that the shared ranking does not change if the probability assessments move on the segment between $(p, q)$ and $(\frac{p+q}{2}, \frac{p+q}{2})$, as if the increase of one term exactly compensates the diminution of the other. Intuitively, beliefs about bargainers' propensities to accept a deal are perfect substitutes with respect to the probability of successful bargaining. Property A.8 violates A.5, but is compatible with A.5$^w$. Moreover, it implies A.6 (Anonymity) but not A.3 (Disagreement indifference). The following characterization holds.

**Theorem 8.** *The shared ranking $\succeq$ satisfies A.1-2-3-4$^w$-5$^w$-8 if and only if it is represented by the copula $W(p, q) = \max(p + q - 1, 0)$.*

The copula $W(p, q) = \max(p + q - 1, 0)$ is known as the Fréchet lower bound, associated with the strongest possibile negative dependence between two marginal distributions.

Therefore, this procedural solution recommends to maximise the probability of successful bargaining when the probability assessments are maximally negatively dependent.

It is worth noticing that the copula $W(p,q)$ is strongly Pareto increasing on the triangle above the diagonal from $(0,1)$ to $(1,0)$, but is zero on the rest of its domain. Hence, the shared ranking $\succeq$ characterized in Theorem 8 is indifferent over all pairs below the diagonal, because it acknowledges that any feasible proposal mapping to a pair $(p,q)$ below the diagonal will be refused for sure and thus it is as good as the default outcome.

### Another class of solutions

Other bargaining solutions in the literature are amenable to our copula-based approach. For example, Blackorby, Bossert, and Donaldson (1994) define a bargaining solution as the (set of) maximisers for a generalised Gini ordering corresponding to distinct levels of inequality aversion, and represented by a quasi-concave, increasing function that is linear on the rank-ordered subsets of $[0,1]^2$. In particular, for $0 \le \alpha \le 1/2$, the family of symmetric copulas

$$C_{\alpha,\alpha}(p,q) = \begin{cases} p + \frac{\alpha}{1-\alpha} q - \frac{\alpha}{1-\alpha} & \text{if } \alpha \le p \le q \le 1 \\ q + \frac{\alpha}{1-\alpha} p - \frac{\alpha}{1-\alpha} & \text{if } \alpha \le q \le p \le 1 \\ 0 & \text{otherwise} \end{cases}$$

defines a continuum of (affine) generalised Gini orderings. Blackorby et al. (1994) interpret $\alpha$ as an index of inequality aversion, whereas our approach views it as an index of stochastic dependence. This family of symmetric copulas includes the egalitarian solution for $\alpha = 0$ and the utilitarian solution for $\alpha = 1/2$. If we drop the analog of A.6 (Anonimity) assumed in Blackorby et al. (1994), this family may be embedded in a more general class of asymmetric two-parameter copulas $C_{\alpha,\beta}$ where inequality aversion or stochastic dependence take different values for $p \gtrless q$; see Nelsen (2006: Exercise 3.8).

## 5 Commentary

### Related literature

**Alternative interpretations.** Using a probability-based language, our approach interprets the product operator in the Nash solution as the consequence of an assumption of stochastic independence between acceptance probabilities. To the best of our knowledge, the utility-based literature offers two alternative interpretations for the product operator.

Roth (1979, Section I.C) frames the bargaining model as a single-person decision problem, where $i$ chooses his claim by maximising his expected utility under the assumption that the demand of the other bargainer $j$ is randomly distributed so that $j$'s utility is uniformly distributed between $u_j(d_j) = 0$ and $u_j(M_j) = 1$; see also Anbar and Kalai (1978). Then the

Nash solution emerges from the independent choices of the two bargainers. From a game-theoretic viewpoint, Roth notes that this approach implies that the two agents' expectations are not mutually consistent. On the other hand, the interpretation is based on an assumption of stochastic independence between the two agents' probability assessments.

A second interpretation is proposed by Trockel (2008). He views the Nash product as a special case for a social welfare function that aggregates the two bargainers' (normalized) utilities into a social ranking. The Nash product evaluates a pair $\mathbf{u} = (\mathbf{u_1}, \mathbf{u_2})$ by the Lebesgue measure of the set of utility pairs that are Pareto-dominated by $\mathbf{u}$. More generally, any copula in Theorem 3 may be interpreted as a social welfare function that evaluates a pair $\mathbf{u} = (\mathbf{u_1}, \mathbf{u_2})$ by a suitable (probability) measure for the feasible Pareto-dominated set.

**Social preferences.** Border and Segal (1997) provide an axiomatisation of the Nash (bargaining) solution related to our approach. The primary interpretation of their setup is that "the two bargainers hire an arbitrator to make choices for them" (p. 1) and that the arbitrator has a preference order $\succsim$ over solutions. Border and Segal (1997) view such preferences in analogy to social choice, where the arbitrator relies on her notions of fairness to formulate a decision rule for any bargaining problem. An ancillary interpretation views the axioms as guidelines that both bargainers should find acceptable before they agree to hire her. Because Border and Segal (1997) view the arbitrator's selection as binding, they highlight the normative interpretation. In our mediator-assisted bargaining, instead, we interpret the procedural solution as her search for a sensible (but not binding) recommendation; for unassisted bargaining, we follow Peters and Wakker (1991: 1787) who argue that "the agreements reached in bargaining games [...] reveal the preferences of the bargainers as a group".

When $\succeq$ is viewed as a social preference relation, one may recast the copula underlying a procedural solution as a social welfare function. A related approach was pioneered by Kaneko and Nakamura (1979), who define and characterise a Nash social welfare function that evaluates the relative increases in individuals' welfare from a state $\delta$ unanimously considered as the worst possible. Kaneko (1980) points out some key conceptual differences between a social welfare function and a bargaining solution. In simple words, their common theme is to derive a solution by maximising a function. When the set of alternatives is not convex, this may lead to set-valued extensions of the Nash solution at the cost of forfeiting its uniqueness.

**Uniqueness.** A copula-based procedural solution may not be unique. This is a consequence of the generality of our approach, that imposes no convexity assumptions on either $A$ or its image in $[0, 1]^2$ via the $P_i$'s mappings. Blackorby et al. (1994: 1162) make the same point about the generalized Gini solutions discussed above: they "are multi-valued solutions (unless attention is restricted to strictly convex problems). The main focus of cooperative bargaining theory has been the characterization of single-valued solutions. [...] relaxing this assumption enlarges the class of solutions considerably. Hence, single-valuedness is not merely an assumption of convenience but, rather, an assumption of substance." Nash (1950: 159) states bluntly: "Convexity makes [the solution] unique". It is not difficult to recover uniqueness for copula-based solutions by adding adequate richness assumptions on the domain or on the bargainers' preferences.

When the solution is not unique, there is a set of alternatives for which the probability of successful bargaining is the same. We concur with Blackorby et al. (1994) who, by analogy with the use of social choice correspondences, argue that the final (unique) recommendation should be made through a random selection. Alternatively, it would be left to the bargainers (who might have discordant preferences) to refine the set of solutions.

### The ordinal Nash solution

Rubinstein, Safra and Thomson (1992) propound a "switch from utility language to alternatives-preference language" for studying Nash bargaining. This paper advances their work using a probability-based language. In particular, they define a notion of *ordinal Nash solution* applicable to a class of bargainers' preferences larger than expected utility; when both agents maximize expected utility, the ordinal Nash solution reduces to the standard Nash solution. We show that the procedural solution based on the Nash (product) copula characterized in Theorem 6 applies to an even larger class of preferences and that it nests the ordinal Nash solution.

Rubinstein et al. (1992: 1173) consider a bargaining problem $(A, \delta; \succeq_1, \succeq_2)$ that satisfies six structural assumptions for $i = 1, 2$: (i) $A$ is a compact set; (ii) $\succeq_i$ is continuous; (iii) $a \succeq_i \delta$ for any $a$ for both $i$ with at least one $a$ such that $a \succ_i$ for both $i$; (iv) the problem is *convex*: for all $x, y \in A$ and $p \in [0, 1]$, there is $a \in A$ (common to both players) such that each player is indifferent between $a$ and the lottery $px \oplus (1 - p)y$; (v) there are no $x, y \in A$ s.t. $x \sim_i y$ for both $i$; (vi) there is a unique best choice $M_i$ in $A$ with $M_i \sim_j \delta$ for $i \neq j$.

The *ordinal Nash solution* is an alternative $a^*$ in $A$ such that, for all $p$ in $(0, 1]$ and $a$ in $A$ and $i = 1, 2$, if $pa \oplus (1-p)\delta \succ_i a^*$ then $pa^* \oplus (1-p)\delta \succeq_j a$. Under expected utility preferences over the set $\mathcal{L}(A)$ of lotteries on $A$, the ordinal Nash solution coincides with the standard Nash solution. They prove its existence for two classes of bargainers' preference relations wider than expected utility, focusing on the first class because they "do not have examples of *convex* problems" for the second one; see p. 1182. The preference relations in this first class satisfy the axiom of reduction of compound lotteries and four properties named DOM, Q, CCE and H; see pp. 1177-1178.

In our setup, bargainers have continuous ordinal preferences on $A$ but need not have preferences over $\mathcal{L}(A)$. To provide enough structure, let $L_p(a)$ denote the lottery $pa \oplus (1-p)\delta$ between an alternative $a$ in $A$ and the default outcome $\delta$. We consider the set $\mathcal{L}(A|\delta) = \{pa \oplus (1-p)\delta : a \in A, p \in [0, 1]\}$, because the ordinal Nash solution requires to deal only with lotteries involving the default outcome $\delta$. Note $\mathcal{L}(A|\delta) \subseteq \mathcal{L}(A)$. Assume that each bargainer $i$

has preferences over $\mathcal{L}(A|\delta)$ such that:

R.1: $a \sim_i L_1(a)$ for any $a$;
R.2: $qL_p(M_i) \oplus (1-q)\delta \sim_i L_{pq}(M_i)$ for any $p, q$ and $i = 1, 2$;
R.3: the sets $\{p \in [0,1] : L_p(M_i) \succeq_i L_q(a)\}$ and $\{p \in [0,1] : L_q(a) \succeq_i L_p(M_i)\}$ are closed in the usual topology for any $p, q$ in $[0,1]$.

The first property embeds $A$ in $\mathcal{L}(A|\delta)$. The second property ensures that reduction of compound lotteries applies for lotteries involving only $M_i$ and $\delta$. The third property states a notion of continuity weaker than (ii) from Rubinstein et al. (1992) and stronger than our B.4 in Section 3.

We claim that the Nash copula reduces to the ordinal Nash solution when bargainers' preferences on $\mathcal{L}(A|\delta)$ satisfy the following two relaxations of the DOM and H assumptions in Rubinstein et al. (1992). We also dispense with Q and CCE altogether, and drop the structural assumptions (iv) and (v).

D$^w$ (Stochastic dominance) for any $a$ in $A$, if $p > q$ then $L_p(a) \succ_i L_q(a)$.

H$^w$ (Homogeneity) if $a \sim_i L_q(M_i)$, then $L_p(a) \sim_i L_{pq}(M_i)$.

The next result states that the procedural solution based on the Nash copula nests the ordinal Nash solution.

**Theorem 9.** *When bargainers' preferences over $\mathcal{L}(A|\delta)$ satisfy D$^w$ and H$^w$, any outcome selected by the Nash copula is an ordinal Nash solution.*

## Testable restrictions

We describe a simple test for our copula-based approach that generates falsifiable predictions. Recall from Section 2 that a *bargaining problem* maps to a subset $B$ in $[0,1]^2$ where each point $\mathbf{p}$ in $B$ corresponds to a pair of (acceptance) probabilities. Then a *solution* is a map that for any problem $B$ selects (at least) one point in $B$. The copula-based approach recommends a solution by maximising a suitable copula $C$ over $B$.

Suppose that $B$ contains a point $\mathbf{p} = (p_1, p_2)$ with $p_1 + p_2 > 1$. The Fréchet lower bound implies $C(\mathbf{p}) \geq p_1 + p_2 - 1 > 0$ for any copula $C$. Similarly, given another feasible point $\mathbf{q} = (q_1, q_2)$, the Fréchet upper bound implies $C(\mathbf{q}) \leq \min(q_1, q_2)$. Therefore, if $\min(q_1, q_2) < p_1 + p_2 - 1$, the point $\mathbf{p}$ must be strictly preferred to $\mathbf{q}$, and $\mathbf{q}$ cannot be in the solution for any copula $C$. More generally, by picking a point $\mathbf{p}^*$ that maximises $p_1 + p_2 - 1$ in $B$, we can formulate the following more stringent test.

**Proposition 10.** *Let $\mathbf{p}^* \in \arg\max_B (p_1 + p_2 - 1)$. Define the quadrant*

$$Q := \left\{\mathbf{q} \in [0,1]^2 : \min(q_1, q_2) \geq p_1^* + p_2^* - 1\right\}.$$

*Then the solution must belong to $B \cap Q$.*

The copula-based approach implies other restrictions. Because a procedural solution must be rationalized by the maximization of a copula, we can obtain only solutions that satisfy independence of irrelevant alternatives; see Peters and Wakker (1991). In particular, the Kalai-Smorodinsky (1975) solution (for short, KS solution) cannot be obtained in our model except in special cases. Formally speaking, this solution violates A.1 as shown by the following simple example. Let $B_1$ be the convex hull in $[0,1]^2$ of the three points $(0,0)$, $(0,1)$, and $(1,0)$. Then the KS solution uniquely prescribes the point $\mathbf{p} = (1/2, 1/2)$. On the other hand, if $B_2$ is the convex hull in $[0,1]^2$ of the four points $(0,0)$, $(0,1)$, $(1/2, 1/2)$, and $(1/2, 0)$, then the KS solution uniquely prescribes the point $\mathbf{q} = (1/3, 2/3)$. As both $\mathbf{p}$ and $\mathbf{q}$ belong to $B_2 \subset B_1$, the first KS solution reveals $\mathbf{p} \succ \mathbf{q}$ while the second one reveals $\mathbf{q} \succ \mathbf{p}$, so that A.1 fails. Although it is possible to rationalize the KS solution as the outcome of a lexicographic maximisation, we find this approach awkward and we do not pursue it here.

**Comparative statics**

There is a small but elegant literature on the comparative statics of the Nash solution. For a typical result, consider Theorem 1 in Kihlstrom et al. (1981): "The utility which Nash's solution assigns to a player increases as his opponent becomes more risk averse." This result compares the solution when the utility $u_i$ of an agent is replaced by a function $v_i$ that is an increasing concave transformation of $u_i$.

The copula-based approach expands the reach of possibile interpretations. Recall from Section 3.1 that, under expected utility, the probability assessment $P_{k(i)}(a)$ by $k$ about $i$ equals the Bernoulli index $u_i(a)$ with $u_i(\delta) = 0$ and $u_i(M_i) = 1$. The same argument in Kihlstrom et al. (1981) shows that the Nash copula assigns a higher (ordinal) utility to the agent $k$ when his probability assessment $P_{k(i)}$ is replaced by an assessment $Q_{k(i)}$ that is an increasing concave transformation of $P_{k(i)}$. Because probability assessments are normalized to $[0,1]$, this implies $Q_{k(i)}(a) \geq P_{k(i)}(a)$ for any $a$. Hence, the Nash copula makes $k = 3 - i$ better off if his assessment about $i$ is more optimistic. Under expected utility, this occurs when $i$ is more risk averse. In the target-based example, this happens when $i$'s target is less demanding in the sense of first-order dominance. In general, whenever $i$ is believed to be more accommodating, the Nash copula ends up rewarding the other bargainer.

Our approach allows for a different strand of comparative statics based on (partial) orderings for copulas. We consider a simple example. Given two copulas $C_1$ and $C_2$, the *concordance* ordering states that $C_1$ is more concordant than $C_2$ if $C_1(\mathbf{p}) \geq C_2(\mathbf{p})$ for all $\mathbf{p}$ in $[0,1]^2$. Suppose that $C_1$ is more concordant than $C_2$. Given $B$, let $\mathbf{p}_i^*$ be a solution under the copula $C_i$, for $i = 1, 2$. Since

$$C_1(\mathbf{p}_1^*) \geq C_1(\mathbf{p}_2^*) \geq C_2(\mathbf{p}_2^*),$$

the joint probability of success at the solution is increasing in the concordance ordering. Agents with concordant targets are more likely to strike a deal.

**Bargaining power**

The economic literature often uses an asymmetric version of the Nash solution as a reduced form to account for differences in the bargaining power of the agents. Given a Nash bargaining problem $(S, d)$, the *asymmetric Nash solution* is defined as the maximiser of the product $(u_1 - d_1)^a (u_2 - d_2)^{1-a}$, for $a$ in $[0, 1]$. As $a$ increases from 0 to 1, the asymmetric solution increasingly favours the first agent, with symmetry holding at $a = 1/2$. The popularity of the asymmetric Nash solution is due to its technical convenience, without much concern for its foundations or interpretation; see Harsanyi and Selten (1972) for a notable exception.

Our approach allows for asymmetric copulas, but the function $N_a(p, q) = p^a q^{1-a}$ fails D.2 in Section A.1 and thus is not a copula. One may seek copulas with similar properties. For instance, by Theorem 2.1 in Liebscher (2008), for any copula $C$ the expression

$$C^a(p, q) = p^a q^{1-a} C(p^{1-a}, q^a) = N_a(p, q) \cdot C(p^{1-a}, q^a)$$

defines an asymmetric copula that bears an obvious relationship with $N_a$. However, there are limitations to this approach.

Let $\Delta_B(C) = \sup_{(p,q)\in B} |C(p, q) - C(q, p)|$ be the degree of asymmetry for a function $C$ over $B$. For example, we have $\Delta_B(N_a) = 1$ at $a = 0$ and $a = 1$: the family of asymmetric Nash solutions exhibits maximal asymmetry. When $B = [0, 1]^2$, it is known that $\Delta_B(C) \leq 1/3$ for any copula $C$ (Nelsen, 2007); moreover, $\Delta_B(C) \leq 1/5$ if $C$ is also quasi-concave (Alvoni and Papini, 2007). Therefore, no copula can mimic the extreme asymmetry allowed by $N_a$. Continuing our example, $\Delta_B(N_a) = 0$ at $a = 1/2$ and it is increasing in $|a - 1/2|$: because $\Delta_B(N_a) = 1/3$ for $|a - 1/2| \sim 0.219$, no copula can replicate $N_a$ for $a$ outside of the interval $(.281, .719)$.

There is a clear distinction between the bargaining problem $(A, \delta; \succeq_1, \succeq_2)$ and its image $B$ in $[0, 1]^2$. The acceptance probabilities map the first problem to $B$, and the choice of a copula $C$ on $B$ is a mere tool to combine the acceptance probabilities into a joint probability of success. Because bargaining power should be traced back to the bargaining problem, it does not affect the functional form of the copula. For example, consider the standard case where two agents bargain over a cake of unit size and let $A = \{(a, 1 - a) : a \in [0, 1]\}$ with $\delta = (0, 0)$. Suppose $P_{k(i)}(a) = a^{\gamma_i}$, where the parameter $\gamma_i \geq 0$ denotes the bargaining strength of $i$ and $\gamma_1 + \gamma_2 > 0$. Increasing $\gamma_i$ makes $P_{k(i)}(a)$ less accomodating; then the shape of

$$B = \left\{(p_1, p_2) \in [0, 1]^2 : \; p_1 = a^{\gamma_1} \text{ and } p_2 = (1 - a)^{\gamma_2}, \text{ for } 0 \leq a \leq 1\right\}$$

takes into account how the bargaining strengths affect the acceptance probabilities. Assuming the Nash copula, the probability of successful bargaining is $a^{\gamma_1} \cdot (1 - a)^{\gamma_2}$ and we recover the same shares $a_1^* = \gamma_1/(\gamma_1 + \gamma_2)$ and $a_2^* = \gamma_2/(\gamma_1 + \gamma_2)$ generated by the asymmetric Nash solution.

# A Appendix

## A.1 Basics on bivariate copulas

Copulas are functions that link multivariate distributions to their one-dimensional marginal distributions; see Nelsen (2006). They are used to model different forms of statistical dependence and construct families of distributions exhibiting them. We summarize some basic notions about bivariate copulas.

A (bivariate) copula is a function $C : [0,1]^2 \to [0,1]$ that satisfies two properties:

D.1) for any $p, q$ in $[0,1]$, $C(p,0) = C(0,q) = 0$, $C(p,1) = p$, and $C(1,q) = q$;
D.2) for any $p_1 > q_1$ and $p_2 > q_2$ in $[0,1]$, $C(p_1,p_2) + C(q_1,q_2) \geq C(p_1,q_2) + C(q_1,p_2)$.

Property D.2 is usually called 2-increasingness or supermodularity. We use this latter name. The combination of D.1 and D.2 implies that $C(p,q)$ is increasing in each argument; see Lemma 2.1.4 in Nelsen (2006). When the weak inequality in D.2 is replaced by a strict one, the copula $C$ is strictly supermodular and strictly increasing in each argument.

A characterization from Sklar (1959) elucidates how the copula connects a bivariate distribution to its univariate marginals.

**Theorem 11.** *Let $(X,Y)$ be a random vector with marginal distributions $F(x)$ and $G(y)$. The following are equivalent:*

i) *$H(x,y)$ is the joint distribution function of $(X,Y)$;*
ii) *there exists a copula $C(p,q)$ such that $H(x,y) = C[F(x), G(y)]$ for all $x, y$.*

*If $F(x)$ and $G(y)$ are continuous, then $C(p,q)$ is unique. Otherwise, $C(p,q)$ is uniquely defined on the cartesian product $Ran(F) \times Ran(G)$ of the ranges of the two marginal distributions. Conversely, if $C(p,q)$ is a copula and $F(x)$ and $G(y)$ are distribution functions, then the function $H(x,y)$ defined above is a joint distribution function with margins $F(x)$ and $G(y)$.*

A prominent example of a copula is the product $\Pi(p,q) = p \cdot q$ associated with stochastic independence. Other important examples are $W(p,q) = \max(p+q-1, 0)$ and $M(p,q) = \min(p,q)$. These two copulas are respectively known as the Fréchet lower and upper bound, because $W(p,q) \leq C(p,q) \leq M(p,q)$, for any copula $C$ and any $(p,q)$ in $[0,1]^2$. Intuitively, the copula $M$ describes the strongest possible positive dependence between $X$ and $Y$, given the marginal distributions $F$ and $G$. Similarly, $W$ captures the strongest possible negative dependence.

## A.2 Logical independence of A.2-3-$4^w$-$5^w$

We show that, given A.1, the four axioms A.2-3-$4^w$-$5^w$ in Theorem 3 are logically independent. Recall that A.1 implies that the preference relation $\succeq$ is representable by a real-valued function $V : [0,1]^2 \to [0,1]$, unique up to positive affine transformations and linear with respect to $\oplus$. We provide a counterexample for each property, omitting obvious quantifiers.

**A.2 (Non-triviality)** Consider $V(p,q) = k$, for some constant $k$ in $[0,1]$. Then A.3 holds because $V(p,0) = k = V(0,q)$; A.$4^w$ holds because $pV(1,1)+(1-p)V(0,1) = k = V(p,1)$, and similarly for the second relation. And A.$5^w$ holds because $(1/2)V(\mathbf{p}\vee\mathbf{q})+(1/2)V(\mathbf{p}\wedge\mathbf{q}) = k = (1/2)V(\mathbf{p})+(1/2)V(\mathbf{q})$. But A.2 does not hold because $V(1,1) = k = V(0,0)$.

**A.3 (Disagreement indifference)** Consider $V(p,q) = p$. Then A.2 holds because $V(1,1) = 1 > 0 = V(0,0)$; A.4$^w$ holds because $pV(1,1)+(1-p)V(0,1) = p = V(p,1)$, and $pV(1,1)+(1-p)V(1,0) = 1 = V(1,p)$. And A.5$^w$ holds because

$$\frac{1}{2}V(\mathbf{p} \vee \mathbf{q}) + \frac{1}{2}V(\mathbf{p} \wedge \mathbf{q}) = \frac{1}{2}(p_1 \vee q_1) + \frac{1}{2}(p_1 \wedge q_1) = \frac{1}{2}p_1 + \frac{1}{2}q_1 = \frac{1}{2}V(\mathbf{p}) + \frac{1}{2}V(\mathbf{q})$$

However, $A.3$ does not hold: for $p > 0$ and any $q$, we have $V(p,0) = p > 0 = V(0,q)$.

**A.4$^w$ (Weak consistency)** Consider $V(p,q) = p^2 q$. Then A.2 holds because $V(1,1) = 1 > 0 = V(0,0)$; A.3 holds because $V(p,0) = 0 = V(0,q)$. And A.5$^w$ holds because the first mixed derivative of $V$ is positive, and hence $V$ is supermodular. But $A.4$ does not hold because $pV(1,1)+(1-p)V(0,1) = p > p^2 = V(p,1)$.

**A.5$^w$ (Weak complementarity)** Consider the function

$$V(p,q) = \begin{cases} \min\left(p, q, \frac{1}{3}, p+q-\frac{2}{3}\right) & \text{if } \frac{2}{3} \leq p+q \leq \frac{4}{3} \\ \max(p+q-1, 0) & \text{otherwise,} \end{cases}$$

borrowed from Exercise 2.11 in Nelsen (2006). Then A.2 holds because $V(1,1) = 1 > 0 = V(0,0)$; A.3 holds because $V(p,0) = 0 = V(0,q)$. And A.4$^w$ holds because $pV(1,1)+(1-p)V(0,1) = p = V(p,1)$, and similarly for the second relation. However, A.5$^w$ does not hold: let $\mathbf{p} = (1/3, 2/3)$ and $\mathbf{q} = (2/3, 1/3)$, so that $\mathbf{p} \vee \mathbf{q} = (2/3, 2/3)$ and $\mathbf{p} \wedge \mathbf{q} = (1/3, 1/3)$. Then $V(\mathbf{p} \vee \mathbf{q}) + V(\mathbf{p} \wedge \mathbf{q}) - V(\mathbf{p}) - V(\mathbf{q}) = -1/3 < 0$, contradicting supermodularity.

## A.3 Proofs

### Proof of Theorem 2

We start with two lemmata. For simplicity, we fix $i$ throughout this proof and we drop subscripts when there is no risk of confusion; for instance, we write $\succeq$ instead of $\succeq_{k(i)}$ or $M$ instead of $M_i$.

**Lemma A.1.** *Given $p \in (0,1)$, $L_{p_n} \to L_p$ in $A \cup \mathcal{L}$ if and only if $p_n \to p$.*

*Proof.* ($\Rightarrow$) Let $\varepsilon > 0$ be such that $(p-\varepsilon, p+\varepsilon) \subset [0,1]$. By B.4, the set $I = \{a \in A \cup \mathcal{L} : L_{p+\epsilon} \succ a \succ L_{p-\epsilon}\}$ is open because it is an intersection of open sets and $L_p \in I$ by B.3. Thus $L_{p_n} \to L_p$ implies that there exists $N$ such that $L_{p_n} \in I$ for all $n \geq N$; that is, $L_{p+\epsilon} \succ L_{p_n} \succ L_{p-\epsilon}$. Therefore, B.3 implies $p - \epsilon < p_n < p + \epsilon$ for all $n \geq N$, and thus $p_n \to p$.
($\Leftarrow$) It suffices to show that for any open neighborhood $\mathcal{O}$ of $L_p$, there is $N$ such that $L_{p_n} \in \mathcal{O}$ for all $n \geq N$. Let $\mathcal{O}$ be an open neighborhood of $L_p$; then $\mathcal{O} \cap \mathcal{L}$ is open in $\mathcal{L}$ since $\mathcal{O}$ is open in the disjoint union topology. But $p_n \to p$ implies $L_{p_n} \to L_p$ in $\mathcal{L}$, because $d(L_{p_n}, L_p) = |p_n - p| \to_n 0$. Therefore, there exists $N$ such that $L_{p_n} \in \mathcal{O} \cap \mathcal{L}$ for $n \geq N$, which implies $L_{p_n} \in \mathcal{O}$. □

**Lemma A.2.** *For every $a \in A$, the set $W_a^{\mathcal{L}} = \{p \in [0,1] : L_p \prec a\}$ of strictly worse lotteries and the set $B_a^{\mathcal{L}} = \{p \in [0,1] : L_p \succ a\}$ of strictly better lotteries are open in the natural topology over $[0,1]$.*

*Proof.* Fix $a \in A$. Clearly, $W_a^{\mathcal{L}}$ is open if and only its complement $\overline{W}_a^{\mathcal{L}}$ is closed. Suppose by contradiction that $\overline{W}_a^{\mathcal{L}}$ is not closed. Then there exists a sequence $\{p_n\} \subset \overline{W}_a^{\mathcal{L}}$ such that $p_n \to p$ with $p \notin \overline{W}_a^{\mathcal{L}}$; then $p \in W_a^{\mathcal{L}}$ and thus $L_p \prec a$. However, $L_{p_n} \succsim a$ for all $n$ and, by Lemma A.1, $L_{p_n} \to L_p$ on $A \cup \mathcal{L}$. Hence, the set $W_a = \{y \in A \cup \mathcal{L} : y \succsim a\}$ is not closed in $A \cup \mathcal{L}$, contradicting B.4. The proof for $B_a^{\mathcal{L}}$ is analogous. □

**Theorem 2.** *Suppose that the preorder $\succeq_{k(i)}$ satisfies B.2-3-4. Then the benchmarking procedure uniquely extends the assessment $\hat{P}_{k(i)}$ on $\mathcal{L}_i$ to a continuous assessment $P_{k(i)}$ on $A \cup \mathcal{L}_i$ with $P_{k(i)}(\delta) = 0$ and $P_{k(i)}(M_i) = 1$, that represents $\succeq_{k(i)}$. Moreover, the restriction of $P_{k(i)}$ to $A$ is continuous and, if B.1 holds, also represents $\succeq_i$.*

*Proof.* We prove first that, for every $a$ in $A$, there exists a unique value $p$ such that $a \sim L_p$. Given $a$, the two open subsets $W_a^{\mathcal{L}}$ and $B_a^{\mathcal{L}}$ in $[0,1]$ are disjoint; therefore, the set $[0,1] \setminus \left[W_a^{\mathcal{L}} \cup B_a^{\mathcal{L}}\right]$ cannot be empty because otherwise the connected interval $[0,1]$ would be covered by the disjoint union of two open sets. We pick a value $p$ from $[0,1] \setminus \left[W_a^{\mathcal{L}} \cup B_a^{\mathcal{L}}\right]$. Since $\succeq$ is a total preorder, we have $a \sim L_p$.

Second, we show that such $p$ is unique. Suppose $[0,1] \setminus \left[W_a^{\mathcal{L}} \cup B_a^{\mathcal{L}}\right]$ contains two values $p > q$. By B.3, we would have $a \sim L_p \succ L_q \sim a$, contradicting the assumption that $\succeq$ is a preorder.

Define the extension $P$ on $A \cup \mathcal{L}$ by $P = \hat{P}$ on $\mathcal{L}$ and $P(a) = p$ when $a \sim L_p$ on $A$. We show that $P$ represents $\succeq$ on $A \cup \mathcal{L}$. Choose $x \succ y$ from $A \cup \mathcal{L}$: if $P(y) \geq P(x)$, then B.3 would imply the contradiction $y \sim L_{P(y)} \succeq L_{P(x)} \sim x$. (Some equivalences in the last sentence hold as equalities if $x$ or $y$ are in $\mathcal{L}$.)

Finally, we prove that $P$ is continuous on $A \cup \mathcal{L}$. It suffices to show that the sets $W_\alpha = \{x \in A \cup \mathcal{L} : P(x) < \alpha\}$ and $B_\alpha = \{x \in A \cup \mathcal{L} : P(x) > \alpha\}$ are open for every $\alpha$ in $[0,1]$. Consider only $W_\alpha$, because the argument for $B_\alpha$ is analogous. Clearly, $W_0 = \emptyset$ is open. So assume $\alpha$ in $(0,1]$. Then

$$W_\alpha = \{x \in A \cup \mathcal{L} : P(x) < \alpha\} = \{x \in A \cup \mathcal{L} : P(x) < P(L_\alpha)\} = \{x \in A \cup \mathcal{L} : x \prec L_\alpha\}$$

is an open set in $A \cup \mathcal{L}$ by B.4. □

## Proof of Theorem 3

**Theorem 3.** *The shared ranking $\succeq$ satisfies A.1-2-3-4$^w$-5$^w$ if and only if there exists a unique copula $C : [0,1]^2 \to [0,1]$ that represents $\succeq$, in the sense that*

$$\mathbf{p} \succeq \mathbf{q} \quad \text{if and only if} \quad C(\mathbf{p}) \geq C(\mathbf{q})$$

*Proof.* Necessity is obvious. We prove sufficiency. By A.1, the Mixture Space Theorem (Herstein and Milnor, 1953) implies that there exists a unique (up to positive affine transformations) function $V : [0,1]^2 \to \mathbb{R}$ that represents $\succeq$ and is linear with respect to $\oplus$. By A.2, $V(1,1) - V(0,0) > 0$. Apply the appropriate positive affine transformation and consider the (unique) function $C : [0,1]^2 \to [0,1]$ defined by

$$C(p,q) = \frac{V(p,q) - V(0,0)}{V(1,1) - V(0,0)}.$$

We show that $C$ satisfies the two properties D.1-2 given in Appendix A.1.

By A.3, we have $C(p,0) = C(0,p) = C(0,0) = 0$, for any $p$ in $[0,1]$. Moreover, clearly $C(1,1) = 1$. By A.4$^w$ and linearity, for any $p$ in $[0,1]$, we get $C(p,1) = C(p(1,1) \oplus (1-p)(0,1)) = pC(1,1) + (1-p)C(0,1) = p$; a similar argument shows that $C(1,p) = p$. This proves D.1.

By A.5$^w$ and the linearity of $C$, for all $\mathbf{p}, \mathbf{q}$ in $[0,1]^2$, it follows that

$$\frac{1}{2}C(\mathbf{p} \vee \mathbf{q}) + \frac{1}{2}C(\mathbf{p} \wedge \mathbf{q}) = C\left(\frac{1}{2}(\mathbf{p} \vee \mathbf{q}) \oplus \frac{1}{2}(\mathbf{p} \wedge \mathbf{q})\right) \geq C(\frac{1}{2}\mathbf{p} \oplus \frac{1}{2}\mathbf{q}) = \frac{1}{2}C(\mathbf{p}) + \frac{1}{2}C(\mathbf{q}),$$

so $C$ is supermodular and D.2 holds. $\square$

### Proof of Theorem 6

We begin with two lemmata.

**Lemma A.3.** *Suppose that the shared ranking $\succeq$ satisfies A.1-2-3-4. Then*

$$(p, q) \sim pq(1,1) \oplus (1 - pq)(0,0)$$

*for all $(p, q)$ in* $[0,1]^2$.

*Proof.* By A.4, $(p, q) \sim p(1, q) \oplus (1-p)(0, q)$ and $(1, q) \sim q(1,1) \oplus (1-q)(1,0)$. By A.3, $(0, q) \sim (1, 0) \sim (0, 0)$. Using A.1 (mixture independence) repeatedly, we have $(p, q) \sim p\,(q(1,1) \oplus (1-q)(0,0)) \oplus (1-p)(0,0) \sim pq(1,1) \oplus (1-pq)(0,0)$. $\square$

**Lemma A.4.** *If the shared ranking $\succeq$ satisfies A.1-2-3-4, then it satisfies A.5.*

*Proof.* Consider A.5. Let $\mathbf{p} \bowtie \mathbf{q}$ in $[0,1]^2$. Assume without loss of generality $p_1 > q_1$ and $p_2 < q_2$. Using Lemma A.3 and the reduction of compound lotteries implied by mixture independence (see e.g. Property M5 in Fishburn, 1982: 11), we have

$$\begin{aligned}
\frac{1}{2}(\mathbf{p} \vee \mathbf{q}) \oplus \frac{1}{2}(\mathbf{p} \wedge \mathbf{q}) &\sim \frac{1}{2}(p_1, q_2) \oplus \frac{1}{2}(q_1, p_2) \\
&\sim \frac{1}{2}\left[p_1q_2(1,1) \oplus (1 - p_1q_2)(0,0)\right] \oplus \frac{1}{2}\left[q_1p_2(1,1) \oplus (1 - q_1p_2)(0,0)\right] \\
&\sim \left(\frac{p_1q_2 + q_1p_2}{2}\right)(1,1) \oplus \left(1 - \frac{p_1q_2 + q_1p_2}{2}\right)(0,0).
\end{aligned}$$

Similarly, we find

$$\frac{1}{2}\mathbf{p} \oplus \frac{1}{2}\mathbf{q} \sim \left(\frac{p_1p_2 + q_1q_2}{2}\right)(1,1) \oplus \left(1 - \frac{p_1p_2 + q_1q_2}{2}\right)(0,0).$$

Moreover, $(p_1 - q_1)(q_2 - p_2) > 0$ implies $p_1q_2 + q_1p_2 > p_1p_2 + q_1q_2$. By A.2, we apply Theorem 4 in Herstein and Milnor (1953: 295) to obtain

$$\left(\frac{p_1q_2 + q_1p_2}{2}\right)(1,1) \oplus \left(1 - \frac{p_1q_2 + q_1p_2}{2}\right)(0,0) \succ \left(\frac{p_1p_2 + q_1q_2}{2}\right)(1,1) \oplus \left(1 - \frac{p_1p_2 + q_1q_2}{2}\right)(0,0)$$

and thus

$$\frac{1}{2}(\mathbf{p} \vee \mathbf{q}) \oplus \frac{1}{2}(\mathbf{p} \wedge \mathbf{q}) \succ \frac{1}{2}\mathbf{p} \oplus \frac{1}{2}\mathbf{q}.$$

$\square$

**Theorem 6.** *The shared ranking $\succeq$ on $[0,1]^2$ satisfies A.1-2-3-4 if and only if it is represented by the copula $\Pi(p, q) = p \cdot q$.*

*Proof.* Necessity is obvious. We prove sufficiency. Clearly, A.4 implies A.$4^w$. Moreover, A.1-2-3-4 implies also A.5 by Lemma A.4. Then Theorem 3 implies that $\succeq$ is represented by a unique copula $C(p,q)$. Using D.1 from A.1, $C(1,q) = q$ and $C(0,q) = 0$ for any $q$ in $[0,1]$. Because A.4 (Consistency) requires

$$p(1,q) \oplus (1-p)(0,q) \sim (p,q)$$

for any $p,q$ in $[0,1]$, we obtain $C(p,q) = pC(1,q) + (1-p)C(0,q) = pq$. □

## Proof of Theorem 8

**Theorem 8.** *The shared ranking $\succeq$ satisfies A.1-2-3-$4^w$-$5^w$-8 if and only if it is represented by the copula $W(p,q) = \max(p+q-1,0)$.*

*Proof.* Necessity is obvious. We prove sufficiency. Theorem 3 implies that $\succeq$ is represented by a unique copula $C(p_1,p_2)$. Because it is known that $W(p_1,p_2)$ is the only quasi-convex copula (see f.i. Example 3.27 in Nelsen, 2006), it suffices to show that A.8 implies $C(\alpha\mathbf{p} + (1-\alpha)\mathbf{q}) \leq \max\{C(\mathbf{p}), C(\mathbf{q})\}$ for any $\alpha$ in $(0,1)$ and $\mathbf{p},\mathbf{q}$ in $[0,1]^2$.

Assume without loss of generality $q_1 + q_2 \leq p_1 + p_2$. Given $\mathbf{p} = (p_1,p_2)$, let $\overline{\mathbf{p}} = \left(\frac{p_1+p_2}{2}, \frac{p_1+p_2}{2}\right)$ denote its symmetrization lying on the main diagonal. By A.8, $\mathbf{p} \sim \overline{\mathbf{p}}$ implies $C(\mathbf{p}) = C(\overline{\mathbf{p}})$. Because $C(p_1,p_2)$ is increasing on the main diagonal, we obtain

$$C\left(\alpha\mathbf{p} + (1-\alpha)\mathbf{q}\right) = C\left(\overline{\alpha\mathbf{p} + (1-\alpha)\mathbf{q}}\right) \leq C\left(\overline{\mathbf{p}}\right) = C(\mathbf{p}),$$

and thus $C$ is quasi-convex. □

## Proof of Theorem 9

We begin with two lemmata. The first lemma states the existence of a function $P_i$ on $\mathcal{L}(A|\delta)$ that represents $i$'s preferences. Note that, because $k$'s assessment $P_{k(i)}$ represents $i$'s preferences on $\mathcal{L}(A|\delta)$, $P_{k(i)} = P_i$. After noting that R.3 assumes the content of Lemma A.2, R.1 implies B.2 and $D^w$ implies B.3, the proof is analogous to Theorem 2.

**Lemma A.5.** *The preference relation of bargainer $i$, $\succeq_i$, satisfies R.1, R.3, and $D^w$ if and only if there is a function $P_i$ that represents $\succeq_i$ on $\mathcal{L}(A|\delta)$ such that $P_i(\delta) = 0$, $P_i(M_i) = 1$, and $P_i$ is linear w.r.t. $\oplus$ over $\mathcal{L}_i$.*

**Lemma A.6.** *Under $H^w$, the function $P_i$ is linear w.r.t. $\oplus$ over $\mathcal{L}(A|\delta)$.*

*Proof.* For $a$ in $A$ and $p$ in $[0,1]$, we need to show that $P_i(L_p(a)) = pP_i(a) + (1-p)P_i(\delta) = pP_i(a)$. By the linearity of $P_i$ over $\mathcal{L}_i$, $a \sim_i L_{P_i(a)}(M_i)$. By $H^w$, $L_p(a) \sim_i L_{pP_i(a)}(M_i)$. Write $P$ for $P_i(L_p(a))$ and consider the lottery $PM_i \oplus (1-P)\delta$ in $\mathcal{L}_i$. By the linearity of $P_i$ over $\mathcal{L}_i$, $P_i\left[PM_i \oplus (1-P)\delta\right] = P$. Because $P_i(\cdot)$ represents $\succeq_i$ on $\mathcal{L}(A|\delta)$, this implies $L_P(M_i) \sim_i L_p(a)$. By transitivity this implies $L_P(M_i) \sim_i L_{pP}(M_i)$. Finally, $\text{DOM}^w$ yields $P_i(L_p(a)) = pP_i(a)$. □

**Theorem 9.** *Under $D^w$ and $H^w$, any outcome selected by the Nash copula is an ordinal Nash solution.*

*Proof.* Let $a^* \in \arg\max_{a \in A} P_1(a)P_2(a)$; that is, $P_1(a^*)P_2(a^*) \geq P_1(a)P_2(a)$ for any $a$ in $A$. Note that $P_i(a^*) \geq P_i(\delta) = 0$ by (iii). Let $p$ in $(0,1]$ and $a$ in $A$. Suppose $L_p(a) \succ_1 a^*$. Because $P_1$ represents $\succeq_1$ by Lemma A.5 and it is linear on $\mathcal{L}(A|\delta)$ by Lemma A.6, we have $pP_1(a) > P_1(a^*) \geq 0$ and thus $P_1(a) > 0$. Hence,

$$pP_1(a)P_2(a^*) > P_1(a^*)P_2(a^*) \geq P_1(a)P_2(a)$$

implies $pP_2(a^*) > P_2(a)$, and thus $L_p(a^*) \succ_2 a$. □

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